8

Hooke's law

When you bend a wooden stick the reaction grows notably stronger the further you go—until it perhaps breaks with a snap. If you release the bending force before it breaks, the stick straightens out again and you can bend it again and again without it changing its reaction or its shape. That is what we call elasticity.

In elementary mechanics the elasticity of a spring is expressed by Hooke's law which says that the amount a spring is stretched or compressed beyond its relaxed length is proportional to the force acting on it. In continuous elastic materials Hooke's law implies that strain is a linear function of stress. Some materials that we usually think of as highly elastic, for example rubber, do not obey Hooke's law except under very small deformations. When stresses grow large, most materials deform more than predicted by Hooke's law and in the end reach the elasticity limit where they become plastic.

Robert Hooke (1635–1703). English biologist, physicist and architect (no verified contemporary portrait exists). In physics he worked on gravitation, elasticity, built telescopes, and the discovered diffraction of light. His famous law of elasticity goes back to 1660. First stated in 1676 as a Latin anagram *ceiiinosssttuv*, he revealed it in 1678 to stand for *ut tensio sic vis*, meaning "as is the extension, so is the force".

The elastic properties of continuous materials are determined by the underlying molecular level but the relation is complicated, to say the least. Luckily, there are broad classes of materials that may be described by a few material parameters. The number of such parameters depends on the how complex the internal structure of the material is. We shall almost exclusively concentrate on structureless, isotropic elastic materials, described by just two material constants, Young's modulus and Poisson's ratio.

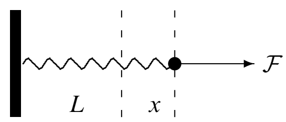
In this chapter, the emphasis will be on matters of principle. We shall derive the basic equations of linear elasticity, but only solve them in the simplest possible cases. In chapter 9 we shall solve these equations in generic cases of more practical interest.

8.1 Young's modulus and Poisson's ratio

Massless elastic springs obeying Hooke's law are a mainstay of elementary mechanics. If a spring of relaxed length L is anchored at one end and pulled by some external "agent" at the other with a force $\mathcal{F}$, its length is increased to $L + x$. Hooke's law states that there is proportionality between force and extension,

$$\mathcal{F} = kx. \tag{8.1}$$

The constant of proportionality, k, is called the *spring constant*. Newton's Third Law guarantees of course that the spring must act back on the external "agent" with a force $-kx$.



A spring of relaxed length L anchored at the left and pulled towards the right by an external force $\mathcal{F}$ will be stretched by $x = \mathcal{F}/k$.

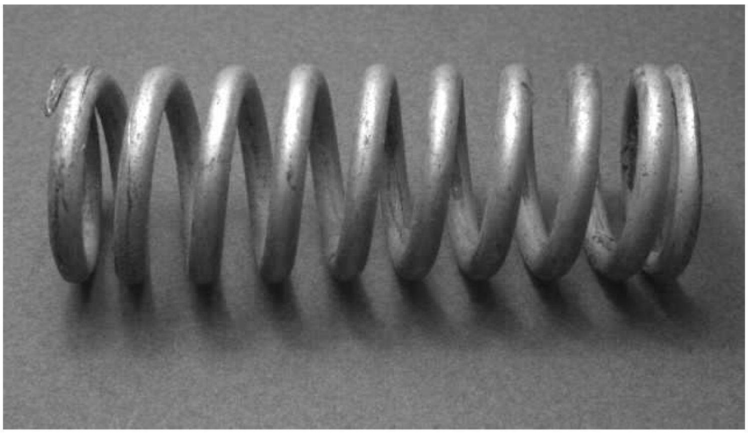
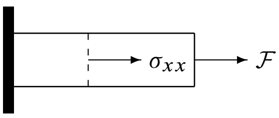


Figure 8.1. Springs come in many shapes. Here a relaxed coiled spring which responds elastically under compression as well as stretching. Reproduced under Wikimedia Commons License.

Young's modulus

Thomas Young (1773–1829). English physician, physicist and egyptologist. He observed the interference of light and was the first to propose that light waves are transverse vibrations, explaining thereby the origin of polarization. He contributed much to the translation of the Rosetta stone.

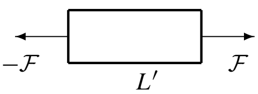
Real springs, such as the one pictured in figure 8.1, are physical bodies with mass, shape and internal structure. Almost any solid body, anchored at one end and pulled at the other, will react like a spring, when the force is not too strong. Basically, this reflects that interatomic forces are approximately elastic, when the atoms are only displaced slightly away from their positions (problem 8.1). Many elastic bodies that we handle daily, for example rubber bands, piano wire, sticks or water hoses, are long rod-like objects with constant cross section, typically made from homogeneous and *isotropic* material without any particular internal structure. Their uniform composition and simple form make such rods convenient models for real, material springs.



The same tension must act on any cross section of the rod-like spring.

The force $\mathcal{F} = kx$ necessary to extend the length L of a rod-like material spring by a small amount x must be proportional to the area, A, of the spring's cross section. For if we bundle N such such springs loosely together to make a thicker spring of area NA, the total force will have to be Nkx in order to get the same change of length. This shows that the relevant quantity to speak about is not the force itself, but rather the (average) normal stress, $\sigma_{xx} = N\mathcal{F}/NA = kx/A$, which is independent of the number of sub-springs, and thus independent of the area A of the cross section. Since the same force $\mathcal{F}$ must act on any cross section of the rod, the stress must be the same at each point along the spring. Likewise, for a smaller piece of the spring of length $L' < L$, the uniformity implies that it will be stretched proportionally less such that $x'/L' = x/L$. This indicates that the relevant parameter is not the absolute change of length x but rather the relative longitudinal extension or strain $u_{xx} = x/L$, which is independent of the length L of the spring. Consequently, the quantity,

$$E = \frac{\sigma_{xx}}{u_{xx}} = \frac{\mathcal{F}/A}{x/L} = k\frac{L}{A} \tag{8.2}$$



The force acts in opposite directions at the terminal cross sections of a smaller slice of the spring. The extension is proportionally smaller.

must be independent of the length L of the spring, the area A of its cross section, and the extension x (for $|x| \ll L$). It is a material parameter, called the *modulus of extension* or *Young's modulus* (1807). Given Young's modulus we may calculate the actual spring constant,

$$\boxed{k = E\frac{A}{L},} \tag{8.3}$$

for any spring, made from this particular material, of length L and cross section A .

