19.1 Solutions to the Problem Sets

1.1) a) The equation for $F_i(\vec{r},t)$ is

$$\frac{\partial F_i(\vec{r},t)}{\partial t} + \vec{v}\cdot\vec{\nabla}_r F_i = A(F_{i+1} + F_{i-1} - 2F_i)$$

Where $i = i\ mod(4)$ and $A = \frac{n|\vec{v}|a}{2^{\frac{1}{2}}}$ The possible direct and restituting collision are shown in Fig. 19.4.

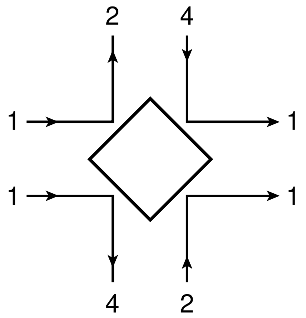

Figure 19.4: Some Direct and Restituting Collisions in the Wind-Tree Model.

b) H-theorem :

$$H(t) = \sum_{i=1}^{4}\int d\vec{r} F_i(\vec{r},t)(\ln(F_i) - 1)$$

$$\begin{aligned}\frac{dH(t)}{dt} &= \sum_{i=1}^{4}\int d\vec{r}\ln(F_i)(\frac{\partial F_i}{\partial t}) \\ &= \sum_{i=1}^{4}\int d\vec{r}(\ln(F_i))\left[-\vec{v}\cdot\vec{\nabla}F_i + A(F_{i+1} + F_{i-1} - 2F_i)\right]\end{aligned}$$

$$= -\sum_{i=1}^{4}\int d\vec{r}\,\vec{v}\cdot\vec{\nabla}\left[F_i(\ln(F_i) - 1)\right] + \sum_{i=1}^{4}\int d\vec{r}\ln(F_i)(F_{i+1} + F_{i-1} - 2F_i)$$

The first term can be made zero by setting F_i zero at the walls of the container, The second term, written out in full is

$$\begin{aligned}&\int d\vec{r}\{\ln(F_1)(F_4 - F_1 + F_2 - F_1)\\&+\ln(F_2)(F_3 - F_2 + F_1 - F_2)\\&+\ln(F_3)(F_4 - F_3 + F_2 - F_3)\\&+\ln(F_4)(F_1 - F_4 + F_3 - F_4)\}\end{aligned}$$

$$\begin{aligned}=\int d\vec{r}\{&\ln(\frac{F_1}{F_4})(F_4 - F_1) + \ln(\frac{F_1}{F_2})(F_2 - F_1)\\&+\ln(\frac{F_2}{F_3})(F_3 - F_2) + \ln(\frac{F_3}{F_4})(F_4 - F_3)\} \leq 0\end{aligned}$$

Therefore

$$\frac{dH}{dt} \leq 0$$

$$\frac{dH}{dt} = 0 \quad iff \qquad F_1 = F_2 = F_3 = F_4$$

c) Write $\frac{d\vec{F}}{dt} = M \cdot \vec{F}$, where

$$M = A\begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

The eigenvalues of M are $A(0, -2, -2, -4)$. As $t \to \infty$, only the zero eigenvalue term remains, and

$$F_i(\infty) = \frac{1}{4}\{F_1(0) + F_2(0) + F_3(0) + F_4(0)\}$$

This is the equilibrium-distribution. The contributions from the other eigenvalues decay as e^{-2At} and e^{-4At}

1.2) The collisions are elastic. Consider the collision diagram : With respect to a space fixed axis (x-axis):

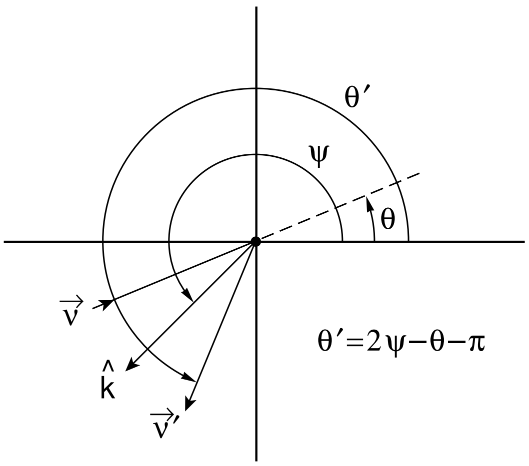

Figure 19.5: The Angular Variables for Problem 1.2.

