The Toda equations and the geometry of surfaces

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Abstract: The Toda equations are a particularly interesting example of a completely integrable Hamiltonian system and were initially studied in a dynamical context. Two particular forms of Toda equations, the "open" and "affine" equations, can be formulated for any simple Lie algebra and solutions to both forms arise naturally in a geometrical context from special types of harmonic maps into certain homogeneous spaces associated to the Lie algebra. Generally speaking, the open case is rather easier to deal with since the solutions correspond to holomorphic curves, but the affine case is in some ways more interesting since it has certain special features not enjoyed by the open case. All these aspects are discussed briefly below.

Introduction

The Toda equations are a remarkable phenomenon with many interesting facets. Since their discovery in the 1960s they have engendered an extensive and varied literature, much of which involves a detailed understanding of a number of different areas of mathematics. As a first attempt at getting to grips with them the reader might like to try the brief but informative article [12], the relevant parts of [21], and the survey article [16]. More exhaustive treatments can be found in [33], [20], and [2].

This article, which is an expanded version of the talk given by the author at the 1st Non-Orthodox School on Nonlinearity and Geometry in Warsaw in September 1995, is intended as an introduction for graduate students and non experts to geometric aspects of the Toda equations from the point of view of the theory of surfaces. Further details of the ideas presented here can be found in [6], [8] and [9]. For a more general but less geometrical discussion of the Toda equations in the context of harmonic maps and extensions of the ideas presented here see [11] and [10].

1 The Toda equations: Hamiltonian formulation

The Toda equations arise in a natural way in the context of Hamiltonian systems. The simplest case is that of a system of particles of equal mass m arranged along a line and joined by identical springs as in Fig. 1. (For a fuller description of this viewpoint see [33]). If y_p denotes the horizontal displacement from the equilibrium position of the p-th particle then by Newton's Law

$$m\frac{d^2 y_p}{dt^2} = f(y_{p+1} - y_p) - f(y_p - y_{p-1}) \tag{1}$$

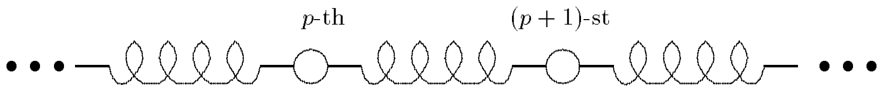


Fig. 1.

where, in the classical case, $f(r) = \kappa r$ with κ denoting Hooke's constant. The case where $f(r)$ is a non-linear function of r was considered numerically by Fermi, Pasta and Ulam [14] in the 1950's in an investigation of the way in which the energy distribution in the system evolves with time. This was developed further in 1967 by Toda [31, 32] who considered the case $f(r) = ae^{\lambda r}$. This turns out to be a particularly interesting case since, as we shall see presently, the resulting Hamiltonian system is completely integrable.

There are various different models of which the most interesting are the following:

(a) the *finite* or *open* case, where there are a finite number n of particles and the ends are fixed (corresponding to $y_0 = y_{n+1} = 0$) as in Fig. 2;

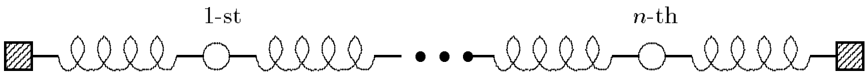


Fig. 2.

(b) the *periodic* or *affine* case where $n+1$ particles are arranged in a circle (corresponding to the case $y_0 = y_{n+1}$) as in Fig. 3;

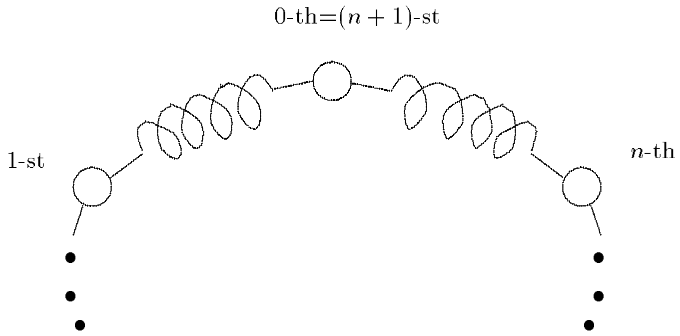


Fig. 3.

(c) the *infinite* case as in Fig. 4.

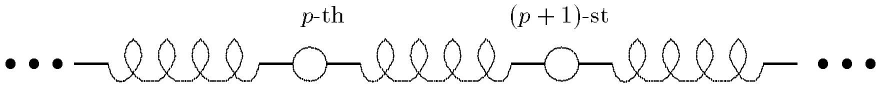


Fig. 4.

For the case considered by Toda, on setting all constants equal to 1 and taking differences of (1) we have

$$\frac{d^2}{dt^2}(y_{p+1} - y_p) = e^{y_p - y_{p-1}} - 2e^{y_{p+1} - y_p} + e^{y_{p+2} - y_{p+1}}. \tag{2}$$

The Hamiltonian in the cases corresponding to Figures 2 and 3 is given by

$$H = \sum_{j=1}^{n} \frac{1}{2} p_j^2 + \sum_{j=0}^{n} e^{q_j - q_{j+1}}, \tag{3}$$

with $q_0 = q_{n+1} = 0$ in the open case and

$$H = \sum_{j=0}^{n} (\frac{1}{2} p_j^2 + e^{q_j - q_{j+1}}), \tag{4}$$

with $q_0 = q_{n+1}$ in the affine case, where $q_j = y_j$ and $p_j = \dfrac{dq_j}{dt} = \dfrac{dy_j}{dt}$. Thus the equations of motion

$$\frac{\partial H}{\partial p_j} = \frac{dq_j}{dt}, \quad \frac{\partial H}{\partial q_j} = -\frac{dp_j}{dt}, \qquad j = 0, \ldots, n+1, \tag{5}$$

are precisely the equations $p_j = \dfrac{dq_j}{dt}$ together with the Toda equations.