Young's modulus characterizes the behavior of the material of the spring, when stretched in one direction. The relation (8.2) also tells us that a unidirectional tension σ_{xx} creates a relative extension,

$$u_{xx} = \frac{\sigma_{xx}}{E}, \tag{8.4}$$

of the material. Evidently, Hooke's law leads to a local linear relation between stress and strain, and materials with this property are generally called *linearly elastic*.

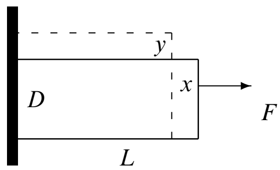
Young's modulus is by way of its definition (8.2) measured in the same units as pressure, and typical values for metals are, as the bulk modulus (2.39), of the order of 10^{11} Pa $=$ 100 GPa $=$ 1 Mbar. In the same way as the bulk modulus is a measure of the incompressibility of a material, Young's modulus is a measure of the *instretchability*. The larger it is, the harder it is to stretch the material. In order to obtain a large strain $u_{xx} \approx 100\%$, one would have to apply stresses of magnitude $\sigma_{xx} \approx E$. Such strains are, of course, not permitted in the theory of small deformations, but Young's modulus nevertheless sets the scale. The fact that the yield stress for metals is roughly a thousand times smaller than Young's modulus, shows that for metals the elastic strain can never become larger than 10^{-3}. This in turn justifies the assumption of small displacement gradients underlying the use of Cauchy's strain tensor.

Material	E [GPa]	ν
Wolfram	411	0.28
Nickel (hard)	219	0.31
Iron (soft)	211	0.29
Plain steel	205	0.29
Cast iron	152	0.27
Copper	130	0.34
Titanium	116	0.32
Brass	100	0.35
Silver	83	0.37
Glass (flint)	80	0.27
Gold	78	0.44
Quartz	73	0.17
Aluminium	70	0.35
Magnesium	45	0.29
Lead	16	0.44

Young's modulus and Poisson's ratio for various isotropic materials (from [Kaye and Laby 1995]). These values are typically a factor 1000 larger than the tensile strength. Single wall carbon nanotubes have been reported with a Young's modulus of up to 1500 GPa [YFAR00].

Example 8.1 [Rope pulling contest]: At company outings, employees often play the game of pulling in teams at each end of a rope. Before the inevitable terminal instability sets in, there is often a prolonged period where the two teams pull with almost equal force $\mathcal{F}$. If the teams each consist of 10 persons, all pulling with about their average weight of 70 kg, the total force becomes $\mathcal{F} = 7000$ N. For a rope diameter of 5 cm, the stress becomes quite considerable, $\sigma_{xx} \approx 3.6$ MPa. If Young's modulus is taken to be $E = 360$ MPa, the rope will stretch by $u_{xx} \approx 1\%$.

Poisson's ratio



A rod-like spring normally contracts in transverse directions when pulled at the ends.

Normal materials will always contract in directions transverse to the direction of extension. If the transverse size, the "diameter" D of a rod-like spring changes by y, the transverse strain becomes of the order of $u_{yy} = y/D$, and will in general be negative for a positive stretching force $\mathcal{F}$. In linearly elastic materials, the transverse strain must also be proportional to $\mathcal{F}$, so that the ratio u_{yy}/u_{xx} will be independent of $\mathcal{F}$. The negative of this ratio,

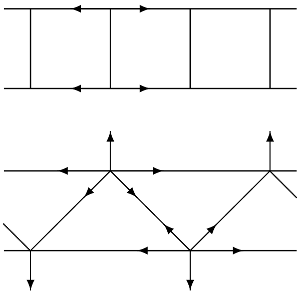
$$\nu = -\frac{u_{yy}}{u_{xx}}, \tag{8.5}$$

is called *Poisson's ratio* (1829)[1]. It is also a material parameter characterizing isotropic materials, and as we shall see below there are no others. Poisson's ratio is dimensionless, and must as we shall see lie between -1 and $+0.5$, although it is always positive for natural materials. Typical values for metals lie around 0.30 (see the margin table).

Whereas longitudinal extension can be understood as a consequence of elastic atomic bonds being stretched, it is harder to understand why materials should contract transversally. The reason is, however, that in an isotropic material there are atomic bonds in all directions, and when bonds that are not purely longitudinal are stretched, they create a transverse tension which can only be relieved by transverse contraction of the material (see the model material below). It is, however, possible to construct artificial materials that expand when stretched (see page 131).


Simeon Denis Poisson (1781–1840). French mathematician. Contributed to electromagnetism, celestial mechanics, and probability theory.

[1]Poisson's ratio is also sometimes denoted σ, but that clashes too much with the symbol for the stress tensor. Later we shall in the context of fluid mechanics also use ν for the kinematic viscosity, a choice which does not clash seriously with the use here.



Ladder with purely transverse rungs (top) and with skew rungs (bottom). The forces acting on the sides balance the transverse contraction forces.

Model material: A ladder constructed from ideal springs (see the margin figure) with rungs orthogonal to the sides will not experience a transverse contraction when stretched. If, on the other hand, some of the rungs are skew (making the ladder unusable), they will be stretched along with the ladder. But that will necessarily generate forces that tend to contract the ladder transversally and these forces either have to be balanced by external forces or relieved by actual contraction of the ladder.