$\vec{v}$ makes an angle θ, $\hat{k}$ makes an angle ψ , $\vec{v'}$ makes an angle θ'.(see Fig. 19.5) We also set $\delta = \theta - \psi$. Then : $\theta + \frac{1}{2}\pi \leq \psi \leq \theta + \frac{3}{2}\pi$

For Γ_- consider a small collision-cylinder of area $db|\vec{v}|\delta t$, $b = a\sin(\delta)$

$$\Gamma_- = n\delta t|\vec{v}| \int_{-a}^{a} dbF(\vec{r}, \vec{v}, t) = n\delta t|\vec{v}|a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\delta\cos(\delta)F(\vec{r}, \vec{v}, t)$$

$$= na\delta t \int_{\theta+\frac{1}{2}\pi}^{\theta+\frac{3}{2}\pi} d\psi|\vec{v}\cdot\hat{k}|F(\vec{r}, \vec{v}, t) = n\delta t \int_{\vec{v}\cdot\hat{k}<0} d\hat{k}B(\vec{v}, \hat{k})F(\vec{r}, \vec{v}, t)$$

$$B(\vec{v}, \hat{k}) = a|\vec{v}\cdot\hat{k}| = B(\vec{v'}, -\hat{k}) = a|\vec{v'}\cdot\hat{k}|$$

Also Γ_+ is given by

$$\Gamma_+ = n\delta t \int_{\vec{v'}\cdot\hat{k}>0} d(-\hat{k})B(\vec{v'}, -\hat{k})F(\vec{r}, \vec{v'}, t)$$

$$= n\delta t \int_{\vec{v}\cdot\hat{k}<0} d\hat{k}B(\vec{v}, \hat{k})F(\vec{r}, \vec{v'}, t)$$

The Lorentz-Boltzmann-equation can now be written as

$$\frac{\partial F(\vec{r}, \vec{v}, t)}{\partial t} + \vec{v}\cdot\vec{\nabla}F(\vec{r}, \vec{v}, t) = na \int_{\theta+\frac{1}{2}\pi}^{\theta+\frac{3}{2}\pi} d\psi|\vec{v}||\cos(\delta)|\{F(\vec{r}, 2\psi-\theta-\pi, t) - F(\vec{r}, \theta, t\}$$

H-theorem : H is defined in the usual way :

$$\frac{dH}{dt} = \int d\vec{r}\int d\vec{v}\frac{\partial F}{\partial t}\ln(F) = \int d\vec{r}\int_0^{2\pi}|\vec{v}|d\theta\int_{\theta+\frac{1}{2}\pi}^{\theta+\frac{3}{2}\pi}d\psi|\vec{v}||\cos(\delta)|\ln(F(\theta))$$
$$\cdot\{F(2\psi-\theta-\pi)-F(\theta)\}$$

Changing variables $\rightarrow (\theta',\psi')$:

$$\theta' = 2\psi-\theta-\pi$$

$$\psi' = \psi-\pi$$

$$\frac{dH}{dt} = na\int d\vec{r}\int_0^{2\pi}d\theta'|\vec{v}|\int_{\theta'+\frac{1}{2}\pi}^{\theta'+\frac{3}{2}\pi}d\psi|\vec{v}||\cos(\psi'-\theta')|\ln(F(2\psi'-\theta'-\pi))$$
$$\cdot\{F(\theta')-F(2\psi'-\theta'-\pi)\}$$

Therefore

$$\frac{dH}{dt} = na\frac{1}{2}\int d\vec{r}\int_0^{2\pi}d\theta|\vec{v}|\int_{\theta+\frac{1}{2}\pi}^{\theta+\frac{3}{2}\pi}d\psi|\vec{v}||\cos(\psi-\theta)|$$
$$\ln(\frac{F(\theta)}{F(2\psi-\theta-\pi)})\cdot$$
$$\{(F(2\psi-\theta-\pi)-F(\theta)\} \leq 0$$

