Rectilinear congruences and Bäcklund transformations: roots of the soliton theory

Robert Prus and Antoni Sym

Institute of Theoretical Physics, Warsaw University, ul. Hoża 69, 00-681 Warsaw, Poland
e-mail: robert@fuw.edu.pl, sym@fuw.edu.pl

Abstract: It is shown that one can speak of the "pre-theory" of solitons. And it is Luigi Bianchi who is a creator of this "pre-theory".

1 "Pre-history" and "pre-theory" of solitons

It is widely accepted that theory (and a great career) of solitons begun in 1965 with the paper [1] by Martin Kruskal and Norman Zabusky. There exists, however, a "pre-history" of the soliton research, i.e. a collection of the various mathematical results obtained in the period 1860–1960 which could be now identified as particular cases of the general soliton setting (see the contribution by Markus Heyerhoff).

It is quite remarkable that a lot of the results of modern soliton theory can be found in the old text-books on differential geometry. Besides monumental "operae" by Gaston Darboux [2] and by Luigi Bianchi [3] one could mention other important works in this context [4–9]. No doubt, a study of old differential geometry is like discovering of remnants of the past civilizations. In both cases one can learn a lot and in both cases one changes views of the past: from disregard to admiration. In particular, a recent progress in the difference (discrete) geometry initiated in the paper by A. Doliwa and P. Santini [10] has been mostly inspired by [4] and [9] (see, for instance, [11] and [12]).

Certainly, apart from the above-mentioned text-books there are numerous old papers containing perfectly soliton-theoretic results. E.g. one can cite a sequence of papers by the outstanding Roumanian geometer Georges Tzitzéica (1873–1939) [3–5] devoted to one of the standard soliton equation known as the Dodd–Bullough equation [16].

Such being the case, one can raise the question: can one speak of "pre-theory" of solitons, i.e. of an explicitly formulated idea of the old geometry which has been successfully applied in the past to generate soliton-theoretic results?

To our best knowledge the answer to this question is positive. More explicitly, we claim that *the only researcher of the past who consciously practiced soliton theory long before Kruskal and Zabusky coined the term "soliton" was a great Italian geometer Luigi Bianchi (1856–1928).*

Guido Fubini — outstanding Italian mathematician and in some sense Bianchi's successor — in the paper "Luigi Bianchi e la Sua Opera Scientifica" ("Luigi Bianchi and His Scientific Work") [17] writes (p. 50):

> Certainly, a great part of the Bianchi's work is based on two following principles: transformations and a theorem of permutability.

In the sequel Fubini lists the transformations. In fact, a major part of this commemoration paper is devoted to a discussion of those transformations. In particular Fubini discusses the so-called a) asymptotic transformations, b) Ribaucour transformations, and c) transformations of Weingarten systems. Frankly, any of the mentioned-above classes of transformations deserve a separate study and thus it is rather out of place to define them now. *Instead, we point out that all the "Bianchi transformations" discussed by Fubini are particular cases of what is called "Bäcklund transformations" in the modern soliton theory [18].* Let us repeat after Fubini that the "permutability property" was a guiding principle for Bianchi in his search for interesting transformations. *This is the main point of our discussion. Today we are perfectly aware that the same "permutability property" is the most characteristic feature of the Bäcklund transformations — one of the most important ingredients of the soliton theory.*

An important comment is in order. In fact the Bäcklund transformations can be defined on different levels [19]:

1. on the level of the so called "scattering data" (the simplest formulation),
2. on the level of the so called "wave functions" (in this case they are known as "Darboux transformations" [18]),
3. on the level of soliton solutions, and
4. on the level of the so called "soliton surfaces".

Almost all the "Bianchi transformations" are the Bäcklund transformations defined on the geometric level of soliton surfaces [20].

