Much noise about nothing

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We formulate a blaah. The method is general and applies in principle to continuous time flows and discrete time maps in arbitrary dimension.

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I. INTRODUCTION

He who establishes his argument by noise and command shows that his reason is weak.

M. de Montaigne

Domenico's edits show up green.

For a hyperbolic, everywhere unstable dynamical system, each step in (Poincaré return) time subdivides the phase space into exponentially growing number of regions, each region labeled by a distinct finite symbol sequence. In the stable manifold directions these regions shrink exponentially.

On the other hand, a physical system cannot have infinite phase space resolution, for any of the following reasons:

1. any real physical system experiences a background noise
2. in any calculation we need to give a numerical predicition with a given accuracy
3. the finite precision of steps in any numerical calculation induces effective noise
4. any set of dynamical equations models nature up to a given finite accuracy

So, where are we to stop in a given calculation of the dynamics of a given hyperbolic flow? Intuitively, as we look at longer and longer orbits, their neighborhoods shrink exponentially, while the variance of the noise induced diffusion grow linearly with time; there has to be a turnover time, at which the noise-induced width overwhelms the exponentially shrinking deterministic dynamics.

Determining this time is not an talthogether trivial matters, as nonlinear dynamics interacts with noise growth in a highly nontrivial way - the noise width for each neighborhood has to be estimated from the orbit-dependent path integrals.

In a triptich of vastly overseen articles [1, 2, 3] the authors completely missed the point by concentration only on the computation of weak-noise corrections to eigenvalues of evaluation operators. What one needs to study are the local Fokker-Planck *eigenfunctions*.

In these notes we scratch at the problem by working out the trivial and well known spectar of a linear map and a circular flow.

II. ONE-DIMENSIONAL SINGLE FIXED POINT

A. One-dimensional linear map

Consider a linear map

$$x_{n+1} = \Lambda x_n + \xi_n \,, \qquad \Lambda \neq 1 \,, \tag{1}$$

with additive white noise of strength D:

$$\langle \xi \rangle \,=\, 0 \,, \qquad \langle \xi_n \xi_m \rangle \,=\, 2D\,\delta_{nm} \,. \tag{2}$$

If $|\Lambda| < 1$, in each iteration the map contracts a trajectory point by factor Λ toward the $x = 0$ fixed point, while the noise smears it across mean width $2D$. This is more precisely described by action of the corresponding discrete time noisy evolution operator on a density of trajectories $\rho(x)$:

$$\mathcal{L}_D \rho(x) = \frac{1}{\sqrt{4\pi D}} \int dy\, e^{-\frac{(x-\Lambda y)^2}{4D}} \rho(y) \,. \tag{3}$$

The continuous time Langevin equation is obtained by adding $-x_n$ to (1),

$$x_{n+1} - x_n = (\Lambda - 1) x_n + \xi_n \,,$$

dividing by δt, rewriting the stability multiplier in terms of the stability exponent, $\Lambda = e^{\lambda \delta t}$ (assuming $\Lambda > 0$), and taking $\delta t \to 0$

$$\frac{dx}{dt} = \lambda x + \hat{\xi} \,, \tag{4}$$

Continuous time Langevin equation (4) leads to the Fokker-Planck equation [4, 5]

$$\partial_t \rho(x,t) + \partial_x [x \rho(x,t)] = D \partial_{xx} \rho(x,t) \tag{5}$$

in this case known as the Ornstein-Uhlenbeck process [6, 7], analyzed to death by everybody and his mother, because it is one of about 2 analytically tractable examples - it is the imaginary time Schrödinger equation for the harmonic oscilator, etc., etc.

$|\Lambda| < 1$ case: The eigenfunctions of (3) are found to be

$$\tilde{\rho}_k(x) = \tilde{N}_k H_k(\beta x) e^{-(\beta x)^2} \,, \qquad \beta = \sqrt{\frac{1 - \Lambda^2}{4D}} \,, \tag{6}$$

where $H_k(y)$ is the kth Hermite polynomial, and N_k is a normalization constant. In this case, the eigenvalues are $-\Lambda^k$, the *same* as in the deterministic case. The zeroth eigenfunction $\rho_0 = N_0 e^{-(\beta x)^2}$ is the natural measure [8] for the Fokker-Planck evolution operator. In the deterministic limit $D \to 0$, such eigenfunctions tend to [5]

$$\rho_k(x) \to (-1)^k \delta^{(k)}(x)\,. \tag{7}$$

$|\Lambda| > 1$ **case:**

$$\rho_k(x) = N_k H_k(\alpha x)\,, \qquad \alpha = \sqrt{\frac{\Lambda^2 - 1}{4D}}\,, \tag{8}$$

with eigenvalues $-\Lambda^{-(k+1)}$.