3.1) Use $k10^r \leq 2^N \leq (k+1)10^r$, if the first digit of 2^N starts with the number k. Take logarithms to base 10. Then

$$\log(k)+r \leq N\log(2) \leq r+\log(k+1)$$

Next notice that modulo 1 we have the relation that :

$$\log(k) \leq N\log(2)(mod1) \leq \log(k+1)$$

Notice that $\log(2)$(to base 10) is an irrational number, so $N\log(2)(mod1)$ for $N=0,1,2,\ldots$ is a dense set of points on the circle of circumference 1, and this dense set of points is uniform on the circle. Therefore the ergodic theorem says that the fraction, f_7 of powers of two that begin with a seven is the fraction of the circumference between log(7) and log(8). Thus $f_7 = \log 8 - \log 7 = 0.058$

3.2) Write

$$\Psi(x,t) = \sum_{n=0}^{\infty} a_n \phi_n(x) e^{i\omega_n t},$$

$$\Psi_N(x,t) = \sum_{n=0}^{N} a_n \phi_n e^{i\omega_n t}$$

If $\Psi(x,t)$ is an almost periodic function of time, then there exists a $T > 0$ such that, for any ϵ :

$$||\Psi(x,t) - \Psi(x,t+T)|| \quad < \quad \epsilon$$

Now pick some arbitrary T for the moment, and write

$$\begin{aligned} ||\Psi(x,t) - \Psi(x,t+T)|| \;&=\; ||\,[\Psi(x,t) - \Psi_N(x,t)] + [\Psi_N(x,t) - \Psi_N(x,t+T)] \\ &\quad +[\Psi_N(x,t+T) - \Psi(x,t+T)]\,|| \end{aligned}$$

By the Cauchy-Schwartz inequality

$$\begin{aligned} ||\Psi(x,t) - \Psi(x,t+T)|| \;&\leq\; ||\Psi(x,t) - \Psi_N(x,t)|| + ||\Psi_N(x,t) - \Psi_N(x,t+T)|| + \\ &\quad +||\Psi_N(x,t+T) - \Psi(x,t+T)|| \end{aligned}$$

Since $\Psi_N(x,t)$ converges to $\Psi(x,t)$, uniformly in t we have that for large enough N

$$||\Psi(x,t) - \Psi_N(x,t)|| < \epsilon$$

and for the same N

$$||\Psi(x,t+T) - \Psi_N(x,t+T)|| < \epsilon$$

Now there exist a T such that

$$||\Psi_N(x,t) - \Psi_N(x,t+T)|| < \epsilon$$

Proof :

Suppose we consider one term in Ψ_N

$$\Psi_N^{(0)} = a_n \phi_n e^{i\omega_n t}$$

Then $\Psi_N^{(0)}(x, t+T) = \Psi_N^{(0)}(x,t)$ if $T = \frac{2\pi}{\omega_n}$.

Suppose we consider two terms in Ψ_N

$$\Psi_N^{(1)}(x,t) = a_{n_1}\phi_{n_1} e^{i\omega_{n_1} t} + a_{n_2}\phi_{n_2} e^{i\omega_{n_2} t}$$

If $\frac{\omega_{n1}}{\omega_{n2}} = \frac{p}{q}$ is a rational number, p and q are integers. Then

$$\Psi_N^{(1)}(t) = a_{n_1}\phi_{n_1} e^{i\omega_{n_1} t} + a_{n_2}\phi_{n_2} e^{i\omega_{n_1} \frac{q}{p} t}$$

and $\Psi_N^{(1)}(t+T) = \Psi_N^{(1)}(t)$ if $T = \frac{2\pi p}{\omega_{n_1}}$

If $\frac{\omega_{n1}}{\omega_{n2}}$ is irrational, then we can approximate it by a rational number :

$$\frac{\omega_{n1}}{\omega_{n2}} = \frac{p}{q} + \delta$$

Where δ is as small as we like. Pick $T = \frac{2\pi p}{\omega_{n_1}}$ and we find that

