Chapter 19

Problem Sets

The problems, and solutions given here were used to expand upon and to illustrate the course material. They are taken from a number of sources as well as from courses given in the Department of Physics at the University of Maryland.

1.1) The Ehrenfest Wind Tree Model : A collection of fixed scatterers ("trees") are placed on a plane at random. The trees are oriented squares with diagonals along the x- and y-directions.(See Fig. 19.1) The number density of trees is n, each side is of length a, and $na^2 \ll 1$. There are moving particles ("wind") that do not interact with each other, but do collide with the trees. The wind particles can move in four directions, labelled 1,2,3,4. Let $F_i(\vec{r}, t) =$ the number of wind particles at $\vec{r}$ moving in direction i at time t.

a) Derive an equation for $F_i(\vec{r}, t)$.

b) Is there an H-theorem? Suppose the system is spatially homogeneous, $F_i(\vec{r}, t) = F_i(t)$, independent of $\vec{r}$.

c) Find a solution $\{F_i(t)\}$ in terms of $\{F_i(0)\}$. What happens if $t \to \infty$ You will need to diagonalize a 4×4-matrix.

1.2) The Lorentz-gas : Replace the trees by hard disks of radius a, and allow the wind-particles to move in all directions on the plane. Derive an equation for $F(\vec{r}, \vec{v}, t)$. Prove an H-theorem.

2.1) Complete the calculation of the tagged-particle diffusion coefficient.

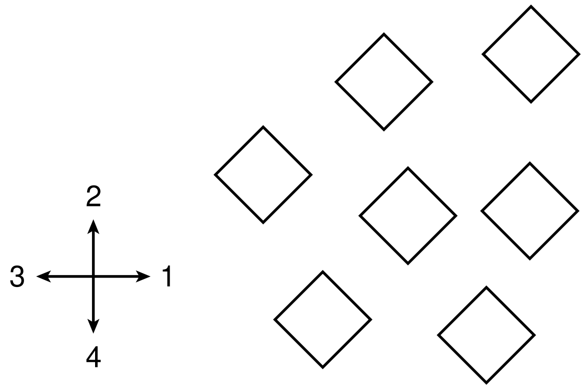

Figure 19.1: The Ehrenfest Wind-Tree Model.

(see notes)

2.2) The Ehrenfest Urn-Model; Consider two urns that contain n balls, that are numbered from 1 to n. Pick a number at random and move the ball with that number to the other urn. Repeat this. If you have access to a computer, find $N_1(t) - N_2(t)$ as a function of time for an initial state where 50 balls are in one urn and zero in the other. How would you characterize the behavior for short and long times ? If you have no access to a computer, formulate the process as a Markov-process and obtain the description of the process analytically. Unless you are very clever you will not be able to solve the equation. For the solution see M. Kac in *Noise and Stochastic Processes,* edited by N. Wax, Dover Books, New York (1954).

3.1) Consider all integral powers of 2 : $2^0, 2^1, 2^2, 2^3, \ldots$ Write down the first digit of each power in a sequence ;

$$1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1 \ldots$$

Calculate the frequency at which a "7" appears in this sequence. (This problem is taken from V.I. Arnold *Mathematical Methods of Classical Mechanics, 2nd. edition,* Springer Verlag, Berlin (1989).) Hint : This is a consequence of the ergodic theorem, and you need to consider that an integer appears whenever :

$$k10^r < 2^m < (k+1)10^r, \tag{19.1}$$

for some integer r.

3.2) Prove Poincare's Recurrence Theorem for quantum systems. That is, consider a Schrodinger wave-function, $\Psi(\vec{r},t)$. Suppose $\{\phi_n(\vec{r})\}$ is a complete set of states and that

$$\Psi(\vec{r},t) = \sum_n a_n e^{i\omega_n t}\phi_n(\vec{r})$$

converges uniformly in time. Suppose also that

$$\int |\Psi|^2 d\vec{r} = 1, \quad \text{and} \quad \int |\phi_n|^2 d\vec{r} = 1$$

Then $\Psi(\vec{r},t)$ is an almost periodic function of time.