There is, however, yet another question to be answered. *Why did Bianchi select as a guiding principle the permutability property?* The answer is relatively simple. One of the earliest (1892) soliton-theoretic results of Bianchi (which is still very much impressive) is the so called "permutability theorem" formulated in the case of the simplest asymptotic transformations [21]. Under specific conditions the permutability property of the underlying transformations implies that one can integrate the associated *nonlinear* and soliton, of course, equation (now identified as the "sine-Gordon" equation (21)) by means of purely *algebraic* operations! This pioneering Bianchi's procedure is standard one in the modern soliton theory. *Later, Bianchi was able to prove other "permutability theorems" related to other "abelian" transformations thus establishing what one could call a "pre-theory of solitons".*

The aim of this paper is to outline a pathway which ultimately led Bianchi to his great discoveries of 1892.

2 Introduction to congruences

In "Differential Line Geometry" one studies a manifold (and its submanifolds) of all straight lines of a given affine (projective) n-space ($n \geq 2$). The subject has a long history and today its importance in differential geometry is rather marginal. Surprisingly enough, the line geometry offers an interesting setting to treat properly Bäcklund transformations! Presumably, the first to notice this possibility was Yu. A. Aminov [22], see also [23].

A few historical comments are in order. One of the most important results of the early Projective Geometry was "duality principle" (Joseph Gergonne) which states that on projective plane points and straight lines are dual (equivalent in some sense) objects. We recall that the projective plane P^2 is an ordinary (affine) plane A^2 supplemented by a line "at infinity". On using any affine coordinates (x, y) on A^2 one can write the equation of a straight line in A^2 as follows:

$$a\,x + b\,y + c = 0 \tag{1}$$

($a = 0$, $b = 0$, $c = 0$ case is excluded), where $a, b, c \in \mathbb{R}$ are constants. Next, one introduces new (seemingly crazy) coordinates (ξ, η, ζ) by $x = \xi/\zeta$ and $y = \eta/\zeta$. Equation (1) is replaced by:

$$a\,\xi + b\,\eta + c\,\zeta = 0. \tag{2}$$

In equation (2) (a, b, c) are interpreted as coordinates (defined modulo non-zero factor) of the straight line. At the same time (ξ, η, ζ) are coordinates of the point of A^2 (again defined modulo non-zero factor). Observe that points and straight lines enter into equation (2) on equal footing. This is the simplest (due to Julius Plücker) argument in favour of the duality principle.

In other words (at least within P^2) one can attempt to replace points by straight lines as the most elementary objects of the space. In fact, J. Plücker was the first (1868) to take the straight line as fundamental element of the space.

Consider now the totality LA^3 of all straight lines in ordinary (affine) space A^3. According to Plücker's idea this set deserves a special study. Similarly, one can study the totality LP^3 of all straight lines in 3-dimensional projective space P^3. In general, one can define LA^n (LP^n) spaces $(n \geq 2)$.

One can show that LA^3 space is in on-to-one correspondence with TP^2 — a tangent bundle of projective plane. This bijection converts LA^3 into differential manifold. In particular, dim $LA^3 = 4$. Similarly, dim $LP^3 = 4$ and LP^3 is diffeomorphic to the so called Klein quadric in P^5 [24].

In both cases one can further study 1- , 2- , and 3- dimensional submanifolds of LA^3 (LP^3) -manifolds. According to Plücker's terminology:

- 2-dimensional submanifolds of LA^3 (LP^3) are called "congruences", and
- 3-dimensional submanifolds of LA^3 (LP^3) are called "complexes".

In other (more familiar) words, a "congruence" is a 2-parameter family of straight lines in A^3 (P^3). We complete this terminology pointing out that 1-parameter family of straight lines can be identified with a "ruled surface" of the classical differential geometry.

3 Kummer's differential geometry of congruences

In the sequel we shall assume the Reader is familiar with standard notions of the classical differential geometry. The classical differential geometry of congruences ([5], [25] and [26]) is less familiar. For the Reader's convenience below we collect the basic notions and results of this theory. The first general setting to treat congruences from *differential standpoint* has been given by E. Kummer in 1860. In a sense we follow Kummer.