The system has N degrees of freedom, where $N = n$ in the open case and $N = n+1$ in the affine case, and is completely integrable in each case in the sense that there are N integrals of the motion which are in involution. That is to say, there are N independent quantities $P_1, \ldots, P_N$ whose Poisson brackets with H and with each other all vanish. In particular, $P_1, \ldots, P_N$ are all constant with respect to time. They can be found very easily from the Lax pair formulation of the equations which in the affine case is as follows. If

$$a_j = \frac{1}{2} e^{\frac{1}{2}(q_{j+1} - q_j)}, \quad b_j = \frac{1}{2} p_j \qquad j = 0, \ldots, n+1, \tag{6}$$

(with $q_0 = q_{n+1}$) it is not hard to check that the equations of motion may be written as

$$\dot{a}_j = a_j(b_{j+1} - b_j), \quad \dot{b}_j = 2(a_j^2 - a_{j-1}^2). \tag{7}$$

Thus, defining matrices L and A by

$$L = \begin{pmatrix} b_0 & a_0 & 0 & & a_n \\ a_0 & b_1 & a_1 & & \\ 0 & a_1 & b_2 & & \\ & & & & \\ & & & b_{n-1} & a_{n-1} \\ a_n & & & a_{n-1} & b_n \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & a_0 & 0 & & -a_n \\ -a_0 & 0 & a_1 & & \\ 0 & -a_1 & 0 & & \\ & & & & \\ & & & 0 & a_{n-1} \\ a_n & & & -a_{n-1} & 0 \end{pmatrix}, \tag{8}$$

it follows that the affine Toda equations may be written

$$\dot{L} = [A, L] = AL - LA. \tag{9}$$

From this it is clear that the coefficients of the characteristic equation of L are all constant, and it may be shown that these are independent and give all the conserved quantities. The open case is similar.

2 The Toda equations: Lie algebra formulation

The ideas of the previous section were extended in the late 1970s by Adler [1], Kostant [20] and Symes [30] who gave a Lie algebra formulation of the equations. Moreover, they showed that for each (simple) Lie algebra there is a corresponding integrable Hamiltonian system and that knowing the set of solutions of the system is equivalent to knowing the weight structure of the fundamental representations of the Lie algebra.

We begin by recalling briefly some basic facts about Lie algebras. We refer the reader to [18, 29, 23] for further details. A real *Lie algebra* $\boldsymbol{g}$ is a vector space with a binary operation called the Lie bracket which assigns to a pair of elements $X, Y \in \boldsymbol{g}$ an element $[X, Y] \in \boldsymbol{g}$ with the property that the bracket is linear in both X and Y, skew-symmetric in the sense that

$$[Y, X] = -[X, Y], \tag{10}$$

and the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{11}$$

holds for all $X, Y, Z \in \boldsymbol{g}$. Complexifying $\boldsymbol{g}$ gives a complex Lie algebra denoted by $\boldsymbol{g}^{\mathbb{C}}$. A Lie algebra is *abelian* if the bracket of any two elements is zero, and *simple* if it is not abelian and has no proper ideals other than $\{0\}$.

Lie algebras arise naturally from Lie groups. If G is a Lie group then the tangent space at the identity may be identified with the set of left invariant vector fields on

G with the Lie bracket given by the bracket of vector fields. Roughly speaking this gives a one-to-one-correspondence between connected (simply connected) Lie groups and Lie algebras. Under this correspondence simple Lie algebras correspond to compact Lie groups which are simple in the sense that their only normal subgroups are finite. It is also easy to check that a homomorphism $\tilde{G} \to G$ which is also a covering map gives rise to an isomorphism of $\tilde{\boldsymbol{g}}$ with $\boldsymbol{g}$. Thus, for example, the standard double covering $SU(2) \to SO(3)$ gives rise to an isomorphism of $\boldsymbol{su}(2)$ with $\boldsymbol{so}(3)$. A connected maximal abelian subgroup is called a maximal torus, denoted by T, and any two maximal tori in G are conjugate. The Lie algebra $\boldsymbol{t}$ of T is a maximal abelian subalgebra of $\boldsymbol{g}$ and $\boldsymbol{t}$ (resp. $\boldsymbol{t}^{\mathbb{C}}$) is called a *Cartan subalgebra* of $\boldsymbol{g}$ (resp. $\boldsymbol{g}^{\mathbb{C}}$). The dimension of $\boldsymbol{t}$ is called the *rank* of $\boldsymbol{g}$.

An important example is that of the special unitary group

$$SU(n+1) = \{A \in GL(n+1;\mathbb{C}) \mid AA^{\dagger} = I, \det A = 1\}, \tag{12}$$

consisting of $(n+1) \times (n+1)$ matrices of determinant $+1$. This has Lie algebra

$$\boldsymbol{su}(n+1) = \{X \in M(n+1;\mathbb{C}) \mid X + X^{\dagger} = 0, \operatorname{trace} X = 0\}, \tag{13}$$

and it is not hard to check that

$$\boldsymbol{su}(n+1)^{\mathbb{C}} = \boldsymbol{sl}(n+1;\mathbb{C}) = \{X \in M(n+1;\mathbb{C}) \mid \operatorname{trace} X = 0\}. \tag{14}$$

In each case the Lie bracket is given by

$$[X,Y] = XY - YX. \tag{15}$$

Moreover, as maximal torus one may take the set of diagonal matrices:

$$T = \{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{n+1}}) \mid \theta_1 + \ldots + \theta_{n+1} = 0\}. \tag{16}$$

In the case of $SU(2)$ this is a circle which corresponds under the covering mentioned above to the circle in $SO(3)$ given by rotations about a coordinate axis in $\mathbb{R}^3$.

