Weak noise perturbation theory

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Physics D15-3, Spring quarter 2000

Midterm exam

due 10:30 AM Thursday, May 25, 2000, Predrag's office, Tech F332

1 Stochastic evolution operator

Our goal here is to develop the field-theoretic formulation for *stochastic* dynamics, given by the discrete Langevin equation [1, 2]

$$x_{n+1} = f(x_n) + \sigma\xi_n\,, \tag{1}$$

where the ξ_n are independent normalized Gaussian random variables. You can think of this as a deterministic trajectory of a particle $x \to f(x) \to f(f(x)) \to \cdots$, kicked at each time step by Gaussian noise of width σ.

As the trajectories have only probabilistic meaning, the dynamics should be formulated in terms of initial probability density $p(x, 0)$. $p(x, n)$ is then the probability of finding a particle at x at time n. We describe the evolution of a system with weak external Gaussian noise (small σ) by introducing the Fokker-Planck evolution operator, a sharply peaked noise distribution function

$$\mathcal{L}_\sigma(y, x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-f(x))^2/2\sigma^2}\,. \tag{2}$$

This is the analogue of the short-time propagator in quantum mechanics; it says that after one time step a particle that started at x spreads into a cloud of width σ, centered on $f(x)$. The classical trajectory is recovered in the $\sigma \to 0$ limit. To keep life simple we shall not take the (time step) $\to$ 0 continuum limit, but restrict the discussion to the case of discrete time step evolution. This is perfectly physical process in case of mechanical, atomic, mesoscopic, etc., systems which receive kicks at discrete time intervals. To make life even simpler, we shall restrict the consideration to a particle moving along a line - d-dimensional generalization is straightforward. A concrete example of such system is given by the map (31) that might describe dynamics of a 1-dimensional repeller. As in quantum mechanics, the propagator for n time steps is given by a path integral

$$\begin{aligned}\mathcal{L}^n_\sigma(y,x) &= \int [dx]\,\exp\left\{-\frac{1}{2\sigma^2}\sum_a [x_{a+1}-f(x_a)]^2\right\} \\ x_n &= y\,, \quad x_0 = x\,, \qquad [dx] = \prod_{a=1}^{n-1}\frac{dx_a}{\sqrt{2\pi\sigma^2}}\,,\end{aligned} \tag{3}$$

with all intermediate positions integrated over.

The goal is to compute the spectrum of this evolution operator. In contrast to quantum mechanics, the eigenvalues describe the decay rates rather than quantum oscillations, but otherwise the computation ahead is based on the same strategy - as in quantum mechanics, the spectrum can be extracted from the trace of the propagator.

In the exam tour guide that follows, I have indicated which steps are exam questions, and which are optional. I do not expect you to get through the whole length of the exam in the time allotted; do as much as feels right. Your task is to work through sect. 2 and then derive at least the "exam question" results.

The references appended are introductions to the theory of stochastic processes, and motivation for this problem. You do not need to consult them as far as this exam is concerned.

2 Stochastic trace formula, steepest descent approximation

We start by calculating the trace of the nth iterate of the stochastic evolution operator $\mathcal{L}_\sigma$ for a one-dimensional analytic map $f(x)$ with additive Gaussian noise σ. This trace is an n-dimensional integral on n points along a discrete periodic chain, so x becomes an n-vector x_a with indices $a, b, \ldots$ ranging from 0 to $n-1$ in a cyclic fashion

$$\begin{aligned}\mathrm{tr}\,\mathcal{L}^n_\sigma &= \int [dx]\,\exp\left\{-\frac{1}{2\sigma^2}\sum_a [x_{a+1}-f(x_a)]^2\right\} \\ x_n &= x_0\,, \qquad [dx] = \prod_{a=0}^{n-1}\frac{dx_a}{\sqrt{2\pi\sigma^2}}\,.\end{aligned} \tag{4}$$