The stresses and strains of stretching

Consider a stretched rod-like object laid out along the x-direction of the coordinate system. The only non-vanishing stress component is a constant tension or pull $P = \mathcal{F}/A$ along x, so that the complete symmetric stress tensor becomes,

$$\sigma_{xx} = P, \qquad \sigma_{yy} = \sigma_{zz} = 0, \qquad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0. \tag{8.6}$$

From eqs. (8.4) and (8.5) we obtain the corresponding diagonal strain components, and adding the vanishing shear strains the complete strain tensor becomes,

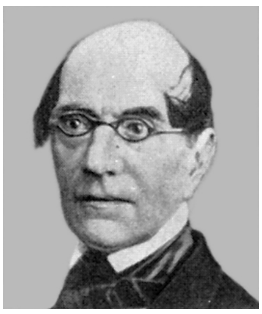
$$u_{xx} = \frac{P}{E}, \qquad u_{yy} = u_{zz} = -\nu\frac{P}{E}, \qquad u_{xy} = u_{yz} = u_{zx} = 0. \tag{8.7}$$

In section 8.3 we shall see that this is actually a feasible deformation which can be represented by a suitable displacement field.

8.2 Hooke's law in isotropic matter

Hooke's law is a linear relation between force and extension, and continuous materials with a linear relation between stress and strain implement the local version of Hooke's law. Since there are six independent strain components and six independent stress components, a general linear relation between them can become quite involved. In isotropic matter where there are no internal material directions defined, the local version of Hooke's law takes the simplest form possible. In the following we shall mostly consider *isotropic homogeneous* matter with the same material properties everywhere. Towards the end of this section we shall, however, briefly touch upon anisotropic materials, such as crystals.

The Lamé coefficients



Gabriel Lamé (1795–1870). French mathematician, engineer and physicist. Worked on curvilinear coordinates, number theory and mathematical physics.

The absence of internal directions in isotropic matter tells us that there are only two tensors available to construct a linear relation between the tensors σ_{ij} and u_{ij}. One is the strain tensor u_{ij} itself, the other is the Kronecker delta δ_{ij} multiplied with the trace of the strain tensor $\sum_k u_{kk}$. The trace is in fact the only scalar quantity that can be formed from a linear combination of the strain tensor components. Consequently, the most general strictly linear tensor relation between stress and strain is of the form (Cauchy 1822; Lamé 1852),

$$\boxed{\sigma_{ij} = 2\mu\, u_{ij} + \lambda\, \delta_{ij} \sum_k u_{kk}.} \tag{8.8}$$

The coefficients λ and μ are material constants, called *elastic moduli* or *Lamé coefficients*. Whereas λ has no special name, μ is called the *shear modulus* or the *modulus of rigidity*, because it controls the magnitude of shear (off-diagonal) stresses. Since the strain tensor is dimensionless, the Lamé coefficients are, like the stress tensor itself, measured in units of pressure. We shall see below that the Lamé coefficients are directly related to Young's modulus and Poisson's ratio.

Written out in full detail, the local version of Hooke's law takes the form,

$$\sigma_{xx} = (2\mu + \lambda)u_{xx} + \lambda(u_{yy} + u_{zz}), \qquad \sigma_{yz} = \sigma_{zy} = 2\mu u_{yz}, \tag{8.9a}$$
$$\sigma_{yy} = (2\mu + \lambda)u_{yy} + \lambda(u_{zz} + u_{xx}), \qquad \sigma_{zx} = \sigma_{xz} = 2\mu u_{zx}, \tag{8.9b}$$
$$\sigma_{zz} = (2\mu + \lambda)u_{zz} + \lambda(u_{xx} + u_{yy}), \qquad \sigma_{xy} = \sigma_{yx} = 2\mu u_{xy}. \tag{8.9c}$$

For $\mu = 0$ the shear stresses vanish and the stress tensor becomes proportional to the unit matrix, $\sigma_{ij} = \lambda\delta_{ij}\sum_k u_{kk}$. This shows that an isotropic elastic material with vanishing shear modulus is in some respects similar to a fluid at rest—although it is definitely not a fluid. It is sometimes convenient to use this observation to verify the result of a calculation by comparison with a similar calculation for a fluid at rest.

In deriving the local version of Hooke's law we have tacitly assumed that all stresses vanish when the strain tensor vanishes. In a relaxed (undeformed) isotropic material at rest with constant temperature, the pressure must also be constant, so that the above relationship should be understood as the extra stress caused by the deformation itself. If the temperature varies across the undeformed body, thermal expansion will give rise to thermal stresses that also must be taken into account.

Finally, it should also be mentioned that the stress and strain tensors in the Euler representation are both viewed as functions of the actual position of a material particle. In the Lagrange representation they are instead viewed as functions of the position of a material particle in the undeformed body. The difference is, however, negligible for slowly varying displacement fields (small displacement gradients), and will be ignored here.

Form invariance of natural laws: The arguments leading to the most general isotropic relation (8.8) depend strongly on our understanding of tensors as geometric objects in their own right. As soon as we have cast a law of nature in the form of a scalar, vector or tensor relation, its validity in all Cartesian coordinate systems is immediately guaranteed. The form invariance of the natural laws under transformations that relate different observers has since Einstein been an important guiding principle in the development of modern theoretical physics.

Young's modulus and Poisson's ratio

Young's modulus and Poisson's ratio must be functions of the two Lamé coefficients. To derive the relations between these material parameters we insert the stresses (8.6) and strains (8.7) for the simple stretching of a material spring into the general relations (8.9) , and get,

$$P = (2\mu + \lambda)\frac{P}{E} - 2\lambda\nu\frac{P}{E}, \qquad 0 = -(2\mu + \lambda)\nu\frac{P}{E} + \lambda\left(-\nu\frac{P}{E} + \frac{P}{E}\right).$$

Solving these equations for E and ν, we obtain

$$\boxed{E = \mu\frac{3\lambda + 2\mu}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}.} \tag{8.10}$$

Conversely, we may also express the Lamé coefficients in terms of Young's modulus and Poisson's ratio,

$$\boxed{\lambda = \frac{E\nu}{(1 - 2\nu)(1 + \nu)}, \qquad \mu = \frac{E}{2(1 + \nu)}.} \tag{8.11}$$

In practice it is Young's modulus and Poisson's ratio that are found in tables, and the above relations immediately allow us to calculate the Lamé coefficients.