It is a remarkable fact that the structure of a simple Lie algebra of rank ℓ may be encoded in a very compact form using either the Cartan matrix $K = (K_{ij})$ which is a non-singular $\ell \times \ell$ matrix with integer coefficients or the Dynkin diagram. The Cartan matrix and the Dynkin diagram determine each other and each of them encodes in an elegant and beautiful way all the essential details of the Lie algebra. This is achieved using the *roots*, which are a system of linear functions on $\boldsymbol{t}^{\mathbb{C}}$, together with a naturally defined inner product on $\boldsymbol{g}$ both of which are derived from the structure of the Lie algebra. From the roots one chooses a special subset with ℓ elements called the *simple roots* which have the property that they form a basis for the dual space $\boldsymbol{t}^*$ of complex-valued linear functions on $\boldsymbol{t}$ and every root can be written as an integral linear combination of the simple roots. The structure of the Lie algebra is then encoded combinatorially in terms of the lengths of the simple roots and the angles between them with the Cartan matrix given by

$$K_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \tag{17}$$

The Dynkin diagram encodes this information in different way. See [29, 25, 34] for details.

For example, in the case of $\boldsymbol{su}(n+1)$, with Cartan subalgebra

$$\boldsymbol{t} = \{\mathrm{diag}(y_0, \dots, y_n) \mid y_0 + \dots + y_n = 0\}, \tag{18}$$

the roots are given by

$$\sigma_i - \sigma_j, \qquad i, j = 0, \dots, n, \tag{19}$$

where

$$\sigma_p(\mathrm{diag}(y_0, \dots, y_n)) = y_p, \qquad p = 0, \dots, n, \tag{20}$$

and the inner product on $\boldsymbol{su}(n+1)$ is given by

$$(X, Y) = -\frac{1}{2}\,\mathrm{trace} XY. \tag{21}$$

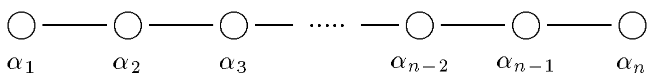
The simple roots are given by

$$\sigma_p - \sigma_{p-1}, \qquad p = 1, \dots, n, \tag{22}$$

Finally the Cartan matrix and Dynkin diagram for $\boldsymbol{su}(n+1)$ are given by

$$K = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \tag{23}$$

and



(24)

respectively.

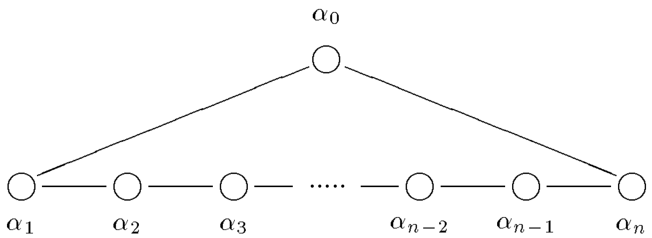
In addition to the Cartan matrix and the Dynkin diagram one may construct in a natural way the extended Cartan matrix and the extended Dynkin diagram. The extended Cartan matrix is an $(\ell+1) \times (\ell+1)$-matrix of rank ℓ containing the Cartan matrix as a submatrix, and the extended Dynkin diagram contains the Dynkin diagram as subdiagram. The extended Cartan matrix $\hat{K} = (\hat{K}_{ij})$, where now $i, j = 0, \dots, n$, may be constructed from the Cartan matrix and has the property that $\hat{K}_{ij} = K_{ij}$, $i, j = 1, \dots, n$ and there are positive integers $m_0, \dots, m_\ell$ with $m_0 = 1$, themselves constructed from the Cartan matrix, such that the rows $R_0, \dots, R_\ell$ satisfy the linear relation

$$m_0 R_0 + \dots + m_\ell R_\ell = 0. \tag{25}$$

For the case of the Lie algebra $\boldsymbol{su}(n+1)$ $(n \geq 2)$ the extended Cartan matrix is given by

$$\hat{K} = \left(\begin{array}{c|cccccc} 2 & -1 & & & & & -1 \\ \hline -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & -1 & 2 & -1 \\ -1 & & & & & -1 & 2 \end{array} \right), \tag{26}$$

with $m_0 = \ldots = m_n = 1$, while the extended Dynkin diagram is



. (27)

In this formulation the Toda equations for a simple Lie algebra $\boldsymbol{g}$ take two forms: the *open* $\boldsymbol{g}$*-Toda equations* given by

$$\frac{d^2\rho_i}{dt^2} + \sum_{j=1}^{\ell} K_{ij} e^{\rho_j} = 0\,, \qquad i = 1, \ldots \ell. \tag{28}$$

and the *affine* $\boldsymbol{g}$*-Toda equations* given by

$$\frac{d^2\rho_i}{dt^2} + \sum_{j=0}^{\ell} \hat{K}_{ij} e^{\rho_j} = 0\,, \qquad i = 0, \ldots \ell, \tag{29}$$

with

$$m_0\rho_0 + \ldots + m_\ell\rho_\ell = 0. \tag{30}$$

Thus the equations derived above for the particles under exponential interactions are precisely the open and affine $\boldsymbol{su}(n+1)$-Toda equations. We note that in this case the Dynkin diagrams strongly mirror the configuration of the springs. In general this similarity is repeated but is not so striking. See [2].