As we are dealing with a path integral on a finite discrete chain, we find it convenient to rewrite the exponent in matrix notation

$$\mathrm{tr}\,\mathcal{L}^n_\sigma = \int [dx]\, e^{-\left[h^{-1}x - f(x)\right]^2/2\sigma^2}\,, \qquad h_{ab} = \delta_{a,b+1}\,, \tag{5}$$

where x and $f(x)$ are column vectors with components x_a and $f(x_a)$ respectively, and h is the left cyclic shift or hopping matrix satisfying $h^n = 1$, $h^{-1} = h^T$. Unless stated otherwise, we shall assume the repeated index summation convention throughout, and that the Kronecker δ function is the periodic one, defined by

$$\delta_{ab} = \frac{1}{n} \sum_{k=0}^{n-1} e^{i2\pi(a-b)k/n} . \tag{6}$$

If the noise is weak, the path integral (4) is dominated by periodic deterministic trajectories. Assuming that the periodic points of given finite period n are isolated and the trajectory broadening σ sufficiently small so that they remain clearly separated, the dominant contributions come from neighborhoods of periodic points; in the *saddle-point approximation* the trace (4) is given by

$$\mathrm{tr}\, \mathcal{L}_\sigma^n \longrightarrow \sum_{x_c \in \mathrm{Fix} f^n} e^{W_c} , \tag{7}$$

where the sum goes over all periodic points $x_c = x_{c+n}$ of period n, $f^n(x_c) = x_c$. The contribution of the x_c neighborhood is obtained by shifting the origin of integration to

$$x_a \to x_a + \phi_a ,$$

where from now on x_a refers to the position of the a-th periodic point, and expanding f in Taylor series around each of the periodic points in the orbit of x_c.

Show that the contribution of the neighborhood of the periodic point x_c is given by

Exam question

$$\begin{aligned} e^{W_c} &= \int [d\phi]\, e^{-\left(\Delta^{-1}\phi - V'(\phi)\right)^2/2\sigma^2} \\ &= |\mathrm{det}\Delta| \int [d\varphi]\, e^{\sum \frac{1}{k} \mathrm{tr}\,(\Delta V''(\phi))^k} e^{-\varphi^2/2\sigma^2} \end{aligned} \tag{8}$$

where the propagator and interaction terms are collected in

$$\Delta_{ab}^{-1}\phi_b = -f'(x_a)\phi_a + \phi_{a+1} , \qquad V(\phi) = \sum_a \sum_{m=2}^{\infty} f^{(m)}(x_a) \frac{\phi_a^{m+1}}{(m+1)!} . \tag{9}$$

We find it convenient to also introduce a bidirectional propagator $C = \Delta\Delta^T$ for reasons that will become apparent below. In the second line of (8) we have changed coordinates,

$$\varphi = \Delta^{-1}\phi - V'(\phi) , \tag{10}$$

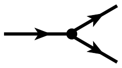
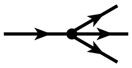
and used the matrix identity $\ln \det M = \mathrm{tr}\ \ln M$ on the Jacobian

$$\frac{1}{\det\left(\Delta^{-1} - V''\right)} = \frac{\det\Delta}{\det\left(1 - \Delta V''\right)} = \det\Delta\, e^{-\mathrm{tr}\ \ln(1-\Delta V'')} \,. \tag{11}$$

The functional dependence of $\phi = \phi(\varphi)$ is recovered by iterating (10)

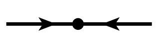
$$\phi_a = \Delta_{ab}\varphi_b + \Delta_{ab} V_b'(\phi) \,. \tag{12}$$

The saddle-point expansion is most conveniently evaluated in terms of Feynman diagrams, which we now introduce. The interaction terms in V and its derivatives can be represented in terms of the vertices

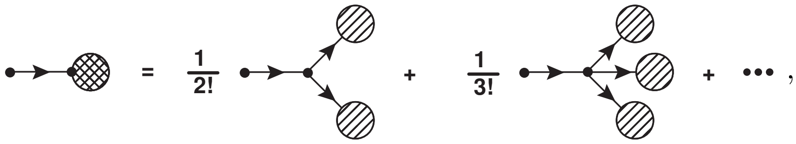
$f^{''}(x_a) \;=$ , $\quad f^{'''}(x_a) \;=$ , $\quad \ldots,$

and the propagators as directed lines

$\Delta_{ab} \;=$

$C_{ab} \;=$ 

The first two derivatives of V may be written



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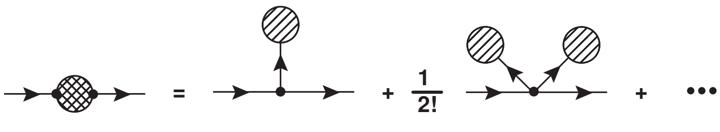
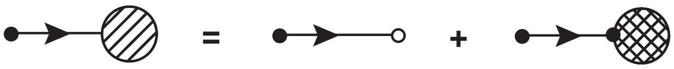


