III. Angular Momentum

1. Consider a general orbital wave function, $\chi(\vartheta, \varphi)$. Starting from the wave mechanics representation of the angular momentum operators show that the components of the orbital angular momentum operators obey the commutation relations,

$$[\mathcal{L}_x\,,\,\mathcal{L}_y] = i\hbar\,\mathcal{L}_z \text{ and cyclic permutations of } xyz \quad , \quad \left[\mathcal{L}_i\,,\,\mathcal{L}^2\right] = 0\,. \tag{1}$$

   For a state of definite $\mathcal{L}_z$ show that the expectation values of $\mathcal{L}_x$ and $\mathcal{L}_y$ are zero.

2. Show that the function $\mathcal{P}_\ell^m(\vartheta) = (1-\cos^2\vartheta)^{|m|/2}\,\frac{d^{|m|}}{d\vartheta^{|m|}}\,P_\ell(\cos\vartheta)$ solves the associated Legendre equation,

$$\sin\vartheta\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\mathcal{P}}{\partial\vartheta}\right) + \left(\ell(\ell+1)\sin^2\vartheta - m^2\right)\mathcal{P} = 0\,. \tag{2}$$

   Derive the recurrsion formula for the expansion coefficients of the associated Legendre functions about $\cos\vartheta = 0$.

3. Consider the Legendre polynomals $\mathcal{P}_\ell(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')$, where $\mathbf{r} = r(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ and $\mathbf{r}' = r'(\sin\vartheta'\cos\varphi', \sin\vartheta'\sin\varphi', \cos\vartheta')$ are two orbital vectors expressed in spherical coordinates. Derive the Addition formula for the spherical harmonics.

$$\mathcal{P}_\ell(\hat{\mathbf{r}}'\cdot\hat{\mathbf{r}}) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell}\mathcal{Y}^*_{\ell m}(\vartheta', \varphi')\mathcal{Y}_{\ell m}(\vartheta, \varphi)\,. \tag{3}$$