Average pressure and bulk modulus

The trace of the stress tensor (8.8) becomes

$$\sum_i \sigma_{ii} = (2\mu + 3\lambda) \sum_i u_{ii}, \tag{8.12}$$

because the trace of the Kronecker delta is $\sum_i \delta_{ii} = 3$. Since the stress tensor in Hooke's law represents the change in stress due to the deformation we find the change in the mechanical pressure (6.19) on page 103 caused by the deformation,

$$\Delta p = -\frac{1}{3}\sum_i \sigma_{ii} = -\left(\lambda + \frac{2}{3}\mu\right)\sum_i u_{ii}. \tag{8.13}$$

The trace of the strain tensor was previously shown in eq. (7.34) to be proportional to the relative local change in density, $\sum_i u_{ii} = \nabla \cdot \boldsymbol{u} = -\Delta\rho/\rho$, and using the definition of the bulk modulus (2.39) $K = \rho dp/d\rho \approx \rho\,\Delta p/\Delta\rho$, one finds the (isothermal) bulk modulus,

$$\boxed{K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}.} \tag{8.14}$$

The bulk modulus equals Young's modulus for $\nu = 1/3$, which is in fact a typical value for ν in many materials. The Lamé coefficients and the bulk modulus are all proportional to Young's modulus and thus of the same order of magnitude.

Inverting Hooke's law

Hooke's law (8.8) may be inverted so that strain is instead expressed as a linear function of stress. Solving for u_{ij} and making use of (8.12) , we get

$$u_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij}\sum_k \sigma_{kk}. \tag{8.15}$$

Introducing Young's modulus and Poisson's ratio from (8.10) , this takes the simpler form

$$\boxed{u_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sum_k \sigma_{kk}.} \tag{8.16}$$

Explicitly, we find for the six independent components

$$u_{xx} = \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E}, \qquad u_{yz} = u_{zy} = \frac{1+\nu}{E}\sigma_{yz}, \tag{8.17a}$$

$$u_{yy} = \frac{\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})}{E}, \qquad u_{zx} = u_{xz} = \frac{1+\nu}{E}\sigma_{zx}, \tag{8.17b}$$

$$u_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E}, \qquad u_{xy} = u_{yx} = \frac{1+\nu}{E}\sigma_{xy}. \tag{8.17c}$$

Evidently, if the only stress is $\sigma_{xx} = P$, we obtain immediately from the correct relations (8.7) for simple stretching.

Positivity constraints

The bulk modulus $K = \lambda + \frac{2}{3}\mu$ cannot be negative, because a material with negative K would expand when put under pressure (see however [WL05]). Imagine what would happen to such a strange material if placed in a closed vessel surrounded by normal material, for example air. Increasing the air pressure a tiny bit would make the strange material expand, causing a further pressure increase followed by expansion until the whole thing blew up. Likewise, materials with negative shear modulus, μ, would mimosa-like pull away from a shearing force instead of yielding to it. Formally, it may be shown (see section 8.4) that the conditions $3\lambda + 2\mu > 0$ and $\mu > 0$ follow from the requirement that the elastic energy density should be bounded from below. Although λ, in principle, may assume negative values, natural materials always have $\lambda > 0$.

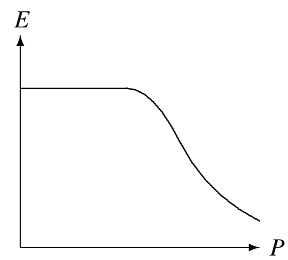
Young's modulus cannot be negative because of these constraints, and this confirms that a rope always stretches when pulled at the ends. If there were materials with the ability to contract when pulled, they would also behave magically. As you begin climbing up a rope made from such material, it pulls you further up. If such materials were ever created, they would spontaneously contract into nothingness at the first possible occasion or at least into a state with a positive value of Young's modulus.

Poisson's ratio $\nu = \lambda/2(\lambda + \mu)$ reaches its maximum $\nu \to \frac{1}{2}$ for $\lambda \to \infty$. The limit $\nu \to \frac{1}{2}$ corresponds to $\mu \to 0$, and in that limit there are no shear stresses in the material which as mentioned in some respects behaves like a fluid at rest. Since the bulk modulus $K = \lambda + \frac{2}{3}\mu$ is positive we have $\lambda > -\frac{2}{3}\mu$, and since Poisson's ratio is a monotonically increasing function of λ, the minimal value is obtained for $\lambda = -\frac{2}{3}\mu$, and becomes $\nu = -1$.

Although natural materials shrink in the transverse directions when stretched, and thus have $\nu > 0$, there may actually exist so-called *auxetic* materials, which expand transversally when stretched, without violating the laws of physics. Composite materials and foams with isotropic elastic properties and negative Poisson's ratio have in fact been made [Lak92, Bau03, HCG&08].

Limits to Hooke's law

Hooke's law for isotropic materials, expressed by the linear relationship between stress and strain, is only valid for stresses up to a certain value, called the *proportionality limit*. Beyond the proportionality limit, nonlinearities set in, and the present formalism becomes invalid. Eventually one reaches a point, called the *elasticity limit*, where the material ceases to be elastic and undergoes permanent deformation without much further increase of stress but instead accompanied with loss of energy to heat. In the end the material may even fracture. Hooke's law is, however, a very good approximation for most metals under normal conditions where stresses are tiny compared to the elastic moduli.



Sketch of how Young's modulus might vary as a function of increased tension. Beyond the proportionality limit, its effective value becomes generally smaller.

* Anisotropic materials

Anisotropic (also called *aeolotropic*) materials have different properties in different directions. In the most general case, Hooke's law takes the form,

$$\boxed{\sigma_{ij} = \sum_{kl} E_{ijkl} u_{kl},} \tag{8.18}$$

where the coefficients E_{ijkl} form a tensor of rank 4, called the *elasticity tensor*.