(14)

with the cross-hatched circle as V and the diagonally filled circle as ϕ. The relation between the fields (12) becomes



(15)

with the small open circle as φ. This recursively generates all tree diagrams ending in φ, which the stochastic averaging of sect. 3 will tie into loop corrections.

As the sum is cyclic, e^{W_c} is the same for all periodic points in a given cycle, independent of the choice of the starting point x_c.

For the leading σ^2 correction that we shall evaluate here we need derivatives of f up to the third.

From the second path integral representation in (8) it follows that Δ can be interpreted as the "free" propagator. As Δ will play a central role in what follows, we write its inverse in its full $[n\times n]$ matrix form:

$$\Delta^{-1} = h^{-1} - \mathbf{f}' = \begin{pmatrix} -f_0^{'} & 1 & & & \\ & -f_1^{'} & 1 & & \\ & & -f_2^{'} & 1 & \\ & & & \ddots & \\ 1 & & & & -f_{n-1}^{'} \end{pmatrix} \tag{16}$$

where $\mathbf{f}'$ is a diagonal matrix with elements $f_a^{'} = f^{'}(x_a)$ a shorthand notation for stability of the map at the periodic point x_a. The determinant of Δ is

$$\det \Delta = \frac{(-1)^n}{\Lambda_c - 1}, \qquad \Lambda_c = \prod_{a=0}^{n-1} f^{'}(x_a)\,, \tag{17}$$

with Λ_c the *stability* of the n cycle going through the periodic point x_c. We shall assume that no cycles is marginally stable, $|\Lambda_c| \neq 1$.

Derive the formula for propagator itself by inverting (16) and using relation $(h\mathbf{f}')^n = \Lambda_c$, (due to the periodicity of the chain): Optional

$$\begin{aligned} \Delta &= -\frac{1}{1-\mathbf{f}'^{-1}h^{-1}}\mathbf{f}'^{-1} = -\sum_{k=0}^{\infty}(\mathbf{f}'^{-1}h^{-1})^k\mathbf{f}'^{-1} \\ &= -\frac{1}{\Lambda_c - 1}\sum_{k=0}^{n-1} h(\mathbf{f}'h)^k \end{aligned} \tag{18}$$

In the full matrix form, the propagator is given by

$$\Delta = \frac{-1}{\Lambda_c - 1} \begin{pmatrix} f_1^{'} \cdots f_{n-1}^{'} & f_2^{'} \cdots f_{n-1}^{'} & f_3^{'} \cdots f_{n-1}^{'} & & \cdots & 1 \\ 1 & f_2^{'} \cdots f_0^{'} & f_3^{'} f_4^{'} \cdots f_0^{'} & & \cdots & f_0^{'} \\ f_1^{'} & 1 & f_3^{'} \cdots f_0^{'} f_1^{'} & & \cdots & f_0^{'} f_1^{'} \\ f_1^{'} f_2^{'} & f_2^{'} & 1 & \ddots & & f_0^{'} f_1^{'} f_2^{'} \\ f_1^{'} f_2^{'} f_3^{'} & f_2^{'} f_3^{'} & f_3^{'} & & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ f_1^{'} \cdots f_{n-2}^{'} & f_2^{'} \cdots f_{n-2}^{'} & \cdots & \cdots & 1 & f_0^{'} \cdots f_{n-2}^{'} \end{pmatrix} \tag{19}$$

or, more compactly,

$$\Delta_{ab} = \frac{-1}{\Lambda_c - 1} \prod_{d=b+1}^{a-1} f^{'}(x_d) \,, \qquad \Delta_{a,a-1} = \frac{-1}{\Lambda_c - 1} \,, \tag{20}$$

where d increases cyclically through the range $b+1$ to $a-1$; for example, if $a = 0$, $a-1 = n-1$. We note that Δ is invertible only for cycles which are not marginal, $|\Lambda_c| \neq 1$.