$\Omega \subset \mathbb{R}^2$ is an open set. $(x^1, x^2) \equiv (x^\alpha) \in \Omega$. Given two smooth functions $\mathbf{r} : \Omega \to V^3$ (V^3 is a vector space of A^3) and $\mathbf{m} : \Omega \to S^2$ (2-dim. sphere). These define a congruence by:

$$\begin{aligned} (x^1, x^2) \quad &\mapsto \text{ a straight line given parametrically as} \\ & t \mapsto \mathbf{r}(x^1, x^2) + t\ \mathbf{m}(x^1, x^2) \end{aligned} \tag{3}$$

For simplicity it is called "(x^1, x^2)-ray".

A parametric surface $(x^1, x^2) \mapsto \mathbf{r}(x^1, x^2)$ is called a "reference surface" of the congruence. Observe a **non-uniqueness** of this representation of the congruence. One can always replace $\mathbf{r}(x^1, x^2)$ by $\mathbf{r}(x^1, x^2) - \lambda(x^1, x^2)\ \mathbf{m}(x^1, x^2)$ (λ - scalar function). For some reasons we put $-\lambda(x^1, x^2)$ instead of $\lambda(x^1, x^2)$.

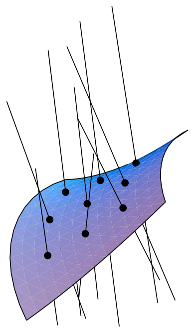

Fig. 1: A congruence and its reference surface.

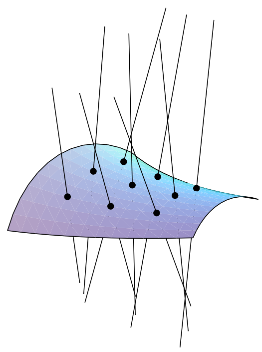

Fig. 2: A normal congruence and its reference surface (normals belong to congruence).

A function $I : \Omega \to \mathbb{R}$ which is invariant with respect to both changes: a change of coordinates (x^α) and a change of the reference surface is called a "congruence invariant".

We introduce two $(0, 2)$ - tensor fields (2 - covariant tensor fields) on Ω:

$$G_{\alpha\beta} := \mathbf{m}_{,\alpha} \cdot \mathbf{m}_{,\beta} \tag{4}$$

$$B_{\alpha\beta} := \mathbf{r}_{,\alpha} \cdot \mathbf{m}_{,\beta} \tag{5}$$

where dot "$\cdot$" denotes Euclidean scalar product and comma means partial differentiation. $G_{\alpha\beta}$ is symmetric while $B_{\alpha\beta}$ is not in general ($B_{(\alpha\beta)}$ is a symmetric part of $B_{\alpha\beta}$). They do depend on "reference surface"!

We put $G = \det\, G_{\alpha\beta}$ $(G > 0)$, $B = \det\, B_{(\alpha\beta)}$ and $\overline{B} = \det\, B_{\alpha\beta}$. The inverse matrix of $G_{\alpha\beta}$ is denoted by $G^{\alpha\beta}$. It is indeed a $(2,0)$-tensor field. Next we put $S = B_{12} - B_{21}$ and $T = \frac{1}{2}\, B_{\alpha\beta}\, G^{\alpha\beta}$.

Now we are in a position to formulate the following fact:

$$I_1 = T^2 - B/G \geq 0 \quad \text{and} \quad I_2 = T^2 - \overline{B}/G \tag{6}$$

are both the congruence invariants.

Consider two polynomials of the second degree:

$$p(\lambda) = B/G - 2\, T\lambda + \lambda^2 \quad \text{and} \quad \overline{p}(\lambda) = \overline{B}/G - 2\, T\lambda + \lambda^2. \tag{7}$$

Let us fix a value $p_0 \in \mathbb{R}$ and a ray (x^1, x^2). Let $p(\lambda) = p_0$ and $\overline{p}(\overline{\lambda}) = p_0$. The points of (x^1, x^2)-ray defined by their position vectors:

$$\mathbf{r}(x^1, x^2) - \lambda\, \mathbf{m}(x^1, x^2) \quad \text{and} \quad \mathbf{r}(x^1, x^2) - \overline{\lambda}\, \mathbf{m}(x^1, x^2) \tag{8}$$

are invariant with respect of both changes: change of coordinates and change of reference surface. I.e. they are defined by the congruence exclusively.

