Affine differential geometry and soliton equations

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Abstract: In the affine differential geometry of Blaschke we study the properties of submanifolds in $\mathbb{R}^n$ which are invariant under the equiaffine transformation group $G = \{\sigma \mid \sigma(x) = Ax + a, A \in SL(\mathbb{R}^n), a \in \mathbb{R}^3\}$. Affine hyperspheres form a very important class of affinely invariant submanifolds in $\mathbb{R}^n$. In this report we will investigate relations between 2-dimensional affine spheres in $\mathbb{R}^3$ and some kinds of soliton equations.

1 The Blaschke metric for surfaces in $\mathbb{R}^3$.

Let $x : \mathbf{M} \longrightarrow \mathbb{R}^3$ be an immersion. Let $\sigma = \{E_1, E_2\}$ be a local basis for **TM** with dual basis $\{\vartheta^1, \vartheta^2\}$. One can define a symmetric 2-form g^σ, locally given by

$$g^\sigma = [E_1(x), E_2(x), d^2x] = [E_1(x), E_2(x), E_iE_j(x)]\vartheta^i \otimes \vartheta^j = g^\sigma_{ij}\vartheta^i \otimes \vartheta^j, \tag{1.1}$$

where [ ] is the standard determinant in $\mathbb{R}^3$, $E_i(x)$ denotes the directional derivative of x; we adopt the Einstein convention in the formula. If τ is another local basis for **TM** we get $g^\sigma = \det(A)g^\tau$ for the transport matrix A from σ to τ. So the property that g^σ is nondegenerate is an affine invariant property of the surface x. We call such a surface a nondegenerate surface. For a nondegenerate surface one can easily verify that

$$(\det(g^\sigma_{ij}))^{-1/4}g^\sigma = (\det(g^\tau_{ij}))^{-1/4}g^\tau := g. \tag{1.2}$$

Thus g is a globally defined (pseudo-) metric on **M** which is equiaffinely invariant. g is called the Blaschke metric for nondegerate surface x in $\mathbb{R}^3$. One can easily generalize the definition of g to nondegenerate hypersurface in $\mathbb{R}^n$ for any dimension n. A surface in Euclidean space $\mathbb{E}^3$ is then nondegenerate if the second fundamental form $h := h_{ij}du^idu^j$ is nondegenerate. The Blaschke metric is now given by

$$g = \det(h_{ij})^{-1/4}h, \tag{1.3}$$

which is conformally equivalent to the second fundamental form h.

2 The Blaschke normal for surfaces in $\mathbb{R}^3$

For any oriented nondegenerate surface $x : \mathbf{M} \longrightarrow R^3$ one can define a unique transversal vector field ξ on x(**M**) satisfying

$$d\xi(v) \in dx(\mathbf{TM}), \qquad [E_1(x), E_2(x), \xi] = 1, \tag{2.1}$$

where $v \in \mathbf{TM}$ and $\{E_1, E_2\}$ is an arbitrary orthonormal basis of $\mathbf{TM}$ with respect to g. It is clear that ξ is equiaffinely invariant. The geometrical meaning of ξ is the following. Let $x(\mathbf{M})$ be a locally strongly convex surface in $\mathbb{R}^3$. Let $p \in \mathbf{M}$. For any small t we consider a plane π_t parallel to $\pi_0 := dx(T_p\mathbf{M})$. π_t intersects $x(\mathbf{M})$ in a plane curve $P(t)$. Let $c(t)$ be the barycentre of $P(t)$. Then $c(t)$ defines a curve through $x(p)$ which is clearly equiaffinely invariant. So $c'(0)$ is an equiaffine invariant up to a nonzero factor. Thus we get an equiaffine invariant line bundle on $x(\mathbf{M})$. Then ξ is the unique section of this line bundle such that $[dx(\cdot), dx(\cdot), \xi]$ is exactly the volume form of the Blaschke metric.

3 Affine sphere in $\mathbb{R}^3$

A nondegenerate surface in $\mathbb{R}^3$ is called an affine sphere, if all ξ meet in one point x_0 in $\mathbb{R}^3 \cup \{\infty\}$. Unlike Euclidean geometry, in the affine case we have infinity many different affine spheres in $\mathbb{R}^3$. It is easy to see that all nondegenerate quadrics in $\mathbb{R}^3$ are affine spheres. Another interesting affine sphere in $\mathbb{R}^3$ is given by the equation $x_1x_2x_3 = 1$ for $x_i > 0$. There are three types of locally strongly convex affine spheres, namely of elliptic type, hyperbolic type or parabolic type, corresponding to the cases that the center x_0 is on the convex side of the surface, on the concave side of the surface or $x_0 = \infty$. To show that how many hyperbolic affine spheres may exist we mention the following conjecture of Calabi, which has been proved by the works of S. Y. Cheng and S. T. Yau [2], T. Sasaki [8], A. M. Li [4]:

CONJECTURE: *Any hyperbolic affine sphere with complete Blaschke metric is asymptotic to a convex cone in $\mathbb{R}^3$ with the center of the affine sphere as vertex. Conversely, for any convex cone there exists a unique hyperbolic affine sphere in $\mathbb{R}^3$ with complete Blaschke metric which is asymptotic to the cone with center in its vertex.*

4 Indefinite affine spheres as solutions of a soliton equation

Let $x : \mathbf{M} \longrightarrow \mathbb{R}^3$ be an affine sphere with indefinite Blaschke metric g. We introduce a local asymptotic coordinate system (u, v) for g such that

$$g = e^{\omega}(du \otimes dv + dv \otimes du). \tag{4.1}$$

We define

$$\Phi := e^{\omega}\frac{[x_u, x_{uu}, \xi]}{[x_u, x_v, \xi]}du^3 := \varphi du^3; \tag{4.2}$$

$$\Psi := e^{\omega}\frac{[x_{vv}, x_v, \xi]}{[x_u, x_v, \xi]}dv^3 := \psi dv^3. \tag{4.3}$$

Though Φ and Ψ are locally defined, they are in fact independent of the choice of the asymptotic coordinate system (u, v). Thus they are globally defined on $\mathbf{M}$.

THEOREM ([7]). *$\{g, \Phi, \Psi\}$ is a complete equiaffinely invariant system for indefinite surfaces in $\mathbb{R}^3$ which determines the surface up to equiaffine transformations in $\mathbb{R}^3$.*

Using this invariant system we can write the structure equations for an indefinite affine sphere as follows:

$$x_{uu} = \omega_u x_u + e^{-\omega}\varphi x_v, \tag{4.4}$$

$$x_{uv} = \tfrac{1}{4}e^{\omega}\xi, \tag{4.5}$$

$$x_{vv} = \omega_v x_v + e^{-\omega}\psi x_u, \tag{4.6}$$

$$\xi_u = -Hx_u, \qquad \xi_v = -Hx_v. \tag{4.7}$$

If we denote by E the matrix (x_u, x_v, ξ), we can rewrite the structure equation as

$$E_u = EU, \qquad E_v = EV, \tag{4.8}$$

where the matrices U and V are given by

$$U = \begin{pmatrix} w_u & 0 & -H \\ e^{-\omega}\varphi & 0 & 0 \\ 0 & \frac{1}{4}e^{\omega} & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & e^{-\omega}\psi & 0 \\ 0 & \omega_v & -H \\ \frac{1}{4}e^{\omega} & 0 & 0 \end{pmatrix}. \tag{4.9}$$

The integrability condition is given by

$$U_v - V_u = [U, V], \tag{4.10}$$

which is equivalent to the equations

$$\omega_{uv} + e^{-2\omega}\varphi\psi + \tfrac{1}{4}He^{\omega} = 0, \tag{4.11}$$

$$\varphi_v = \psi_u = 0, \qquad H = \text{constant}, \tag{4.12}$$

where H is the so-called affine mean-curvature of x. In case $\Phi = \varphi\, du^3 \neq 0$ and $\Psi = \psi\, dv^3 \neq 0$ we can change the asymptotic coordinates (u, v) such that $\varphi = 1$ and $\psi = \varepsilon \in \{1, -1\}$. Thus we get

$$\omega_{uv} + \varepsilon e^{-2\omega} + \tfrac{1}{4}He^{\omega} = 0, \qquad H = \text{constant}. \tag{4.13}$$

Equation (4.13) is a soliton equation. So any indefinite affine sphere with $\Phi \neq 0$ and $\Psi \neq 0$ produces a solution ω to the soliton equation (4.13). Conversely, given any solution ω of the soliton equation we can define two matrices U and V by (4.9), where $\varphi = \lambda \neq 0$ and $\psi = \varepsilon\lambda^{-1}$, which satisfies the integrability conditions (4.10). Thus for any nonsingular initial matrix E_0 we get a unique solution $E = (x_1, x_2, \xi)$ of (4.8) depending on λ. Then by solving $dx_\lambda = x_1 du + x_2 dv$ we get a solution x_λ. It is easy to check that, for any λ, x_λ is a indefinite affine sphere with invariants $g = e^{\omega}(du \otimes dv + dv \otimes du)$, $\Phi = \lambda du^3$ and $\Psi = \varepsilon\lambda^{-1}dv^3$. Thus one solution of (4.13) determines a 1-parameter family of indefinite affine spheres in $\mathbb{R}^3$.

5 Definite affine spheres as solutions of a soliton equation

Let $x : \mathbf{M} \longrightarrow \mathbb{R}^3$ be an affine sphere with positive definite Blaschke metric g. We introduce a local complex coordinate system z for g such that

$$g = e^{\omega}(dz \otimes d\bar{z} + d\bar{z} \otimes dz). \tag{5.1}$$

We define a complex cubic form Φ by

$$\Phi := e^{\omega} \frac{[x_{zz}, x_z, \xi]}{[x_{\bar{z}}, x_z, \xi]} dz^3. \tag{5.2}$$

Though locally defined Φ is in fact independent of the choice of the complex coordinate z, thus it is globally defined on $\mathbf{M}$.