Derive the momentum space (discrete Fourier transform) representation for the free propagator, interaction vertices. (Warning - this might be quite a bit of work, postpone for the last.) Optional

3 Weak noise perturbation expansion

The saddle-point approximation (8) is a discrete path integral on periodic chain of n points which we shall evaluate by standard field-theoretic methods. Separating the quadratic terms we obtain

$$e^{W_c} = \frac{1}{|\Lambda_c - 1|} \int [d\varphi] \, e^{-S_0(\varphi) - S_I(\varphi)} \,, \tag{21}$$

where

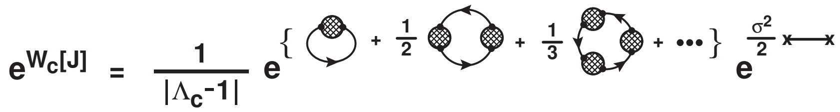
$$S_0(\varphi) = \varphi^2 / 2\sigma^2 \,, \qquad S_I(\varphi) = -\sum_{k=1}^{\infty} \frac{1}{k} \mathrm{tr} \, \left[\Delta V''(\phi(\varphi)) \right]^k \tag{22}$$

The terms collected in $S_I(\varphi)$, linear or higher in φ, are the interaction vertices.

Next introduce a source term J_a and define a partition function

$$
\begin{aligned}
e^{W_c(J)} &= \frac{1}{|\Lambda_c - 1|} \int [d\varphi] e^{-S_0(\varphi) - S_I(\varphi) + J_a \varphi_a} \\
&= \frac{1}{|\Lambda_c - 1|} e^{-S_I(\frac{d}{dJ})} \int [d\varphi] e^{-S_0(\varphi) + J_a \varphi_a} \\
&= \frac{1}{|\Lambda_c - 1|} e^{-S_I(\frac{d}{dJ})}\, e^{\frac{\sigma^2}{2} J^2} . \qquad (23)
\end{aligned}
$$

Here we have used standard formulas for Gaussian integrals together with the normalization (4). In the diagrammatic notation this is



(24)

Expanding

$$
e^{\frac{\sigma^2}{2} J^2} = 1 + \frac{\sigma^2}{2} J_a J_a + \frac{\sigma^4}{8} J_a J_a J_b J_b + \ldots , \qquad (25)
$$

operating on this series with $\exp\{-S_I(\frac{d}{dJ})\}$,

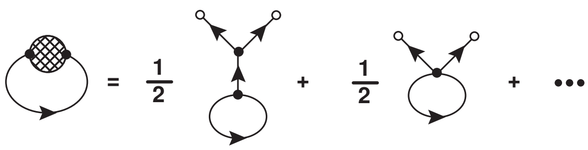
$$
\frac{d}{dJ_a} \frac{d}{dJ_b} e^{\frac{\sigma^2}{2} J_d J_d} \bigg|_{J=0} = \sigma^2 \delta_{ab} , \quad \cdots ,
$$

collecting terms of the same order in σ^2, and setting J_a to zero yields the perturbation expansion

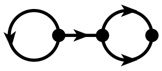
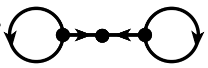
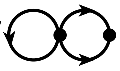
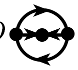
$$
W_c = -\ln |\Lambda_c - 1| + \sum_{k=1}^{\infty} W_{c,2k} \sigma^{2k} . \qquad (26)
$$

In field-theoretic calculations the $W_{c,0}$ term is usually an overall volume term that drops out in the expectation value computations. In contrast, here the $W_{c,0} = -\ln |\Lambda_c - 1|$ term is the classical weight of the cycle which plays the key role both in the classical and stochastic trace formulas.