For isotropic materials the elasticity tensor one may verify from the the expression (8.8) that the elasticity tensor takes the form,

$$E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right). \tag{8.19}$$

In general materials there are more than two material constants, the actual number depending on the intrinsic complexity of the internal structure of the material.

Requiring only that the stress tensor be symmetric, the elasticity tensor connects the six independent components of stress with the six independent components of strain and can in principle contain $(6 \times 6) = 36$ independent parameters. If one further demands that an elastic energy function should exist (see page 135) this (6×6) array of coefficients must itself be symmetric under the exchange $ij \leftrightarrow kl$, i.e. $E_{ijkl} = E_{klij}$, and can thus only contain $(6 \times 7)/2 = 21$ independent parameters. The orientation of a completely asymmetric material relative to the coordinate system takes three parameters (Euler angles), so altogether there may be up to 18 truly independent material constants characterizing the intrinsic elastic properties of such a material, a number actually realized by triclinic crystals [Landau and Lifshitz 1986, Green and Adkins 1960].

Example 8.2 [Cubic symmetry]: In crystals with cubic symmetry the atoms or molecules are arranged in a lattice of cubic cells with identical structure. In the natural coordinate system along the lattice axes, the material has four-fold rotation symmetry (90°) around the each of the three coordinate axes and mirror symmetry in each of the the the three coordinate planes. There are thus $4 \times 3 \times 3 = 36$ elements in the symmetry group (and not an infinity as in isotropic materials). These symmetries demand that the set of equations expressing Hooke's law remains unchanged under change of sign of any of the coordinate axes and under interchange of any two coordinate axes, and this limits their form severely. The diagonal components take the same form as for full isotropy, whereas the off-diagonal components also take the same form but with a different material constant κ for pure shear,

$$\sigma_{xx} = (2\mu + \lambda)u_{xx} + \lambda(u_{yy} + u_{zz}), \qquad \sigma_{yz} = \sigma_{zy} = 2\kappa u_{yz}, \tag{8.20a}$$

$$\sigma_{yy} = (2\mu + \lambda)u_{yy} + \lambda(u_{zz} + u_{xx}), \qquad \sigma_{zx} = \sigma_{xz} = 2\kappa u_{zx}, \tag{8.20b}$$

$$\sigma_{zz} = (2\mu + \lambda)u_{zz} + \lambda(u_{xx} + u_{yy}), \qquad \sigma_{xy} = \sigma_{yx} = 2\kappa u_{xy}. \tag{8.20c}$$

Cubic symmetry thus has three material constants. Notice that this form is only valid in a coordinate system with axes coinciding with the lattice axes.

8.3 Static uniform deformation

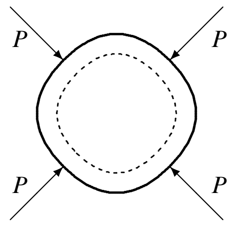
To see how Hooke's law works for continuous systems, we now turn to the extremely simple case of a *static uniform deformation* in which the strain tensor u_{ij} takes the same value everywhere in a body at all times. Hooke's law (8.8) then ensures that the stress tensor is likewise constant throughout the body, so that all its derivatives vanish, $\nabla_k\sigma_{ij} = 0$. From the condition (6.22) for mechanical equilibrium, $f_i + \sum_j \nabla_j\sigma_{ij} = 0$, it follows that $f_i = 0$ so that uniform deformation necessarily excludes body forces. Conversely, in the presence of body forces such as gravity or electromagnetism, there must always be non-uniform deformation of an isotropic material, in fact a quite reasonable conclusion.

Furthermore, at the boundary of a uniformly deformed body, the stress vector $\boldsymbol{\sigma} \cdot \boldsymbol{n}$ is required to be continuous, and this puts strong restrictions on the form of the external forces that may act on the surface of the body. Uniform deformation is for this reason only possible under very special circumstances, but when it applies the displacement field is nearly trivial.

Uniform compression

In a fluid at rest with a constant pressure P, the stress tensor is $\sigma_{ij} = -P\delta_{ij}$ everywhere. If a solid body made from isotropic material is immersed into this fluid, the natural guess is that the pressure will also be P inside the body. Inserting $\sigma_{ij} = -P\delta_{ij}$ into (8.16) and using that $\sum_k \sigma_{kk} = -3P$, the strain may be written,

$$u_{ij} = -\frac{P}{3K}\delta_{ij}. \tag{8.21}$$



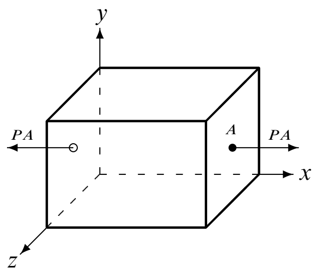
A body made from isotropic, homogenous material subject to a uniform external pressure will be uniformly compressed.

Since $u_{xx} = \nabla_x u_x = -P/3K$, we may immediately integrate this equation (and the similar ones for u_{yy} and u_{zz}) and obtain a particular solution to the displacement field,

$$u_x = -\frac{P}{3K}\,x, \qquad u_y = -\frac{P}{3K}\,y, \qquad u_z = -\frac{P}{3K}\,z. \tag{8.22}$$

The most general solution is obtained by adding an arbitrary small rigid body displacement to this solution.

Note that we arrived at this result by making an educated *guess* for the form of the stress tensor inside the body. It could in principle be wrong but is in fact correct due to a uniqueness theorem to be derived in section 8.4. The theorem guarantees, in analogy with the uniqueness theorems of electrostatics, that provided the equations of mechanical equilibrium and the boundary conditions are fulfilled by the guess (which they are here), there is essentially only one solution to any *elastostatic* problem. The only liberty left is the arbitrary small rigid body displacement which may always be added to the solution.

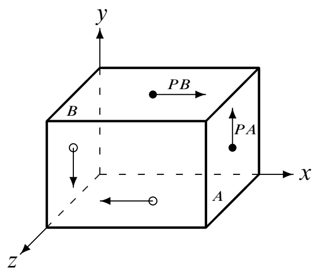


Uniformly rectangular stretched rod under constant longitudinal tension P. The shape of the cross section can be arbitrary, but must be constant throughout the rod, with area A.