This result enables one to define important notions of the "limit points" and "focus points" along a given ray.

The limit points of a given ray are defined by their position vectors:

$$\mathbf{r}(x^1, x^2) - \lambda_i\, \mathbf{m}(x^1, x^2), \qquad (i = 1, 2) \tag{9}$$

where $\lambda_1 \geq \lambda_2$ are roots of the polynomial $p(\lambda)$. Since $I_1 \geq 0$ limit points always exist.

The focus points (or foci) of a given ray are defined by their position vectors:

$$\mathbf{r}(x^1, x^2) - \mu_i\, \mathbf{m}(x^1, x^2), \qquad (i = 1, 2) \tag{10}$$

where $\mu_1 \geq \mu_2$ are roots (provided they are real!) of the polynomial $\overline{p}(\lambda)$. One can show that:

$$I_2 > 0 \quad \Rightarrow \quad \text{there exist two foci,} \tag{11}$$

$$I_2 = 0 \quad \Rightarrow \quad \text{there exists a single focus,} \tag{12}$$

$$I_2 < 0 \quad \Rightarrow \quad \text{no foci.} \tag{13}$$

A generic configuration of limit points and foci of given ray is shown below.

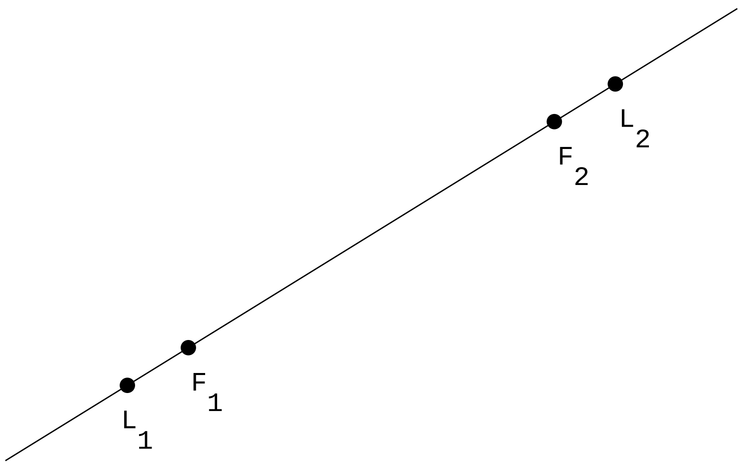

Fig. 3: Limit points (L_i) and foci (F_i) along a given ray. Observe the symmetry of configuration: both L_1L_2 and F_1F_2 have a middle point in common.

From now on, we shall assume that $I_2 > 0$ (two foci exists along any ray of the congruence).

One can show that:

$$|L_1L_2|^2 = 4I_1, \tag{14}$$
$$|F_1F_2|^2 = 4I_2. \tag{15}$$

We have also a general formula (confront Fig. 3):

$$I_2 = I_1 - \frac{1}{4}S^2/G. \tag{16}$$

It is not difficult to prove (W. R. Hamilton, 1830) that $S = 0$ if and only if a congruence is *normal*, i.e. its rays are normals to some surface (Fig. 2).

Thus, for a normal congruence foci always exist since they coincide with limit points.

In general, a totality of F_1-points (F_2-points) constitutes a parametric surface which is called "F_1-focal surface" ("F_2-focal surface") and is denoted by $\mathcal{F}_1$ ($\mathcal{F}_2$) respectively.

The essential fact is that any ray is tangent to both focal surfaces at its foci.