In diagrammatic language, we join all possible pairs of φ vertices, each one giving a $\sigma^2 C$ propagator. Thus the first diagram in (24) is expanded (14,15) to

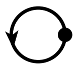


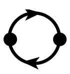
and then the φ vertices joined to form two diagrams contributing at order σ^2. **Show that** the full noise corrections of order σ^2 are given by all connected two-loop diagrams, of form: Exam question

$W_{c,2} = A$  $+ B$  $+ C$  $+ D$  . (27)

Determine the combinatorial factors A, B, C and D. Exam question

Show that that several sub-diagrams may be simplified using (20). These are (no sum on a, b) Exam question

 a $= \Delta_{aa} = \dfrac{-\Lambda_c}{\Lambda_c - 1}\dfrac{1}{f'_a}$ (28)

a  b $= \Delta_{ab}\Delta_{ba} = \dfrac{\Lambda_c}{(\Lambda_c - 1)^2}\dfrac{1 + (\Lambda_c - 1)\delta_{ab}}{f'_a f'_b}$.

Add the terms and **show that** the σ^2 contribution to the trace is given by: Optional

$$
\begin{aligned}
W_{c,2} &= \frac{1}{2}\left[\frac{\Lambda_c}{\Lambda_c - 1}\sum_a \left(\frac{f''^2_a}{f'^2_a} - \frac{f'''_a}{f'_a}\right) C_{aa} \right. \\
&\quad \left. + \frac{\Lambda_c^2 + \Lambda_c}{(\Lambda_c - 1)^2}\sum_{ab}\frac{f''_a}{f'_a}\frac{f''_b}{f'_b} C_{ab} - \frac{\Lambda_c}{\Lambda_c - 1}\sum_{ab}\frac{f''_a}{f'_a} f''_b \Delta_{ab} C_{bb}\right] . \qquad (29)
\end{aligned}
$$

(This is a serious amount of work, so do not get discouraged if you do not get the whole result.)

3.1 Fixed point

The contribution from a fixed point (cycle of length one) is particularly simple, as all the sums and products collapse to a single term, and $f' = \Lambda$. **Show that** Exam question

$$
W_{c,2} = \frac{1}{2}\frac{1}{(\Lambda - 1)^2}\left\{3\left(\frac{f''}{\Lambda - 1}\right)^2 - \frac{f'''}{\Lambda - 1}\right\} . \qquad (30)
$$

Note that depending on the sign of $W_{p,2}$, small amounts of noise can either enhance or inhibit escape from the fixed point; in other words, you can make the system more stable if you kick it with a right kind of random noise!

4 Numerical tests

Optional

To test the above expressions for the trace, compute the required derivatives for the fixed points and some periodic orbits $f^n(x) = x$ for the quartic map

$$f(x) = 20\left(\frac{1}{2^4} - \left(\frac{1}{2} - x\right)^4\right) . \tag{31}$$

and compute the σ^2 contribution to the trace (29). The choice of the map is motivated by requiring that the system be simple (one-dimensional in this case), with non-trivial f'', f''' (hence quartic), with complete binary dynamics (hence a nice repeller), and no diffraction and non-hyperbolic regions in the immediate vicinity of the repeller (where the Gaussian saddle points would be insufficient).

5 Solution

See refs. [3, 4] and [5], all available on Predrag's homepage.

The corect combinatorial factors A, B, C and D in (27) are given in eq. (27) of ref. [3].

Formula (29) is derived in ref. [3], eq. (29). A more ellegant derivation of (29), althogether without Feynman diagrams (!), is given in eq. (25) of ref. [4].

For a direct derivation of (30) see eq. (12) in ref. [4].

References

[1] N. G. Van Kampen. *Stochastic Processes in Physics and Chemistry.* North Holland, Amsterdam, 1981.

[2] A. Lasota and M. MacKey. *Chaos, Fractals, and Noise; Stochastic Aspects of Dynamics.* Springer-Verlag, Berlin, 1994.

[3] P. Cvitanović, C.P. Dettmann, R. Mainieri, and G. Vattay. Trace formulas for stochastic evolution operators: Weak noise perturbation theory. *Journal of Statistical Physics*, 93:981–1003, 1998. `chao-dyn/9807034`.

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[5] P. Cvitanović, C.P. Dettmann, G. Palla, N. Søndergård, and G. Vattay. Spectrum of stochastic evolution operators: Local matrix representation approach. *Physical Review E*, 60:3936–3941, 1999. `chao-dyn/9904027`.