The eigenfunctions (6) and (8) can be seen respectively as the left and the right eigenfunctions of the Fokker-Planck operator with $|\Lambda| > 1$ (or the right and the left eigenfunctions of the same operator with $|\Lambda| < 1$). They are orthonormal:

$$\int dx \rho_k(x) \rho_j(x) = \delta_{kj}\,. \tag{9}$$

A solution of the continuous time Fokker-Planck equation (5) can be expanded in the eigenfunction basis as

$$\rho(x,t) = \sum_{k=0}^{\infty} C_k \psi_k(x) e^{-s_k t} \tag{10}$$

with

$$\tilde{\psi}_k(x) = H_k(\mu x) e^{-(\mu x)^2}\,, \tag{11}$$

$$\mu = \sqrt{\frac{|\lambda|}{2D}}$$

and

$$s_k = -k\lambda \tag{12}$$

in the attracting case ($\lambda < 0$), while

$$\psi_k(x) = H_k(\mu x) \tag{13}$$

with

$$s_k = (k+1)\lambda \tag{14}$$

in the repulsive case ($\lambda > 0$).

B. Deterministic limit

In the deterministic, noiseless limit (3) reduces to the Perron-Frobenius operator:

$$\lim_{D \to 0} \mathcal{L}_D \rho(x) = \mathcal{L}\rho(x) = \int dy \delta(x - \Lambda y) \rho(y)\,. \tag{15}$$

As $D \to 0$, the eigenvalues (8) tend to the deterministic eigenvalues

$$\rho_k(x) \to \frac{x^k}{k!} \tag{16}$$

III. NOISY CIRCLE

Consider next one of the simplest 2-dimensional dynamical systems, a pair of ODE's with a circular limit cycle, together with additive isotropic white noise of strength D:

$$\begin{aligned} \dot{r} &= v - \mu r + \xi_r \\ \dot{\theta} &= \omega + \xi_\theta \end{aligned} \tag{17}$$

where

$$< \xi_r(t)\xi_r(\tau) >= 2D\delta(t-\tau), \qquad < \xi_r(t)\xi_\theta(\tau) >= 0 \tag{18}$$

The corresponding Fokker-Planck equation reads [7]:

$$\partial_t P + \frac{1}{r}\partial_r[(v - \mu r) r P] + \partial_\theta \omega P - \frac{2D}{r}\partial_r(r\partial_r P) - \frac{2D}{r^2}\partial_{\theta\theta} P = 0 \tag{19}$$

The limit cycle $r = v/\mu$ can be either stable or unstable depending on the sign of μ and v. Let us first consider the stable case ($v > 0, \mu > 0$). The first thing to look for is a stationary solution to the asymptotic form of (19):

$$\partial_r[(v - \mu r) r P_\infty] - D\partial_r(r\partial_r P_\infty) = 0 \tag{20}$$

A solution is

$$P_\infty(r) = C e^{\frac{vr}{2D} - \frac{\mu r^2}{4D}} \tag{21}$$

which implies that P_∞ is a Gaussian of width $2\sqrt{D/\mu}$ in the neighborhood of the limit cycle. Notice the similarity of this asymptotic solution with the invariant measure (6) of the one-dimensional single attracting point. The general solution to (19)) is [7]

$$P(r,\theta,t) = e^{-\frac{\mu r^2}{4D}} e^{-s_n^\nu t} r^{|\nu|} L_n^{|\nu|}(r) e^{i\nu\theta} \tag{22}$$

where $L_n^{|\nu|}(r)$ are generalized Laguerre polynomials and both the eigenvalues s_n^ν and the coefficients C_n^ν can be found numerically. Let us now consider the case when the cycle $r = v/\mu$ is unstable, that is when $v < 0, \mu < 0$. The Fokker-Planck equation is still (19) and, as before, the general solution can be found by separation of variables:

$$P(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{\nu=-\infty}^{\infty} A_n^\nu e^{-s_n^\nu t} r^{|\nu|} L_n^{|\nu|}(r) e^{i\nu\theta} \tag{23}$$

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