$$\Psi_N^{(1)}(t+T) = a_{n_1}\phi_{n_1} e^{i\omega_{n_1}(t + \frac{2\pi p}{\omega_{n1}})} + a_{n_2}\phi_{n_2} e^{\frac{i\omega_{n_1} q}{p+q\delta}(t+\frac{2\pi p}{\omega_{n_1}})}$$

$$= a_{n_1}\phi_{n_1} e^{i\omega_{n_1} t} + a_{n_2}\phi_{n_2} e^{i\omega_{n_2}} e^{i2\pi(\frac{qp}{p+q\delta})}$$

For sufficiently small δ, the right hand side is arbitrarily close to $\Psi_N^{(1)}(t)$

Thus we have proved that the sum of two periodic functions is an almost periodic function. We can extend this to prove that the sum of a *finite* number of periodic functions is almost periodic. Thus there exists a T such that $||\Psi_N(t+T) - \Psi_N(t)|| < \epsilon$ If so then $||\Psi(x,t) - \Psi(x,t+T)|| < 3\epsilon$ and $\Psi(x,T)$ is an almost periodic function.

4.1 a) The transformation, B_α

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{x}{\alpha} \\ \alpha y \end{pmatrix} : \; 0 < x < \alpha$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta}(x-\alpha) \\ \beta y + \alpha \end{pmatrix} : \ \alpha < x < 1$$

The inverse transformation, B_α^{-1}

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\alpha \\ \frac{y}{\alpha} \end{pmatrix} : \ 0 < y < \alpha$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha + \beta x \\ \frac{y-\alpha}{\beta} \end{pmatrix} : \ \alpha < y < 1$$

b) Liouville's equation

$$\rho_n(x,y) = \rho_{n-1}(B_\alpha^{-1}(x,y))$$

$$\rho_{n+1}(x,y) = \theta(\alpha - y)\rho_n(\alpha x, \frac{y}{\alpha}) + \theta(y-\alpha)\rho_n(\alpha + \beta x, \frac{y-\alpha}{\beta})$$

Where

$$\theta(x) = \begin{cases} 1 : x > 0 \\ 0 : x < 0 \end{cases}$$

c)

$$W_n(x) \equiv \int_0^1 dy \rho_n(x,y) = \int_0^\alpha dy \rho_n(x,y) + \int_\alpha^1 dy \rho_n(x,y)$$
$$= \int_0^\alpha dy \rho_{n-1}(\alpha x, \frac{y}{\alpha}) + \int_\alpha^1 dy \rho_{n-1}(\alpha + \beta x, \frac{y-\alpha}{\beta})$$

A change of integration variables gives the Boltzmann equation :

$$W_n(x) = \alpha W_{n-1}(\alpha x) + \beta W_{n-1}(\alpha + \beta x)$$

d) Since $\alpha + \beta = 1$, a steady state solution is $W_n(x) = 1$. Construct an H-function

$$H_n = \int_0^1 dx W_n(x) \ln(W_n(x))$$

The proof that $H_{n+1} \leq H_n$ is almost identical to that given for the case $\alpha = \beta = \frac{1}{2}$.

5.1) The dyadic notation uses $x = \sum_{i=0}^{\infty} \frac{a_i}{2^{i+1}}$, $y = \sum_{i=1}^{\infty} \frac{b_{-i}}{2^i}$, to write

$$(x, y) = \ldots b_{-3} b_{-2} b_{-1} . a_0 a_1 a_2 \ldots$$

a) Points of period 1 are possibly

$$P_1^{(1)} = \ldots 1111.1111 \ldots$$

$$P_2^{(2)} = \ldots 0000.0000 \ldots$$

Now $P_1^{(1)} = (1, 1)$ and $P_2^{(1)} = (0, 0)$ are the same points mod 1, so there's *one* point of period 1. Points of period 2 are pattern :

$$\ldots 0101010101.01010101010 \ldots \quad (\frac{1}{3}, \frac{2}{3}) \quad .01$$

