Langevin equation

Chapter 28

Weak noise dynamics

⇓PRIVATE

> In fact I'm Vattay Gábor, just these Indo-Europeans mix up the right order.

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(G. Vattay and P. Cvitanović)

When a deterministic trajectory is smeared out under influence of Gaussian white noise of strength D, the deterministic dynamics is recovered in the weak noise limit $D \to 0$.

⇓PRELIMINARY
⇑PRELIMINARY

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28.1 Weak noise

Consider a deterministic flow

$$\dot{x} = v(x) + \xi(t)\,, \tag{28.1}$$

perturbed by a weak stochastic term $\xi(t)$. The best studied example is the Langevin equation for a Brownian particle, in which one replaces the Newton's equation for force by two counter-balancing forces: random kicks $\xi(t)$ which tend to smear out a particle trajectory, and a damping term which tries to drive the velocity to zero. Assuming that the distribution of kicks is Gaussian (white noise), and the kicks so frequent that they can be assumed uncorrelated, one derives a fluctuation-dissipation theorem that relates the temperature of the background gas to the damping constant and the mass of the Brownian particle.

white noise
noise!white
Fokker-Planck equation

Here we shall consider more general stochastic differential equations in the weak noise limit. The equation (28.1) is meaningless unless we specify the probability distribution for $\xi(t)$'s. If the noise is weak, $|v(x)| > |\xi|$, and uncorrelated in time

$$\left\langle \xi(t)\xi(t') \right\rangle = 2D\delta(t-t')\,, \tag{28.2}$$

it is common to assume that the noise is *white*, that is, specified by a Gaussian distribution after an infinitesimal time step $\delta\tau$,

$$p(\xi,\delta\tau) = \sqrt{\frac{\delta\tau}{4\pi D}} e^{-\frac{\xi^2}{4D}\delta\tau} \tag{28.3}$$

with boundary condition that as $D \to 0$, the distribution tends to the (deterministic) Dirac delta function. The time evolution of $\rho(x,t)$, the probability distribution of x, is governed by the (generalized) Fokker-Planck equation

$$\partial_t \rho + \nabla(\rho v(x)) = D\nabla^2\rho. \tag{28.4}$$

eliminate symbol nabla ∇

(The symbol nabla ∇ denotes the divergence when it acts on vectors and the gradient, when it acts on scalars.) We say that the system is bound if for large values of x the probability distribution vanishes

$$\rho(x,t) \to 0\,, \qquad \text{as } |x| \to \infty\,.$$

In this case the total probability is conserved

$$\int dx\, \rho\rho(x,t) = e^{-\Phi(x,t)/D}. \tag{28.5}$$

The time evolution of Φ is given by

$$\partial_t \Phi + \nabla\Phi v + (\nabla\Phi)^2 - D\nabla v + D\nabla^2\Phi = 0\,.$$

Consider now the weak noise limit and drop the terms proportional to D. The remaining equation

$$\partial_t \Phi + H(x, \nabla\Phi) = 0. \tag{28.6}$$

is reminiscent of the Hamilton-Jacobi equation of classical physics. The function Φ can be interpreted as a noise "action" corresponding to the noise "Hamiltonian"

least action principle

$$H(x,p) = pv(x) + p^2 \,.$$

with the Hamilton equations of motion

replace boldface **D**

$$\begin{aligned} \dot{x} &= \partial_p H = v + 2p, & (28.7)\\ \dot{p} &= -\partial_x H = -Ap\,. & (28.8) \end{aligned}$$

and the "noise"Lagrangian

link to mat of variations A def

$$L(x,\dot{x}) = (\dot{x} - v(x))^2. \tag{28.9}$$

What is the meaning of this Hamiltonian, Lagrangian, action? We can figure out this by taking two points x and x' and asking which is the most probable noisy path that connects them in time T. The probability of a given path $x(t)$ is the same as the probability of the noise sequence $\xi(t)$ which generates the path. This probability is proportional with the product of the noise probability functions along the path, or the stochastic Wiener path integral

normalize

$$P(x,x',t) \sim \exp\left(-\frac{1}{D}\int_0^T dt\xi^2(t)\right). \tag{28.10}$$

The most probable path is the one which maximizes the integral inside of the exponential. If we express the noise as

$$\xi(t) = \dot{x}(t) - v(x(t))$$

the condition leads to the least action principle

of Fermat?

$$\min \int_0^T dt(\dot{x}(t) - v(x(t)))^2 = \min \int_0^T L(x(t),\dot{x}(t))dt, \tag{28.11}$$

like in the classical mechanics. Following the usual mechanics textbook derivation of the path minimizing the action integral, we can convince ourselves, that it is given by the solution of the Hamilton's equations. To a given x, x' and T we have to find the initial p which realizes this path.

Langevin equation
Onsager-Machlup

Commentary

Remark 28.1 Literature. Theory of stochastic processes is a vast subject, spanning over centuries and over disciplines ranging form pure mathematics to impure financial dealings. We enjoyed reading van Kampen classic [28.1], especially his railings against those who blunder carelessly into nonlinear landscapes (with this chapter we join the list of van Kampen's sinners). A more specialized monograph like Risken's [28.2] will do just as well. The "Langevin equation" introduces noise and damping only into the acceleration of Newton's equations; here we are considering more general stochastic differential equations in the weak noise limit. Onsager-Machlup seminal paper [28.16] was the first to introduce variational methods with the Lagrangian (28.9), but for linear stochastic ODE's only (*ie*, no state space varying velocity field $v(x)$).

Résumé

The effect of the noise can be taken into account by adding noise corrections to the classical trace formula.

recheck whether something can be used from chapter/refsQmnoise

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