Uniform stretching

At the beginning of this chapter we investigated the reaction of a rod-like material body stretched along its main axis, say the x-direction, by means of a tension $\sigma_{xx} = P$ acting uniformly over its constant cross section. If there are no other external forces acting on the body, the natural guess is that the only non-vanishing component of the stress tensor is $\sigma_{xx} = P$ throughout the body, because that gives no surface stresses on the sides of the rod. Inserting this into (8.15) , we obtain as before the strains (8.7) . The corresponding displacement field is again found by integrating $\nabla_x u_x = u_{xx}$ etc, and we find the particular solution

$$u_x = \frac{P}{E}\,x, \qquad u_y = -\nu\frac{P}{E}\,y, \qquad u_z = -\nu\frac{P}{E}\,z. \tag{8.23}$$



Clamped rectangular slab under constant shear stress $\sigma_{xy} = \sigma_{yx} = P$. The upper clamp is acted upon by an external force $\mathcal{F}_x = PB$ where B is the area of the clamp, while the force on the lower is $\mathcal{F}_x = -PB$. The symmetry of the stress tensor demands a clamp force $\mathcal{F}_y = PA$ on the right hand side and a force $\mathcal{F}_y = -PA$ on the left hand side where A is the area of that clamp.

It describes a simple dilatation along the x-axis and a compression towards the x-axis in the yz-plane.

If the rod-like spring is clamped on the sides by a hard material, the boundary conditions are instead $u_y = u_z = 0$ on the sides. In that case the only non-vanishing constant strain is u_{xx}, and the solution is obtained along the same lines as above (see problem 8.4).

Uniform shear

Finally we return to the example from page 99 of a clamped rectangular slab of homogeneous, isotropic material subject to shear stress along one side (here the x-direction). As we argued there, the shear stress $\sigma_{xy} = P$ must be constant throughout the material. What we did not appreciate at that point was that the symmetry of the stress tensor demands that also $\sigma_{yx} = P$ everywhere in the material. As a consequence, there must also be shearing forces acting on the ends of the slab whereas the remaining sides are free (see the margin figure).

Assuming that there are no other stresses, the only strain component becomes $u_{xy} = P/2\mu$, and using that $2u_{xy} = \nabla_x u_y + \nabla_y u_x$, we find a particular solution

$$u_x = \frac{P}{\mu}\, y, \qquad u_y = u_z = 0. \tag{8.24}$$

In these coordinates the displacement in the x-direction vanishes for $y = 0$ and grows linearly with y. Notice that each infinitesimal "needle" besides the deformation is also rotated by a small angle $\phi = \frac{1}{2}(\nabla_x u_y - \nabla_y u_x) = -P/2\mu$ around the z-axis (see problem 7.8).

∗ Finite uniform deformation in the Euler representation

In section 7.5 on page 120 we discussed the extension of the strain tensor formalism to strongly varying displacement fields. We found that in the Euler representation where the displacement field is a function of the actual coordinates $\boldsymbol{x}$, the correct definition of the strain tensor is nonlinear,

$$u_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i - \sum_k \nabla_i u_k \nabla_j u_k\right). \tag{8.25}$$

General arguments again ensure that a linear relationship[2] between stress and strain for a homogeneous, isotropic material is given by eq. (8.8) . We shall for simplicity assume that the elastic coefficients are constant.

Under these assumptions it follows as before that a constant stress tensor implies a constant strain tensor. In the three cases discussed above, the strain tensors are calculated from the stress tensors in the same way as before. Differences first arise in the calculation of the Eulerian displacement fields. For uniform compression the strain tensor is again $u_{ij} = -(P/3K)\delta_{ij}$, and since the displacement field must be a uniform scaling we find the solution,

$$u_x = Ax, \qquad u_y = Ay, \qquad u_z = Az, \qquad \text{with } A = 1 - \sqrt{1 + \frac{2P}{3K}} \tag{8.26}$$

For $|P| \ll K$ this turns into the solution (8.22) . Notice that the solution breaks down for a dilatation with $P < -3K/2$. The reason for the singular behavior for $P = -3K/2$ is that it yields $\boldsymbol{u} = \boldsymbol{x}$, implying that all material particles originally were found at the same position, $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{u} = \boldsymbol{0}$. Similar behavior is found for uniform stretching and uniform shear.

8.4 Elastic energy

The work performed by the external force in extending a spring further by the amount dx is $dW = \mathcal{F}\,dx = kx\,dx$. Integrating this expression, we obtain the total work $W = \frac{1}{2}kx^2$, which is identified with the well-known expression for the elastic energy, $\mathcal{E} = \frac{1}{2}kx^2$, stored in a stretched or compressed spring. Calculated per unit of volume $V = AL$ for a rod-like spring, we find the density of elastic energy in the material

$$\varepsilon = \frac{\mathcal{E}}{V} = \frac{kx^2}{2V} = \frac{1}{2}Eu_{xx}^2 = \frac{\sigma_{xx}^2}{2E}. \tag{8.27}$$

where $u_{xx} = x/L$ and $\sigma_{xx} = \mathcal{F}/A$. Poisson's ratio ν does not appear, because the transverse contraction given by Poisson's ratio can play no role in building up the elastic energy of a stretched spring, when there are no forces acting on the sides of the spring.

[2] If linearity is not required the most general stress tensor of an isotropic material will be a linear combination of the three matrices $\mathbf{1}$, $\boldsymbol{u}$, and $\boldsymbol{u}^2$ with coefficients that may depend on the three scalar invariants of the strain tensor $\boldsymbol{u}$. Such materials are called *hyperelastic* [Doghri 2000] and may, for example, be used to describe the elasticity of rubber.