Hence, we have the following configuration ($\mathbb{T}_{F_1}\mathcal{F}_1$ and $\mathbb{T}_{F_2}\mathcal{F}_2$ are tangent planes to $\mathcal{F}_1$ ($\mathcal{F}_1$) at F_1 (F_2)):

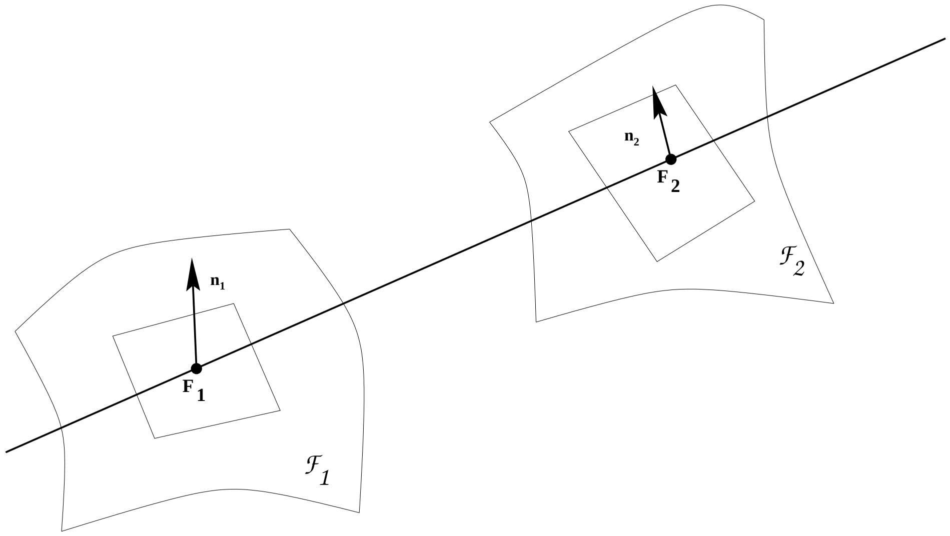

Fig. 4: A standard configuration of focal surfaces $(\mathcal{F}_i)$ and their tangent planes. $\mathbf{n}_i$ are unit normal vectors to $\mathcal{F}_i$ at F_i.

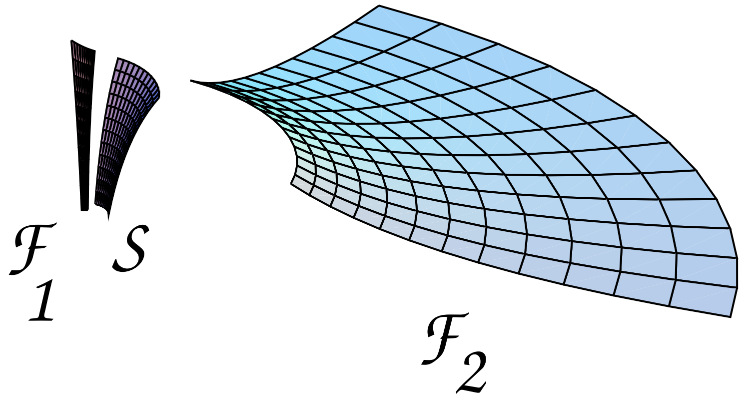

Fig. 5: A particular case of focal surfaces $(\mathcal{F}_i)$ for normal congruence defined by S.

From the Figure 4 we infer that $\mathbf{n}_1 \cdot \mathbf{n}_2 = \cos\sigma$ defines an angle σ between two tangent planes $\mathbb{T}_{F_1}\mathcal{F}_1$ and $\mathbb{T}_{F_2}\mathcal{F}_2$. One can prove the following important formula:

$$\sin\sigma = \frac{|F_1F_2|}{|L_1L_2|}. \tag{17}$$

Obviously, for any normal congruence $\mathbb{T}_{F_1}\mathcal{F}_1$ and $\mathbb{T}_{F_2}\mathcal{F}_2$ are orthogonal.

It turns out that many important results of the more advanced theory of congruences have their roots in the earlier theory of normal congruences. E.g. on selecting a surface which defines a normal congruence as a reference surface one can show that its focal surfaces are given by formulae (10) where:

$$\mu_i = -1/k_i, \qquad (i = 1, 2) \tag{18}$$

and k_i are the so called *principal curvatures* at a point $\mathbf{r}(x^1, x^2)$.