4.1) Consider the following generalization of the baker's transformation. Divide up a unit-square into two pieces, of area α, and of area β, such that $\alpha+\beta=1$. In I stretch x by a factor $\frac{1}{\alpha}$ and contract y by α. In II stretch x by a factor $\frac{1}{\beta}$ and contract y by β. Put II on top of I. This process is repeated many times, so that the original unit-square gets mixed throughout the entire square. (See Fig. 19.2).

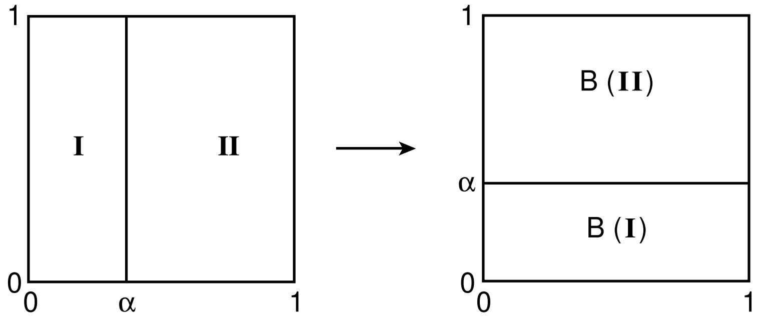

Figure 19.2: A Slightly Generalized Baker's Transformation.

a) Write down the equations of motion for this transformation and for its inverse.

b) Derive the Liouville equation for this system

c) Integrate over y and derive a Boltzmann-equation

d) Show that the Boltzmann-equation has an H-theorem and an equilibrium solution.

5.1) Using the dyadic representation of a point (x, y) in the unit-square :

a) Find all points of period 1,2,3 and 4 under the baker's transformation. A point of period n satisfies

$$(x, y) = B^n(x, y)$$

b) How may points of period n are there ?

5.2) The version of Liouville's equation for a discrete map is called the Frobenius-Perron equation. and can be written

$$\rho_n(\Gamma) = \int d\Gamma' \rho_{n-1}(\Gamma')\delta(\Gamma - M(\Gamma'))$$

Where n denotes the n^{th} step, M is the map and Γ a phase-point.

a) Use this to derive the Liouville equation for the baker's transformation.

b) Consider the one-dimensional map $M(x)$:(See Fig. 19.3)

$$M(x) = 2x \quad : 0 < x < \frac{1}{2}$$

$$= 2(1 - x) \quad : \frac{1}{2} < x < 1$$

Find $\rho_{n+1}(x)$ in terms of $\rho_0(x)$. What happens if $n \to \infty$?

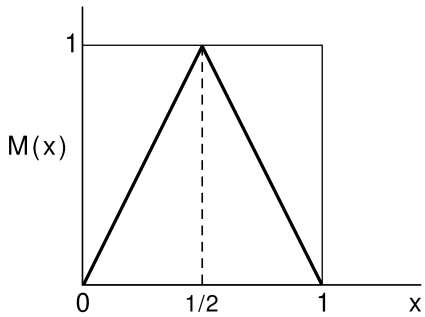

Figure 19.3: The Tent Map. See problem 5.2.

6.1) Construct a 1-d map that would provide a dynamical realization of at random walk on a one dimensional lattice with probability α of going to the right and probability $\beta = 1 - \alpha$ of going to the left .

6.2) Suppose that the map has ergodic properties and compute the Lyapunov exponent for the map.

6.3) Suppose $\alpha = \beta = \frac{1}{2}$. Compute the diffusion-coefficient of the map, the escape-rate from a lattice of length L, the Lyapunov exponent on the repeller, and the K-S entropy on the repeller. You may use the escape rate-formula, and suppose that $L \gg 1$.

7.1) Show that the diffusion coefficient calculated by means of Takagi functions remains the same even if the exact velocity of the moving particle is used instead of the simplified velocity given in Chapter 16. You will need to define a new "Takagi function", but otherwise the calculation is straightforward.

7.2) Compute and plot the jump function $J_{25}(x)$ for the map discussed in Chapter 16.

8.1) Derive the expression for the change in the radius of curvature due to collisions with a circular scatterer, Eq. (17.2).

8.2) Solve the equilibrium extended Lorentz-Boltzmann equation, Eq. (17.21), and obtain Eq. (17.22) for $na^2 << 1$.