Observe that in the case of the affine $\boldsymbol{su}(n+1)$-Toda equations zero is a solution. This is not true in general but observe that because the extended Cartan matrix is singular there does exist a constant solution. As a consequence it is possible to change the unknowns by the addition of constant factors to obtain a modified version of the equations for which zero is a solution. The reformulated version of the eqations in this case can be most neatly expressed (cf. [6]) in Lie algebraic terms as

$$\frac{d^2\Omega}{dt^2} + \sum_{k=0}^{\ell} m_k e^{\alpha_k(\Omega)} \alpha_k^{\#} = 0. \tag{31}$$

Here Ω is a function taking values in $i\boldsymbol{t}$ (where $i = \sqrt{-1}$), $\alpha_1, \ldots, \alpha_\ell$ is a set of simple roots, the highest root is $-\alpha_0 := m_1\alpha_1 + \ldots + m_\ell\alpha_\ell$ with $m_k \in \mathbb{Z}$ and $m_0 := 1$, and $\alpha_k^{\#}$ is the dual (with respect to the inner product) of α_k defined by

$$(\alpha_k^{\#}, v) = \alpha_k(v), \qquad v \in \boldsymbol{t}. \tag{32}$$

Moreover, the two forms of the Toda equations are related by

$$\alpha_i(\Omega) = \rho_i + \lambda_i, \qquad i = 0, \ldots, \ell \tag{33}$$

for some suitable constants λ_i.

For example, in the case of $\boldsymbol{su}(n+1)$, with Cartan subalgebra

$$\mathbf{t} = \{\mathrm{diag}(y_0, \ldots, y_n) \mid y_0 + \ldots + y_n = 0\},$$

we have $\Omega = \mathrm{diag}(w_0, \ldots, w_n)$ with $w_0 + \ldots + w_n = 0$, and the highest root is

$$-\alpha_0 = \sigma_n - \sigma_0 = \alpha_1 + \ldots + \alpha_n. \tag{34}$$

3 The 2-dimensional Toda equations

There is a corresponding 2-dimensional version of the Toda equations in which $\dfrac{d^2}{dt^2}$ is replaced by the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{35}$$

For reasons which are largely dictated by geometrical considerations described in the next section, we will sometimes use a complex coordinate $z = x + iy$ and replace Δ by $4\partial^2/\partial z\partial\bar{z}$.

The two forms of the 2-dimensional Toda equations for a simple Lie algebra $\boldsymbol{g}$ are thus given by

$$\Delta\rho_i + \sum_{j=1}^{\ell} K_{ij} e^{\rho_j} = 0\,, \qquad i = 1, \ldots \ell, \tag{36}$$

and

$$\Delta\rho_i + \sum_{j=0}^{\ell} \hat{K}_{ij} e^{\rho_j} = 0\,, \qquad i = 0, \ldots \ell. \tag{37}$$

In the case where $\boldsymbol{g} = \boldsymbol{su}(2)$ these may be written

$$\Delta\rho - 2e^{-2\rho} = 0 \tag{38}$$

in the open case and as the sinh-Gordon equation

$$\Delta\rho + 2\sinh\rho = 0 \tag{39}$$

in the affine case.

We note the following very important facts:

(a) the 2-dimensional open Toda equations are conformally invariant;

(b) the 2-dimensional affine Toda equations are NOT conformally invariant.

To see these facts consider a complex change of variable $w = f(z)$ with $f'(z) = e^{g(z)}$, under which the Laplacian Δ is replaced by $e^{g(z)+\overline{g(z)}}\Delta$ and $\rho_i(z, \bar{z})$ is replaced by $\tilde{\rho}_i(w, \bar{w}) + g(z) + \overline{g(z)}$. Then (36) becomes

$$\Delta\tilde{\rho}_i + \sum_{j=0}^{\ell} K_{ij} e^{\tilde{\rho}_j} = 0\,, \qquad i = 0, \ldots \ell, \tag{40}$$

which establishes (a). However, we cannot make a corresponding change in case (b) and still preserve the condition

$$m_0\rho_0 + \ldots + m_\ell\rho_\ell = 0. \tag{41}$$

Indeed, the only conformal changes of coordinates preserving this condition are given by multiplication by $e^{i\theta}$. These are crucial facts since they indicate that, whereas the coordinates in the open case can be changed conformally, in the affine case the coordinate is, in some sense, fixed. We shall see the significance of this geometrically in the next section.

Finally we note that for the 2-dimensional case solutions come in S^1-families. For if $\{\rho_i\}$ is a set of solutions then so is $\{\rho_i^\lambda\}$, where

$$\rho_i^\lambda(z) = \rho_i(\lambda z), |\lambda| = 1. \tag{42}$$

Although we shall not make much of this fact here, it is an important observation geometrically and also leads to a fuller understanding of the construction and nature of the solutions in terms of loop algebras. See [6] for further details.

4 Geometrical aspects: basic questions

We begin by asking some obvious questions about the 2-dimensional Toda equations:

Question I Do the Toda equations arise naturally geometrically and if so how?

Question II For each geometrical situation in which the Toda equations arise, what is the geometric meaning of the unknowns?

We observe first of all that since the Toda equations arise in the dynamical context from a variational problem it is likely that the same will be true in a geometrical context. Thus we might expect that geometrical interpretations of the equations may be found in terms of minimal surfaces, surfaces of constant mean curvature or something similar. This is indeed the case and in the next section we give some particular examples. However, first we give the extent of the present understanding of the answers to the above questions.

We suppose in what follows that G is a compact simple Lie group with Lie algebra $\boldsymbol{g}$, that T denotes its maximal torus, and that U is an open subset of $\mathbb{C}$. We recall that the homogeneous space G/T admits a G-invariant complex structure and is often referred to as a (generalized) *flag manifold*.