THEOREM ([7]).$\{g, \Phi\}$ *gives a complete equiaffinely invariant system for definite surfaces in* $\mathbb{R}^3$ *which determines the surface up to equiaffine transformations in* $\mathbb{R}^3$.

Using this invariant system we can write the structure equations for definite affine sphere as follows:

$$x_{zz} = \omega_z x_z + e^{-\omega}\varphi x_{\bar{z}}, \tag{5.3}$$

$$x_{z\bar{z}} = \tfrac{1}{4}e^{\omega}\xi, \tag{5.4}$$

$$x_{\bar{z}\bar{z}} = \omega_{\bar{z}} x_{\bar{z}} + e^{-\omega}\bar{\varphi} x_z, \tag{5.5}$$

$$\xi_z = -H x_z, \qquad \xi_{\bar{z}} = -H x_{\bar{z}}. \tag{5.6}$$

If we denote by E the complex matrix $(x_z, x_{\bar{z}}, \xi)$, we can rewrite the structure equations as

$$E_z = EU, \qquad E_{\bar{z}} = EV, \tag{5.7}$$

where the matrices U and V are given by

$$U = \begin{pmatrix} \omega_z & 0 & -H \\ e^{-\omega}\varphi & 0 & 0 \\ 0 & \frac{1}{4}e^{\omega} & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & e^{-\omega}\bar{\varphi} & 0 \\ 0 & \omega_{\bar{z}} & -H \\ \frac{1}{4}e^{\omega} & 0 & 0 \end{pmatrix}. \tag{5.8}$$

The integrability conditions are given by

$$U_{\bar{z}} - V_z = [U, V], \tag{5.9}$$

which is equivalent to the equation

$$\omega_{z\bar{z}} + e^{-2\omega}|\varphi|^2 + \tfrac{1}{4}He^{\omega} = 0, \tag{5.10}$$

$$\varphi_{\bar{z}} = 0, \qquad H = \text{constant}, \tag{5.11}$$

where H is the so-called affine mean-curvature of x. In a neighborhood $U \subset \mathbf{M}$ with $\Phi = \varphi\, dz^3 \neq 0$ we can change the complex coordinate z such that $\varphi = 1$. Thus we get

$$\omega_{z\bar{z}} + e^{-2\omega} + \tfrac{1}{4}He^{\omega} = 0, \qquad H = \text{constant}. \tag{5.12}$$

Equation (5.12) is a soliton equation. So any definite affine sphere yields a solution ω to the soliton equation (5.12). Conversely, given any solution ω of the soliton equation we can define two matrices U and V by (5.8), where $\varphi = e^{i\vartheta}$ for some $\vartheta \in \mathbf{R}$, which satisfies the integrability conditions (5.9), thus for any nonsingular complex initial matrix $E_0 = (a, b, c)$ with $a = \bar{b}$ and $c = \bar{c}$ we get a unique solution $E = (x_1, x_2, \xi)$ of (5.7) depending on ϑ. Then by solving $dx_\vartheta = x_1 dz + x_2 d\bar{z}$ we get a solution x_ϑ. It is easy to check that, for any ϑ, x_ϑ is a definite affine sphere with invariants $g = e^\omega(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$ and $\Phi = e^{i\vartheta} dz^3$. Thus one solution of (5.12) determines a 1-parameter family of definite affine spheres in $\mathbb{R}^3$.

6 Some global results

We note that the solutions of the soliton equation (5.12) cannot produce all definite affine spheres in $\mathbb{R}^3$, because for a general affine sphere Φ may have zero points. So in general we have to solve globally the equation

$$\triangle\omega + 4|\varphi|^2 e^{-2\omega} + He^\omega = 0, \qquad \varphi_{\bar{z}} = 0. \tag{6.1}$$

Fortunately we can solve this equation in some case.

THEOREM ([9]). *Given any no vanishing holomorphic cubic form Φ on a compact Riemann surface M_g with genus $g \geq 1$ and any constant $H < 0$, there exists a unique metric $G = e^\omega |dz|^2$, compatible with the Riemann structure on M_g, such that ω solves equation* (6.1).

Combine this with a theorem of Simon-Wang mentioned in Section 5 we get

THEOREM ([9]). *For any given non-vanishing holomorphic cubic form Φ on a compact Riemann surface M_g there exists a unique affine sphere $\mathbf{M}$ (up to affine transformations) in $\mathbb{R}^3$ with complete Blaschke metric which admits an action of a discrete equiaffine subgroup $\mathbf{\Gamma}$ such that (i) $\mathbf{M}/\mathbf{\Gamma} = \mathbf{M}_g$, (ii) the cubic form of x defined by* (5.2) *is the given form Φ.*

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