4 Bianchi's pathway

Here we describe chronology of mathematical events of the second half of XIX century which ultimately led to the Bianchi's formulation [21] of the Bäcklund transformation of the so called "sine-Gordon" equation (25) which in a sense encodes a geometry of pseudospherical surfaces in E^3.

a) In 1861 J. Weingarten introduces a class of surfaces in E^3 ("Weingarten or W-surfaces") defined by the condition that each of the principal curvatures is a function of the other. This category of surfaces includes in particular minimal surfaces, surfaces of constant mean curvature, pseudospherical surfaces, etc.

b) Obviously, any congruence defines two maps $\mathcal{F}_1 \xrightarrow{\beta} \mathcal{F}_2$ and its inverse $\mathcal{F}_2 \xrightarrow{\beta^{-1}} \mathcal{F}_1$. In 1872 A. Ribaucour proves the following two theorems on *normal* congruences:

   I) β maps asymptotic lines of $\mathcal{F}_1$ onto asymptotic lines of $\mathcal{F}_2 \Leftrightarrow$ A surface defining a normal congruence is W - surface.

   II) β maps curvature lines of $\mathcal{F}_1$ onto curvature lines of $\mathcal{F}_2 \Leftrightarrow$ A surface defining a normal congruence is a special W-surface for which the constraint $1/k_1 - 1/k_2 = c = const.$ holds. Moreover, in this case both focal surfaces $\mathcal{F}_i$ are pseudospherical surfaces of Gaussian curvature $K = -1/c^2$.

c) It is not clear whether L. Bianchi was aware of the Ribaucour results. In any case, in his habilitation thesis [27] he introduces a notion of "complementary surfaces" which is nothing else but a couple of focal surfaces of some normal congruences and the major part of the thesis is devoted to study a map $\beta : \mathcal{F}_1 \to \mathcal{F}_2$ where both $\mathcal{F}_1$ and $\mathcal{F}_2$ are pseudospherical (the second Ribaucour theorem).

d) In 1883 two important results have been obtained. Firstly, G. Darboux in a sequence of short notes [28] discusses an analytic counterpart of "complementary transformation". In Bianchi's formulation it assumes a form:

$$\varphi_{,u} + \theta_{,v} = \sin\varphi\cos\theta, \tag{19}$$
$$\varphi_{,v} + \theta_{,u} = -\cos\varphi\sin\theta. \tag{20}$$

G. Darboux shows that if $\theta(u, v)$ satisfies the "sine-Gordon" equation:

$$\theta_{,uu} - \theta_{,vv} = \sin\theta\cos\theta \tag{21}$$

then φ defined (modulo a constant of integration) by (19–20) is also a solution of the "sine-Gordon" equation.

We point out: it was neither Bianchi nor Bäcklund who was the *first* to write down the (special) Bäcklund transformation for the "sine-Gordon" equation. It was a great French geometer Gaston Darboux (1847–1917).

In the same 1883 A. V. Bäcklund (1845–1922) published a paper which contains a generalization of the Bianchi's complementary transformations of a given pseudospherical surface into another one (of the same constant negative Gaussian curvature). Bäcklund's generalization contains a real *parameter* [29].

e) In a very important paper of 1885 [30] Bianchi gave analytic formulation of Bäcklund (geometric) transformation which is now known under the name "Bäcklund transformation for the sine-Gordon equation". It reads:

$$\begin{aligned} \varphi_{,u} + \theta_{,v} &= (\sin\varphi\cos\theta + \sin\sigma\cos\varphi\sin\theta)/\cos\sigma, & (22) \\ \varphi_{,v} + \theta_{,u} &= -(\cos\varphi\sin\theta + \sin\sigma\sin\varphi\cos\theta)/\cos\sigma. & (23) \end{aligned}$$

Obviously, $\sigma = 0$ corresponds to Darboux formulae (19–20).