Answers to Question I **Local** The Toda equations are integrability conditions for the existence of certain G-frames (called *Toda frames*) $F : U \to G$. These integrability conditions are the well-known Maurer–Cartan equations which are also sometimes called "zero curvature equations". Thus, locally, the solutions of the Toda equations correspond to certain special maps $F : U \to G$. This aspect is well understood. (See [6], [27]). These local maps project down to give maps of U into G/T which have a global interpretation.

Global The solutions of the Toda equations are local descriptions of certain globally defined harmonic maps $\psi : S \to G/T$, where S is a surface and U is essentially a conformal coordinate neighbourhood on S. Geometrically they are certain (branched) conformal minimal immersions (except in the case where G has rank 1). The precise conditions for such harmonic maps are discussed in Section 3; in particular the maps are holomorphic in the open case, and non-holomorphic in the affine case. This aspect is understood in basic terms (see [6, 27]) but many interesting questions remain unanswered, particularly with regard to singularities.

The harmonic maps $\psi : S \to G/T$ which arise have the striking property that, if K is a subgroup of G which contains T and $\pi : G/T \to G/K$ is the natural projection, the map $\phi = \pi\psi : S \to G/K$ is also harmonic [3]. Thus a more refined version of Question I would ask for a geometrical interpretation of such maps for each such K. The answer in this case is partially understood for some G and some K, essentially where G is one of the classical groups or the exceptional group G_2. (See [6, 8, 9, 13]).

Answers to Question II The geometric meaning of the unknowns is understood completely for some G and some K, essentially where G/K is either a sphere or a projective space over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. (See [9] for details of the real case).

5 Geometrical aspects: examples

We now illustrate these ideas with some particularly interesting and instructive examples. We restrict our attention to the simplest case, namely that where the Lie algebra is $\boldsymbol{su}(n+1)$. For examples of other Lie algebras see [6].

In these examples, we consider harmonic maps of surfaces which for simplicity one may think of in terms of immersions of surfaces. We consider only local questions relating to simply connected open subsets of surfaces (although in some cases these lead to results of a global nature [26, 15, 6]) and for simplicity we restrict ourselves to cases where there are no "singularities". This means that, not only will we assume that the immersions themselves have no singularities but that there are no "higher-order singularities" such as umbilical points. Dealing the global aspects and in particular incorporating the singularities into the theory are interesting but as yet unresolved problems.

Before giving the examples we list a number of characteristic features of the general geometrical situation which may be clearly seen in the examples:

(i) Geometrical considerations give rise to expressions in terms of conformal local coordinates which are essentially unique in the affine case.

(ii) There is an essentially canonically defined local G-frame (the Toda frame), which is a local lift of a globally defined map into G/T.

(iii) The Toda equations appear as an integrability condition for the Toda frame.

(iv) The number of geometrical invariants naturally occuring is equal to the rank of G and these form a complete set of invariants.

We give two important examples. The first, surfaces of constant mean curvature (CMC surfaces) in $\mathbb{R}^3$, gives a good insight into the geometry of the Toda equations. The second, harmonic maps into $\mathbb{C}P^n$, is particularly instructive in helping to explain the general case.

Example 1. CMC surfaces in $\mathbb{R}^3$.

Let S be an immersed surface in $\mathbb{R}^3$. Then, in a neighbourhood of a point which is not an umbilic, one may choose isothermal coordinates x, y (uniquely up to multiplication by a constant) so that the coordinate curves are lines of curvature.

Suppose now that S has constant mean curvature c. There are two cases to consider. First consider the minimal case where $c = 0$. In this case there is a function $\alpha(x, y)$ such that the principal curvatures are given by

$$\kappa_1 = e^{\alpha}, \qquad \kappa_2 = -e^{\alpha}, \tag{43}$$

and hence the Gaussian curvature K is given by

$$K = \kappa_1\kappa_2 = -e^{2\alpha}. \tag{44}$$

Next consider the case where $c \neq 0$. In this case there is a function $\alpha(x, y)$ such that the principal curvatures are given by

$$\kappa_1 = \frac{c}{2}(1 + e^{\alpha}), \qquad \kappa_2 = \frac{c}{2}(1 - e^{\alpha}), \tag{45}$$

and hence the Gaussian curvature K is given by

$$K = \kappa_1 \kappa_2 = \frac{c^2}{4}(1 - e^{2\alpha}). \tag{46}$$

In both cases there is a corresponding adapted $SO(3)$-frame F given by the unit tangent vectors e_1, e_2 to the coordinate curves and the unit normal N and if the metric is given by $ds^2 = e^{2\omega}(dx^2 + dy^2)$ then the Maurer–Cartan equations for F are the Gauss and Codazzi–Mainardi equations which take the form

$$\Delta(2\omega) = -2e^{2\omega}K \qquad \text{(Gauss)}. \tag{47}$$

$$\left.\begin{array}{ll} (\alpha + 2\omega)_x & = 0, \\ (\alpha + 2\omega)_y & = 0. \end{array}\right\} \qquad \text{(Codazzi–Mainardi)}. \tag{48}$$

From the latter we see that $\alpha + 2\omega$ is constant and this constant is zero for a unique choice of scaling of the coordinates. This then fixes the coordinates up to multiplication by ± 1. Thus, writing the Codazzi–Mainardi equations as $\alpha + 2\omega = 0$, the Gauss equation reduces to

$$\Delta(2\omega) - 2e^{-2\omega} = 0 \tag{49}$$

in the case $c = 0$, and to

$$\Delta(2\omega) + c^2 \sinh 2\omega = 0 \tag{50}$$

in the case $c \neq 0$.