$$\ldots 1010101010.10101010101 \ldots \quad (\frac{2}{3}, \frac{1}{3}) \quad .10$$

in addition to the point of period 1

Points of period 3 are

$$\ldots 0100100100.10010010010 \ldots \quad (\frac{4}{7}, \frac{1}{7}) \quad .100$$

$$\ldots 0010010010.01001001001 \ldots \quad (\frac{2}{7}, \frac{2}{7}) \quad .010$$

$$\ldots 1001001001.00100100100 \ldots \quad (\frac{1}{7}, \frac{4}{7}) \quad .001$$

$$\ldots 0110110110.11011011011 \ldots \quad (\frac{6}{7}, \frac{3}{7}) \quad .110$$

$$\ldots 1101101101.10110110110 \ldots \quad (\frac{5}{7}, \frac{5}{7}) \quad .101$$

$$\ldots 1011011011.01101101101 \ldots \quad (\frac{3}{7}, \frac{6}{7}) \quad .011$$

b) The number of points of period n (including all sub periods) is $2^n - 1$. The minus 1 is a result of the fact that (0,0) and (1,1) are counted as one point. Check period 1 : 1 , period 2 : 3 , period 3 : 7 .

5.2

a) For a two dimensional system the Frobenius-perron equation is :

$$\rho_n(x,y) = \int dx' \int dy' \rho_{n-1}(x',y')\delta(x - M_x(x',y'))\delta(y - M_y(x',y'))$$

For the baker's transformation, $M(x,y) = B(x,y)$.

$$\begin{aligned} \rho_n(x,y) &= \int_0^{\frac{1}{2}} dx' \int_0^1 dy' \rho_{n-1}(x',y')\delta(x-2x')\delta(y-\frac{y'}{2}) + \\ &+ \int_{\frac{1}{2}}^1 dx' \int_0^1 dy' \rho_{n-1}(x',y')\delta(x-(2x'-1))\delta(y-(\frac{y'+1}{2})) \end{aligned}$$

Notice that for $0 < y' < 1$ the delta function $\delta(y - \frac{y'}{2})$ will only be non-zero for $y < \frac{1}{2}$. Similarly the delta-function $\delta(y - (\frac{y'+1}{2}))$ will only be non-zero for $\frac{1}{2} < y < 1$. The Liouville equation for the baker's transformation then is

$$\rho_n(x,y) = \begin{cases} \rho_{n-1}(\frac{x}{2}, 2y) \text{ for } 0 < y < \frac{1}{2} \\ \rho_{n-1}(\frac{x+1}{2}, 2y-1) \text{ for } 1 > y > \frac{1}{2} \end{cases}$$

b) We start from the F-P equation for a 1-d map :

$$\rho_n(x) = \int_0^1 dx' \rho_{n-1}(x')\delta(x - M(x')) = \sum_{x_0'} \frac{1}{|M'(x_0')|}\rho_{n-1}(x_0')$$

Where the sum is over all x_0' which satisfy $M(x_0') = x$: $x_0^{(1)} = \frac{x}{2}$ and $x_0^{(2)} = \frac{2-x}{2}$, $|M'(x_0)| = 2$.

$$\rho_n(x) = \frac{1}{2}\left[\rho_{n-1}(\frac{x}{2}) + \rho_{n-1}(\frac{2-x}{2})\right]$$

An equilibrium solution, ρ_{eq} = constant. For this example one can use

$$\rho_n(x) = \int_0^1 dy \rho_0(y)\delta(x - M^n(y))$$

The slope of each line of $M^n(y)$ is 2^n,

$$\rho_n(x) = \frac{1}{2^n}\sum_i \rho_0(x_i) = \sum_i \rho_0(x_0)\Delta x_i \approx \int_0^1 \rho_0(x)dx = \text{constant}, \qquad (19.2)$$

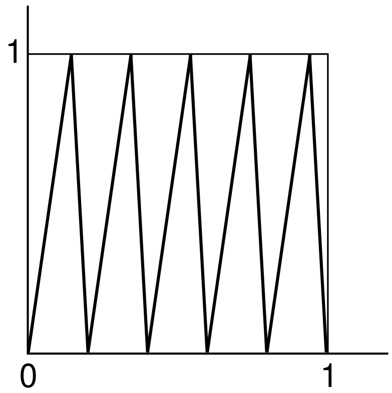

Figure 19.6: The Iterated Tent Map. See Solution 5.2.

provided that $\rho_0(x)$ is a smooth function of x.(See Fig. 19.6) A number of students provided this nice argument.