Deformation work

In the general case, strains and stresses vary across the body, and the calculation becomes more complicated. To determine the general expression for the elastic energy density we use the expression derived in section 7.4 for the virtual work performed under an infinitesimal change δu_{ij} of the strain field,

$$\delta W_{\text{deform}} = \int_V \boldsymbol{\sigma} : \delta \boldsymbol{u} \, dV. \tag{8.28}$$

where $\boldsymbol{\sigma} : \delta \boldsymbol{u} = \sum_{ij} \sigma_{ij} \delta u_{ij}$. The volume V is the actual volume of the body, not the volume of the undeformed body, but the difference between integrating over the actual volume and the undeformed volume is negligible for a slowly varying displacement field.

To build up a complete strain field, one must divide the process into infinitesimal steps and add the work for each step. Here one should remember that the stress tensor depends on the strain tensor, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u})$ or $\sigma_{ij} = \sigma_{ij}(\{u_{kl}\})$, and therefore also changes along the road (in the six-dimensional space of possible symmetric deformation tensors). For the final result to be independent of the road along which the strain is built up, we must require that the cross derivatives are equal,

$$\boxed{\frac{\partial \sigma_{ij}}{\partial u_{kl}} = \frac{\partial \sigma_{kl}}{\partial u_{ij}}.} \tag{8.29}$$

When this condition is fulfilled the work spent in building up the deformation may be interpreted as an energy stored in the deformation of the body.

Denoting the deformation energy density $\varepsilon = \varepsilon(\boldsymbol{u})$, the total deformation energy becomes

$$\mathcal{E} = \int_V \varepsilon \, dV, \tag{8.30}$$

and the stress-strain relation is derived from the energy density,

$$\sigma_{ij} = \frac{\partial \varepsilon}{\partial u_{ij}}. \tag{8.31}$$

Evidently, this stress tensor satisfies the condition (8.29) .

Energy in general linear materials

If the stress tensor is a general linear function of the strain tensor as in eq. (8.18) ,

$$\sigma_{ij} = \sum_{kl} E_{ijkl} u_{kl} \tag{8.32}$$

the above condition translates into the symmetry of the elasticity tensor,

$$E_{ijkl} = E_{klij}. \tag{8.33}$$

When this condition is fulfilled, the energy of a strain in its own stress field may now be built up by adding infinitesimal strain changes, starting from zero stress. On average each infinitesimal strain change only meets half the final stress, so that the total energy density becomes,

$$\boxed{\varepsilon = \frac{1}{2} \sum_{ijkl} E_{ijkl} u_{ij} u_{kl}.} \tag{8.34}$$

Not unsurprisingly, it is a quadratic polynomial in the strain tensor components. It may also be written compactly as $\varepsilon = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{u}$.

Isotropic materials

For isotropic materials with σ_{ij} given by Hooke's law (8.8) the energy density simplifies to,

$$\varepsilon = \mu \sum_{ij} u_{ij}^2 + \frac{1}{2}\lambda \left(\sum_i u_{ii}\right)^2 = \mu \mathrm{Tr}\left(\boldsymbol{u}^2\right) + \frac{1}{2}\lambda \mathrm{Tr}(\boldsymbol{u})^2 \tag{8.35}$$

The energy density must be bounded from below, for if it were not, elastic materials would be unstable, and an unlimited amount of energy could be obtained by increasing the state of deformation. From the boundedness, it follows immediately that the shear modulus must be positive, $\mu > 0$, because otherwise we might make one of the off-diagonal components of the strain tensor, say u_{xy}, grow without limit and extract unlimited energy. The condition on λ is more subtle because the diagonal components of the strain tensor are involved in both terms. For a uniform deformation with $u_{ij} = k\delta_{ij}$ the energy density becomes $\varepsilon = \frac{3}{2}(3\lambda + 2\mu)k^2$, implying that $3\lambda + 2\mu > 0$, i.e. that the bulk modulus (8.14) is positive. In problem 8.7 it is shown that there are no stronger conditions.

The energy density may also be expressed as a function of the stress tensor,

$$\varepsilon = \frac{1+\nu}{2E} \sum_{ij} \sigma_{ij}^2 - \frac{\nu}{2E} \left(\sum_i \sigma_{ii}\right)^2 = \frac{1+\nu}{2E} \mathrm{Tr}\left(\boldsymbol{\sigma}^2\right) - \frac{\nu}{2E} \mathrm{Tr}(\boldsymbol{\sigma})^2. \tag{8.36}$$

Under uniform stretching where the only non-vanishing stress is σ_{xx}, we have $\mathrm{Tr}\left(\boldsymbol{\sigma}^2\right) = \mathrm{Tr}(\boldsymbol{\sigma})^2 = \sigma_{xx}^2$, and all the dependence on Poisson's ratio ν cancels out, bringing us back to the energy density in a rod-like spring (8.27) .

Total energy in an external field

In an external gravitational field, a small displacement $\boldsymbol{u}(\boldsymbol{x})$ will lead to a change in gravitational potential energy of a material particle (defined in eq. (2.33) on page 31),

$$d\mathcal{E} = \big(\Phi(\boldsymbol{x}) - \Phi(\boldsymbol{X})\big)\, dM = \big(\Phi(\boldsymbol{x}) - \Phi(\boldsymbol{x} - \boldsymbol{u}(\boldsymbol{x}))\big) dM \approx -\rho \boldsymbol{g} \cdot \boldsymbol{u}\, dV, \tag{8.37}$$

where in the last step we have expanded the potential to first order in $\boldsymbol{u}$ and used that $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$. Since the elastic energy is of second order in the displacement gradients, we should for consistency also expand the potential to second order, but for normal bodies the second order contribution to the gravitational energy density is, however, much smaller than the elastic energy density (see problem 8.8).

Adding the gravitational contribution, the total energy of a linearly elastic body in a conservative external force field $\boldsymbol{f} = \rho \boldsymbol{g}$ is accordingly,

$$\mathcal{E} = -\int_V \boldsymbol{f} \cdot \boldsymbol{u}\, dV + \int_V \tfrac{1}{2} \boldsymbol{\sigma} : \boldsymbol{u}\, dV, \tag{8.38}$$

where $\boldsymbol{\sigma} : \boldsymbol{u} = \sum_{ij} \sigma_{ij} u_{ji}$.