Thus the integrability condition is the open $\boldsymbol{so}(3)$-Toda equation in the case $c = 0$ and the affine $\boldsymbol{so}(3)$-Toda equation in the case $c \neq 0$. In each case we note that there is a single geometric invariant involved, namely the function $e^{2\omega}$ in the expression for the metric. We recall that $\boldsymbol{su}(2) = \boldsymbol{so}(3)$ so that as noted in Section 3, (49) and (50) are the open and affine $\boldsymbol{su}(2)$-Toda equations. Geometrically these are just the conditions for the Gauss map of the immersion to be antiholomorphic in the minimal case and harmonic in the case $c \neq 0$. Indeed, the Gauss map, which is globally defined, is a map into G/T with $G = SO(3)$ (or $SU(2)$) and by the Ruh–Vilms Theorem it is harmonic if and only if the surface has constant mean curvature [28].

The S^1-families of solutions mentioned at the end of Section 3 correspond in this case to S^1-families of CMC surfaces. These are well known in the case of minimal surfaces and have been studied in more detail recently for the case of non-minimal CMC surfaces by Pinkall and Sterling [26] who used loop algebra techniques to describe all CMC tori in $\mathbb{R}^3$. Explicit formulae for these in terms of theta functions were given by Bobenko in [4]. For further details of these aspects of surfaces in $\mathbb{R}^3$ corresponding to soliton type equations see [24].

Example 2. Harmonic maps into $\mathbb{C}P^n$.

We first recall that the complex projective space $\mathbb{C}P^n$ is the space of all complex lines through the origin in $\mathbb{C}^{n+1}$. It is a compact complex manifold and [19] has a natural Kähler metric called the Fubini–Study metric inherited from the standard metric on $\mathbb{C}^{n+1}$.

Let $\phi : S \to \mathbb{C}P^n$ be a harmonic map. Then ϕ gives rise to sequence $\phi_p : S \to \mathbb{C}P^n$ of harmonic maps with $\phi_0 = \phi$. This sequence is called the *harmonic sequence* of ϕ and may be described locally by a sequence of functions $\{f_p\}$ as follows:

$$\phi_p = [f_p]\,, \quad f_p : U \to \mathbb{C}^{n+1} \setminus \{0\}, \tag{51}$$

where

$$\begin{aligned} \frac{\partial f_p}{\partial z} &= f_{p+1} + \frac{\partial}{\partial z} \log |f_p|^2 \, f_p, \\ \frac{\partial f_p}{\partial \bar z} &= -\frac{|f_p|^2}{|f_{p-1}|^2} \, f_{p-1}, \\ \langle f_{p+1}, f_p \rangle &= 0. \end{aligned} \tag{52}$$

The simplest case is the *superminimal* case, where the sequence is finite. Relabelling, if necessary, we can write this as $\phi_0, \ldots, \phi_n$. In this case ϕ_0 is a holomorphic curve, $\phi_0, \ldots, \phi_n$ is its Frenet frame (see [17]), and the elements are necessarily mutually orthogonal. In this case the solutions can be written out algebraically. For example in the case where S is the Riemann sphere the curves can be written in terms of polynomials.

The next simplest case is the *orthogonally periodic* case. In this case any $n+1$ consecutive elements are mutually orthogonal and the sequence is periodic in the sense that, for all p, $\phi_{n+1+p} = \phi_p$. Elements of such a sequence are said to be *superconformal*. For example, if $\phi : S \to \mathbb{C}P^n$ is a harmonic map then (see [7]) ϕ is conformal if and only if any three consecutive elements in the harmonic sequence are mutually orthogonal so that all conformal harmonic maps into $\mathbb{C}P^2$ are either superminimal or superconformal. In general, however, the sequence is infinite in both directions and, apart from adjacent elements being orthogonal by construction, there are no orthogonality properties.

The integrability conditions for (52), namely

$$\frac{\partial^2 f_p}{\partial z \partial \bar z} = \frac{\partial^2 f_p}{\partial \bar z \partial z}, \tag{53}$$

are then easily seen to reduce to

$$\frac{\partial^2}{\partial z \partial \bar z} \log |f_p|^2 = \frac{|f_{p+1}|^2}{|f_p|^2} - \frac{|f_p|^2}{|f_{p-1}|^2}, \tag{54}$$

and writing

$$|f_p| = e^{\omega_p}, \tag{55}$$

this gives (cf. (49))

$$2\frac{\partial^2\omega_p}{\partial z\partial\bar{z}} + e^{2(\omega_p-\omega_{p-1})} - e^{2(\omega_{p+1}-\omega_p)} = 0. \tag{56}$$

Since [5] the forms $(|f_p|^2/|f_{p-1}|^2)|dz|^2$ are globally defined, it is more natural to rewrite (55) in terms of the quotients $|f_p|^2/|f_{p-1}|^2$, in which case using (56) we obtain

$$\frac{\partial^2\alpha_p}{\partial z\partial\bar{z}} - e^{2\alpha_{p-1}} + 2e^{2\alpha_p} - e^{2\alpha_{p+1}} = 0, \tag{57}$$

where $\alpha_p = \omega_p - \omega_{p-1}$. These are called the A_∞-Toda equations, and in terms of the discussion in Section 1 they are the 2-dimensional analogue of the case of a doubly infinite line of of springs. In the superminimal case where ϕ_0 is holomorphic and ϕ_n is anti-holomorphic we have to take $\omega_{-1} = \omega_{n+1} = 0$, while in the superconformal case we have the periodicity condition $\omega_p = \omega_{n+p}$.

In all cases the functions α_p have geometric meaning. Indeed the metric ds_p^2 and Kähler angle θ_p of ψ_p are given by

$$ds_p^2 = (e^{2\alpha_p} + e^{2\alpha_{p+1}})|dz|^2, \qquad (\tan\frac{1}{2}\theta_p)^2 = e^{2(\alpha_p-\alpha_{p+1})}. \tag{58}$$

Moreover, in the superminimal and orthogonally periodic cases these invariants determine ψ up to holomorphic isometries of $\mathbb{C}P^n$. (See [7]).