The following diagram summarizes the chronology of the mathematical results leading finally to the important formulae (22–23):

	Level of pseudospherical surfaces	Level of the solutions to the sine - Gordon equation
$\sigma = 0$	1879 L. Bianchi [27]	1883 G. Darboux [28]
σ - arbitrary	1883 A. V. Bäcklund [29]	1885 L. Bianchi [30]

In the same paper [30], L. Bianchi for the first time points out a possibility to interpret Bäcklund results within the Kummer's theory of a general (not necessarily normal) congruence.

This idea is fully developed in the paper [31] published in 1887.

f) The starting point of this research is a question: do rectilinear congruences exist in E^3 for which both distances of (real) foci and of limit points are constant (do not depend on rays).

The answer is positive. In particular, one can show the following theorem:

If $k = const$ denotes a distance between foci and R ($R \geq k$) denotes a distance between limit points, then, necessarily, both focal surfaces are pseudospherical of the Gaussian curvature $K = -1/R^2$.

So both parameters (k, R) admit a nice geometric interpretation:

Due to formula (17):

$$\frac{k}{R} = \sin\sigma \tag{24}$$

defines an angle between tangent planes of focal surfaces at focal points, while:

$$K = -\frac{1}{R^2} \tag{25}$$

defines a common value of Gaussian curvature for both focal pseudospherical surfaces.

g) The culminating point of this line of research was a beautiful paper by Luigi Bianchi on "permutability theorem" [21].

Here we follow the formalism presented in the first edition (1894) of the Bianchi's "Lezioni di Geometria Differenziale" [32].

First of all by a simple transformation and having renamed variables involved we replace formulae (22–23) by:

$$\frac{\partial(\varphi-\omega)}{\partial u} = \frac{1+\sin\sigma}{\cos\sigma}\sin(\varphi+\omega), \tag{26}$$

$$\frac{\partial(\varphi+\omega)}{\partial v} = \frac{1-\sin\sigma}{\cos\sigma}\sin(\varphi-\omega). \tag{27}$$

Here $\omega = \omega(u, v)$ is a given solution to the sine-Gordon equation (21) rewritten in the "light-cone" coordinates:

$$\omega_{,uv} = \sin\omega\cos\omega \tag{28}$$

while $\varphi = \varphi(u, v, \sigma, c)$ (c — constant of integration) is its Bäcklund transform. We write formally:

$$\omega \xmapsto{(\sigma,c)} \varphi \tag{29}$$

On performing a long calculation Luigi Bianchi was able to prove that for Bäcklund transformations (29) the following permutability theorem holds:

For any two choices (σ_1, c_1), (σ_2, c_2) one can find (σ_1, c_1') and (σ_2, c_2') such that $(\sigma_2, c_2) \circ (\sigma_1, c_1)$ and $(\sigma_1, c_1') \circ (\sigma_2, c_2')$ when applied to ω give rise to the same solution to (28).

The most important consequence of this theorem is an existence of an amazingly effective method to integrate the "sine-Gordon" equation.

Suppose that for a given solution ω we are able to compute all its transforms $\varphi(u,v,\sigma,c)$. Let us select (σ_1, c_1) and put $\omega_1 = \varphi(u,v,\sigma_1,c_1)$. Bianchi proves that $\Omega = \Omega(u,v,\sigma,c,\sigma_1,c_1)$ $(\sigma_1 \neq \sigma)$ given by:

$$\tan\left(\frac{\Omega-\omega}{2}\right) = \frac{\cos\left(\frac{\sigma_1+\sigma}{2}\right)}{\sin\left(\frac{\sigma_1-\sigma}{2}\right)} \tan\left(\frac{\omega_1 - \varphi(u,v,\sigma,c)}{2}\right) \tag{30}$$

constitutes a four-parameter family of exact (!) solutions to the sine-Gordon equation (28). Observe you do not have to solve any differential equation!!!