Notice that the Boltzmann equation for the baker's transformation

$$W_n(x) = \frac{1}{2}(W_{n-1}(\frac{1}{2}x) + W_{n-1}(\frac{x+1}{2}))$$

is the F-P equation for the map $x' = 2x(mod1)$. An argument similar to this holds for the (generalized) baker's transformation from problem 4.1 . Here we see the two ingredients for the approach to equilibrium in these systems :

1) A map that is chaotic (exponential separation of close points) in the variable of interest, in this case x.

2) An initial state distribution which is sufficiently smooth.

In these 1-d maps the fact that the map is not invertible (i.e. there is no unique antecedent point for x, only two points) corresponds to the choice of the fast and slow variables in the 2-d baker's transformation.

6.1) A map that can be used to represent a random-walk is the following

If we write $x_n = [x_n] + \tilde{x}_n$, where $[x_n]$ = integral point of x_n (the closest integer less than or equal to x_n), $\tilde{x}_n$= fractional part of x_n. The lines of the map are :

$$x_{n+1} = [x_n] + 1 + \frac{1}{\alpha}\tilde{x}_n \quad 0 \leq \tilde{x}_n < \alpha$$

$$x_{n+1} = [x_n] - 1 + \frac{1}{\beta}(\tilde{x}_n - \alpha) \quad \alpha \leq \tilde{x}_n < 1$$

The map has the property that regions of length α are uniformly mapped to the next interval on the right, and regions of length β are uniformly mapped to the left.(See Fig. 19.7.)

6.2) Lyapunov-exponents ;

$$|\frac{dx_n}{dx_0}| = |\frac{dx_n}{dx_{n-1}}||\frac{dx_{n-1}}{dx_{n-2}}|\dots|\frac{dx_1}{dx_0}|$$

$$(\frac{1}{\alpha})^{n_2}(\frac{1}{\beta})^{n_1}$$

n_1 is the number of steps where particles move to the right. n_2 is the number of steps where particles move to the left. Write

$$|\frac{dx_n}{dx_0}| = e^{n\lambda_n(x_0)} = e^{n_1 \ln(\frac{1}{\alpha}) + n_2 \ln(\frac{1}{\beta})}$$

then

$$\lambda(x_0) = \lim_{n\to\infty} \lambda_n(x_0) = \lim_{n\to\infty}((\frac{n_1}{n})\ln(\frac{1}{\alpha}) + (\frac{n_2}{n})\ln(\frac{1}{\beta}))$$

For an ergodic system : $\frac{n_1}{n}$= fraction of time a particle is in the regions of length α, $\frac{n_2}{n}$= fraction of time a particle is in the regions of length β, in the limit that $n \to \infty$. So $\frac{n_1}{n} = \alpha$ and $\frac{n_2}{n} = \beta$.

$$\lambda(x_0) = \lambda = \alpha \ln(\frac{1}{\alpha}) + \beta \ln(\frac{1}{\beta})$$

6.3)

$$\gamma = D(\frac{\pi}{L})^2 = \lambda - h_{KS}$$

$$\lambda = \frac{1}{2}\ln(2) + \frac{1}{2}\ln(2) = \ln(2)$$

$D = \frac{1}{2}$ from random-walk theory .

$$\gamma = \frac{\pi^2}{2L^2}$$

$$h_{KS} = \ln(2) - \frac{\pi^2}{2L^2}$$

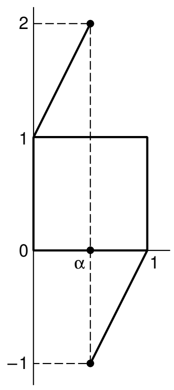

Figure 19.7: The Map for Problem 6.1.