As can be seen from the discussion above, the superminimal and superconformal cases correspond to the open and affine $\boldsymbol{su}(n+1)$-Toda equations respectively. In both cases it turns out that locally, for a complex coordinate z on S, which is essentially unique in the affine case, the frame $F : U \to SU(n+1)$ given by $F = (F_0, \ldots, F_n)$ where $F_p = f_p/|f_p|$ satisfies

$$F^{-1}\frac{\partial F}{\partial z} = \frac{\partial\Omega}{\partial z} + e^{\Omega}B_0e^{-\Omega}, \quad B_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}. \tag{59}$$

(B_0 nilpotent with the 1's just below the diagonal) in the open case and

$$F^{-1}\frac{\partial F}{\partial z} = \frac{\partial\Omega}{\partial z} + e^{\Omega}Be^{-\Omega}, \quad B = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}. \tag{60}$$

(B semisimple of order $n+1$) in the affine case, where $\Omega = \mathrm{diag}(\omega_0, \ldots, \omega_n)$. Moreover, the Maurer–Cartan equations

$$da^i_j + a^i_k \wedge a^k_j = 0, F^{-1}dF = (a^i_j), \tag{61}$$

which are the integrability conditions for the matrix of 1-forms $F^{-1}dF$ and which are essentially just

$$\frac{\partial^2 F}{\partial z \partial \bar{z}} = \frac{\partial^2 F}{\partial \bar{z} \partial z},$$

are precisely the conditions for $\omega_0, \ldots, \omega_n$ to be a set of solutions of the open or affine $\boldsymbol{su}(n+1)$-Toda equations. Thus, in particular, any set of solutions to either the open or affine $\boldsymbol{su}(n+1)$-Toda equations determines a frame $F = (F_0, \ldots, F_n)$. In the open case $[F_0]$ is a holomorphic curve with $[F_0], \ldots, [F_n]$ its Frenet frame; in the affine case the maps $[F_p]$, $p = 0, \ldots, n$, form an orthogonally periodic harmonic sequence and hence are superconformal. We note that in both cases, although the frame F is not globally defined, it descends to a globally defined map $\tilde{\psi} : S \to SU(n+1)/T^n$ of S into the flag manifold $SU(n+1)/T^n$ defined by setting $\tilde{\psi}(z)$ equal to the flag determined by $\psi_0(z), \ldots, \psi_n(z)$. We shall have more to say about this example in the next section.

6 The general case

The strategy for understanding the Toda equations for an arbitrary simple Lie algebra $\boldsymbol{g}$ is to establish a natural correspondence between solutions of the $\boldsymbol{g}$-Toda equations in a domain U in $\mathbb{C}$ and certain special maps of U into G/T. We describe this briefly below, illustrating the ideas throughout with the case where $\boldsymbol{g} = \mathbf{su}(n+1)$, and leave the reader to find detailed arguments for general $\boldsymbol{g}$ in [27] in the open case and [6] in the affine case. We give no indication here of how to deal with the questions of how to identify the harmonic maps into G/K arising in this way (as indicated in Section 4) and to relate the unknowns in the Toda equations to geometric invariants of these maps. These are the subject of ongoing research.

The first step is to note that if G is a compact simple Lie group with Lie algebra $\boldsymbol{g}$ then G/T actually depends on $\mathbf{g}$ rather than G, and to observe that G/T has some nice symmetry properties. Indeed, G/T is an m-symmetric space. This means that at each point p of G/T there is a geodesic symmetry of G/T of order m having p as an isolated fixed point. In the case $m = 2$ an m-symmetric space is just a symmetric space in the usual sense (see [18, 19]) and the geodesic symmetry is just the geodesic symmetry of order two for a symmetric space, such as rotation through π in the case of S^2. The symmetry comes from a canonically constructed automorphism τ called the *Coxeter automorphism* of the Lie algebra which gives rise to an inner automorphism of G also denoted by τ. Furthermore $m = 1 + \text{height of highest root of } \boldsymbol{g}$. In the case $\boldsymbol{g} = \boldsymbol{su}(n+1)$, $G = SU(n+1)$ we have $m = n+1$ and τ is given by conjugation by $\mathrm{diag}\{1, \zeta, \ldots, \zeta^n\}$, where $\zeta = \exp(2\pi i/m)$.

Under the action of τ the complexification $\boldsymbol{g}^{\mathbb{C}}$ of $\boldsymbol{g}$ splits as a direct sum

$$\boldsymbol{g}^{\mathbb{C}} = \mathcal{M}_0 \oplus \ldots \oplus \mathcal{M}_m \tag{62}$$

of eigenspaces with $\mathcal{M}_k$ the ζ^k-eigenspace, where $\zeta = \exp(2\pi i/m)$. In particular, $\mathcal{M}_0$ is the Cartan subalgebra $\boldsymbol{t}^{\mathbb{C}}$. In the case $\boldsymbol{g} = \boldsymbol{su}(n+1)$, $G = SU(n+1)$ we have $m = n+1$ and τ is given by conjugation by $\mathrm{diag}\{1, \zeta, \ldots, \zeta^n\}$. In this case the subspace $\mathcal{M}_k$ has all elements in the (i,j)-th place equal to zero whenever $i \not\equiv j + k \bmod (n+1)$.

The decomposition (6.1) leads to a corresponding decomposition

$$T(G/T)^{\mathbb{C}} = [\mathcal{M}_1] \oplus \ldots \oplus [\mathcal{M}_m] \tag{63}$$

of the complexified tangent bundle $T(G/T)^{\mathbb{C}}$ of G/T as a direct sum of subbundles.