Thus the only difficulty in this technology is to solve the *first order system (26–27) for a given* ω. For instance, on selecting $\omega = 0$ we are in a position to generate the ladder of exact solutions to (28) which *nowadays* are called N - soliton solutions to the sine–Gordon equation.

Acknowledgements. This work was partially supported by the Polish State Committee for Scientific Research (KBN grant 2 PO3B 185 09).

REFERENCES

[1] N. Zabusky and M. Kruskal: *Phys. Rev. Lett.* **15, 6** (1965), 240.

[2] G. Darboux: *Leçons sur la théorie générale des surfaces I–IV,* Gautier-Villars, Paris, 1887–1896.

[3] L. Bianchi: *Lezioni di Geometria Differenziale, terza editione,* Zanichelli, Bologna, 1924.

[4] L. P. Eisenhart: *Transformations of surfaces,* Princeton University Press, Princeton, 1923

[5] L. P. Eisenhart: *A Treatise on the Differential Geometry of Curves and Surfaces,* Ginn and Company, Boston, 1909.

[6] G. Tzitzéica: *Géométrie Différentielle Projective des Réseaux,* Bucarest, 1923.

[7] S. P. Finikov: *Theory of congruences,* (in Russian) Gostekhizdat, Moscow-Leningrad, 1950.

[8] E. P. Lane: *A Treatise on Projective Differential Geometry,* University of Chicago Press, Chicago, 1942.

[9] E. P. Lane: *Projective Differential Geometry of Curves and Surfaces,* University of Chicago Press, Chicago, 1932.

[10] A. Doliwa and P. M. Santini: *J. Math. Phys.* **36** (1995), 1259.

[11] A. Doliwa and P. M. Santini: *Phys. Lett. A* **233** (1997), 365.

[12] J. Cieśliński, A. Doliwa and P. M. Santini: *Phys. Lett. A* **235** (1997), 480.

[13] G. Tzitzéica: *Comptes Rendus* **144** (1907), 1257.

[14] G. Tzitzéica: *Rendiconti del Circolo Matematico di Palermo* **25** (1908) 180.

[15] G. Tzitzéica: *Comptes Rendus* **150** (1910), 955.

[16] R. K. Dodd and R. K. Bullough: *Proc. R. Soc.* **A351** (1976), 449.

[17] G. Fubini: *Annali di Matematica* **62** (1929), 45.

[18] V. B. Matveev and M. A. Salle: *Darboux Transformations and Solitons,* Springer, Berlin, 1991.

[19] A. Sym: in *Geometric Aspects of the Einstein Equations and Integrable Systems.* Lecture Notes in Physics, **239** (1985), 154.

[20] A. Bobenko: in *Harmonic Maps and Integrable Systems,* eds. A. Fordy and J. Wood, Vieweg, 1994.

[21] L. Bianchi: *Rend. Acc. Naz. Lincei* **5** (1892), 3.

[22] Yu. A. Aminov: *Dokl. Akad. Nauk SSSR* **236,3** (1977), 521.

[23] K. Tenenblat and Ch. L. Terng: *Annals of Math.* **111** (1980), 477.

[24] V. Hlavaty: *Differential Line Geometry,* P. Noordhoff LTD, Groningen, 1953.

[25] A. R. Forsyth: *Lectures on the Differential Geometry of Curves and Surfaces,* Cambridge University Press, Cambridge, 1912.

[26] A. Goetz: *Introduction to Differential Geometry,* Addison-Wesley, Reading, 1970.

[27] L. Bianchi: *Annali Scuola Norm. Sup.* **(1) 2** (1879), 285.

[28] G. Darboux: *Comptes Rendus* **97** (1883), 848, 892, 946.

[29] A. V. Bäcklund: *Lunds Universitets Arsskrift* **19** (1883).

[30] L. Bianchi: *Annali di Mat.* **13** (1885), 177.

[31] L. Bianchi: *Annali di Mat.* **15** (1887–88), 161.

[32] L. Bianchi: *Lezioni di Geometria Differenziale, terza editione,* Enrico Spoerri, Pisa, 1894.