∗ Uniqueness of elastostatics solutions

To prove the uniqueness of the solutions to the mechanical equilibrium equations (6.22) for linearly elastic materials we assume that we have found two solutions $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ that both satisfy the equilibrium equations and the boundary conditions. Due to the linearity, the difference between the solutions $\boldsymbol{u} = \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}$ generates a difference in strain $u_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)$ and a difference in stress $\sigma_{ij} = \sum_{kl} E_{ijkl} u_{kl}$. Under the assumption that the body forces are identical for the two fields (as is the case in constant gravity) the change in stress satisfies the equation, $\sum_j \nabla_j \sigma_{ij} = 0$, and by means of Gauss' theorem we obtain,

$$0 = \int_V \sum_{ij} u_i \nabla_j \sigma_{ij} \, dV = \oint_S \sum_{ij} u_i \sigma_{ij} \, dS_j - \int_V \sum_{ij} \sigma_{ij} \nabla_j u_i \, dV.$$

Here the surface integral vanishes because of the boundary conditions, which either specify the same displacements for the two solutions at the surface, i.e. $u_i = 0$, or the same stress vectors, i.e. $\sum_j \sigma_{ij} dS_j = 0$. Using the symmetry of the stress tensor we get,

$$0 = \sum_{ij} \sigma_{ij} \nabla_j u_i = \sum_{ij} \sigma_{ij} u_{ij} = \sum_{ijkl} E_{ijkl} u_{ij} u_{kl}.$$

The integrand is of the same form as the energy density (8.34) , which is always assumed to be positive definite. Consequently, the integral can only vanish if the strain tensor for the difference field vanishes everywhere in the body, i.e. $u_{ij} = 0$.

We have thus shown that given the boundary conditions, there is essentially only one solution to the equations of mechanical equilibrium in linear elastic materials. Although the two displacement fields may, in principle, differ by a small rigid body displacement, they will give rise to identical deformations everywhere in the body. Therefore, if we have somehow guessed a solution satisfying the equations of mechanical equilibrium and the boundary conditions, it will necessarily be the right one.

Problems

8.1 Two particles interact with a smooth distance dependent force $\mathcal{F}(r)$. Show that the force obeys Hooke's law in the neighborhood of an equilibrium configuration.

8.2 **(a)** Show that we may write (8.8) in the form

$$\sigma_{ij} = 2\mu \left(u_{ij} - \frac{1}{3}\delta_{ij} \sum_k u_{kk} \right) + K\delta_{ij} \sum_k u_{kk}. \tag{8.39}$$

(b) Show that first term gives no contribution to the average pressure.

8.3 A displacement field is given by

$$u_x = \alpha(x + 2y) + \beta x^2,$$
$$u_y = \alpha(y + 2z) + \beta y^2,$$
$$u_z = \alpha(z + 2x) + \beta z^2,$$

where α and β are 'small'.

(a) Calculate the divergence and curl.
(b) Calculate Cauchy's strain tensor.
(c) Calculate the stress tensor in a linear elastic medium with Lamé coefficients λ and μ for the special case $\beta = 0$.

8.4 A beam with constant cross section is fixed such that its sides cannot move. One end of the beam is also held in place while the other end is pulled with a uniform tension P. Determine the strains, stresses and the displacement field in the beam.

8.5 A rectangular beam with its axis along the x-axis is fixed on the two sides orthogonal to the y-axis but left free on the two sides orthogonal to the z-axis. The beam is held fixed at one end and pulled with a uniform tension P the other. Determine the strains, stresses and the displacement field in the beam.

8.6

(a) Show that Cauchy's strain tensor always fulfills the condition,

$$\nabla_y^2 u_{xx} + \nabla_x^2 u_{yy} = 2\nabla_x \nabla_y u_{xy}. \tag{8.40}$$

(b) Assume now that the only non-vanishing components of the stress tensor are σ_{xx}, σ_{yy} and $\sigma_{xy} = \sigma_{yx}$. Formulate Cauchy's equilibrium equations for this stress tensor in the absence of volume forces.

(c) Show that the following stress tensor is a solution to the equilibrium equations,

$$\sigma_{xx} = \nabla_y^2 \phi, \qquad \sigma_{yy} = \nabla_x^2 \phi, \qquad \sigma_{xy} = -\nabla_x \nabla_y \phi, \tag{8.41}$$

where ϕ is an arbitrary function of x and y.

(d) Calculate the strain tensor in terms of ϕ in an isotropic elastic medium, and show that the condition (8.40) implies that ϕ must satisfy the biharmonic equation,

$$\nabla_x^4 \phi + \nabla_y^4 \phi + 2\nabla_x^2 \nabla_y^2 \phi = 0. \tag{8.42}$$

(e) Determine a solution of the displacement field (u_x, u_y, u_z), when $\phi = xy^2$ and Young's modulus is set to $E = 1$. Hint: begin by integrating the diagonal elements of the strain tensor, and afterwards add to extra terms to u_x to get the correct off-diagonal elements.

* **8.7** Show that one may write the energy density (8.34) in the following form

$$\epsilon = \frac{1}{2}[\lambda - 2\mu(3\alpha^2 - 2\alpha)]\left(\sum_i u_{ii}\right)^2 + \mu \sum_{ij}\left(u_{ij} - \alpha \sum_k u_{kk}\delta_{ij}\right)^2 \tag{8.43}$$

where α is arbitrary. Use this to argue that $3\lambda + 2\mu > 0$ and that this is the strictest condition on λ.

* **8.8** Estimate the ratio of the second order gravitational energy density (8.37) to the elastic energy density for a body of size L, and show that it is normally tiny.

8.9 Calculate the Euler field for a beam subject to finite uniform stretching P. Discuss the singularity.

8.10 Calculate the Euler field for a slab subject to finite uniform shear P. Discuss the singularity.

8.11 Show that the field of uniform compression (8.22) is a superposition of uniform stretching fields (8.23) in the three coordinate directions.