Next we recall (see [19]) that G/T admits an integrable G-invariant complex structure and hence there is another important decomposition of the complexified tangent bundle $T(G/T)^{\mathbb{C}}$ of G/T

$$T(G/T)^{\mathbb{C}} = T^{1,0}G/T \oplus T^{0,1}G/T \tag{64}$$

into the $(1,0)$- and $(0,1)$-components. These are the $\pm i$-eigenbundles of the automorphism giving the complex structure and not surprisingly can be described entirely in terms of the Lie algebra. Indeed, consider the decomposition

$$\boldsymbol{g}^{\mathbb{C}} = \boldsymbol{t}^{\mathbb{C}} \oplus \boldsymbol{g}_+ \oplus \boldsymbol{g}_-, \boldsymbol{g}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \boldsymbol{g}^\alpha \tag{65}$$

of the Lie algebra $\boldsymbol{g}^{\mathbb{C}}$, where $\boldsymbol{g}^\alpha$ denotes the root space corresponding to the root α and Δ_+, Δ_- denote the subsets of positive and negative roots. For example, in the case where $\boldsymbol{g} = \boldsymbol{su}(n+1)$, this corresponds to the decomposition of $\boldsymbol{sl}(n+1;\mathbb{C})$ into diagonal, strictly lower triangular and strictly upper triangular matrices. The $(1,0)$- and $(0,1)$-components of $T(G/T)^{\mathbb{C}}$ are then given by

$$T^{1,0}G/T = [\bigoplus_{\alpha \in \Delta_+} \boldsymbol{g}^\alpha], T^{0,1}G/T = [\bigoplus_{\alpha \in \Delta_-} \boldsymbol{g}^\alpha]. \tag{66}$$

The importance of this is that a function $f : S \to G/T$ of a Riemann surface S into G/T is holomorphic if and only if $df(T^{1,0}S) \subseteq T(G/T)^{(1,0)}$.

Now suppose that U is a coordinate neighbourhood on a Riemann surface with coordinate z. Then a frame $F : U \to G$ is a *Toda frame* if

$$F^{-1}dF(T^{1,0}U) \subseteq \mathcal{M}_0 \oplus \mathcal{M}_1. \tag{67}$$

For example the frame $F : U \to SU(n+1)$ of Example 2 in Section 5 is a Toda frame since from (5.16) and (5.17)

$$F^{-1}\frac{\partial F}{\partial z} = A_0 + A_1 \tag{68}$$

with

$$A_0 = \frac{\partial \Omega}{\partial z}, A_1 = e^{\Omega} B_0 e^{-\Omega}, \tag{69}$$

in the open case and

$$A_0 = \frac{\partial \Omega}{\partial z}, A_1 = e^{\Omega} B e^{-\Omega}, \tag{70}$$

in the affine case, where $\Omega = \mathrm{diag}\,(\omega_0, \ldots, \omega_n)$.

A frame $F : U \to G$ defines in a natural way a map $\psi : U \to G/T$. Geometrically the condition (6.6) may be interpreted in terms of ψ as follows. For each point $p \in S$ the symmetry of order m of G/T at $\psi(p)$ maps $d\psi(TS)$ into itself and is given by a rotation through $2\pi/m$.

A map $\psi : S \to G/T$ of a Riemann surface S into G/T is *τ-primitive* if

(i) $d\psi(T^{1,0}S) \subseteq [\mathcal{M}_1]$;

(ii) for each $p \in S$ and all non-zero $v \in T_p^{1,0}S$, $d\psi_p(v)$ is cyclic except, perhaps, for a discrete set of points p. (See [11], [6]).

This second condition is a non-singularity condition. In the $SU(n+1)$ case it amounts to $\det(d\psi_p(v)) \neq 0$.

A Toda frame $F : U \to G$ is an *open Toda frame* if

$$F^{-1} dF(T^{1,0}U) \subseteq \mathcal{M}_0 \oplus (\mathcal{M}_1 \cap \boldsymbol{g}_+), \tag{71}$$

which implies that ψ is holomorphic, and an *affine Toda frame* if ψ is τ-primitive.

One then proves the following theorem:

THEOREM. *There are bijective correspondences*

$$\left\{ \begin{array}{c} \textit{holomorphic maps} \\ \psi : U \to G/T \\ \textit{with } d\psi(T^{1,0}U) \subseteq [\mathcal{M}_1] \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{open Toda frames} \\ \textit{plus coordinate} \end{array} \right\},$$

$$\left\{ \begin{array}{c} \tau\textit{-primitive maps} \\ \psi : U \to G/T \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{affine Toda frames} \\ \textit{plus coordinate} \end{array} \right\}.$$

Using this one may then establish natural correspondences between, on the one hand, solutions of the open $\boldsymbol{g}$-Toda equations in a domain U in $\mathbb{C}$ and holomorphic maps $\psi : U \to G/T$ with $d\psi(T^{1,0}U) \subseteq [\mathcal{M}_1]$ and, on the other, solutions of the affine $\boldsymbol{g}$-Toda equations in a domain U in $\mathbb{C}$ and τ-primitive maps of U into G/T. This amounts to showing (see [27], [6]) that the $\boldsymbol{g}$-Toda equations are the integrability conditions for a Toda frame in each case. Moreover, in the affine case, given an affine Toda frame F on U one shows that there is a corresponding essentially unique complex coordinate z on U and a smooth map $\Omega : U \to i\boldsymbol{t}$ such that

$$F^{-1} \frac{\partial F}{\partial z} = \frac{\partial \Omega}{\partial z} + \mathrm{Ad}(\exp \Omega)(B), \tag{72}$$

where $B \in T$ is a canonically defined element of order m, and $\mathrm{Ad} : G \to \mathrm{Aut}\,\boldsymbol{g}$ is the adjoint representation.

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