The driven, damped pendulum.

Tomas Bohr
Physics Institute
The Technical University of Denmark

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In this note we shall study some properties of the driven, damped pendulum in the form

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 \sin x = F \cos \omega t \tag{1}$$

which can also be written as a system of first order equations

$$\begin{aligned} \dot{x} &= y &(2)\\ \dot{y} &= -\gamma y - \omega_0^2 \sin x + F \cos \omega t &(3) \end{aligned}$$

or, as an *autonomous* system

$$\begin{aligned} \dot{x} &= y &(4)\\ \dot{y} &= -\gamma y - \omega_0^2 \sin x + F \cos z &(5)\\ \dot{z} &= \omega &(6) \end{aligned}$$

which is of the general form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{7}$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{f}$ is the right-hand side of (4).

For $\gamma > 0$ this is a *dissipative* system and, as shown in Strogatz p. 312-313, the volume contraction in the 3 dimensional phase space is given as

$$\dot{V} = \int_V \nabla \cdot \mathbf{f} dx dy dz = -\gamma \tag{8}$$

which means that the equation is *uniformly area contracting* and that a volume in phase space simply decays as

$$V(t) = V(0) e^{-\gamma t} \tag{9}$$

no matter where in phase space we are.

1. Near resonance: linear theory

Let us first look at the linear oscillator obtained from (1) by expanding $\sin x$ to lowest order in x and neglecting the damping:

$$\ddot{x} + \omega_0^2 x = F \cos \omega t \tag{10}$$

which has the general solution

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F \cos \omega t}{\omega_0^2 - \omega^2} \tag{11}$$

valid as long as $\omega \neq \omega_0$. For $\omega = \omega_0$ we get

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F t \sin \omega t}{2\omega} \tag{12}$$

which contains an aperiodic secular term growing without bounds when $t \to \infty$. Note that the phase lag of the response is 0 when $\omega < \omega_0$ and π when $\omega > \omega_0$ and $\pi/2$ precisely at $\omega = \omega_0$.

The divergence of the amplitude can be stopped by putting in dissipation:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = F \cos \omega t \tag{13}$$

As long as $0 < \gamma^2 < 4\omega_0^2$ this can be solved as

$$x(t) = e^{-\gamma t/2}(A \cos \omega_1 t + B \sin \omega_1 t) + F \frac{(\omega_0^2 - \omega^2) \cos \omega t + \gamma \omega \sin \omega t}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \tag{14}$$

where $\omega_1 = \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}$. After a time long compared to $1/\gamma$ the transient term will disappear and the response becomes periodic with the period of the forcing $T = 2\pi/\omega$. This response can be written as

$$x(t) = r \cos(\omega t - \phi) \tag{15}$$

where

$$r = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} \tag{16}$$

and

$$\tan \phi = \frac{\gamma \omega}{\omega_0^2 - \omega^2} \tag{17}$$

Now the response remains finite at resonance, of the order of $F/\gamma\omega$.

2. Near resonance: non-linear theory

Nonlinearity will also remove the divergent response and in addition give rise to hysteresis. Let us first rederive the "two-timing equation" Strogatz p. 224 (7.6.53)) by averaging (exercise 7.6.25 on p. 239).

Assume that we have the general non-linear, perhaps forced problem

$$\ddot{x} + x + \epsilon h(x, \dot{x}, t) = 0 \tag{18}$$

which we shall write as

$$\dot{x} = y \tag{19}$$
$$\dot{y} = -x - \epsilon h(x, y, t) \tag{20}$$

and let us write the solution as

$$x(t) = r(t)\cos(t + \phi(t)) \tag{21}$$
$$y(t) = -r(t)\sin(t + \phi(t)) \tag{22}$$

This is an ansatz and not, at this stage, an approximation. When $\epsilon = 0$ this is true for any constant r and ϕ. Differentiating $x(t)$ and comparing to the form of $y(t)$ using (19) gives us the relation

$$\dot{r}\cos\theta = r\dot{\phi}\sin\theta \tag{23}$$

where $\theta = t + \phi$. From (20) we now get, by multiplying, respectively with $\sin\theta$ and $\cos\theta$

$$\dot{r} = \epsilon h(r(t)\cos(t + \phi(t)), -r(t)\sin(t + \phi(t)), t)\sin(t + \phi(t)) \tag{24}$$
$$r\dot{\phi} = \epsilon h(r(t)\cos(t + \phi(t)), -r(t)\sin(t + \phi(t)), t)\cos(t + \phi(t)) \tag{25}$$

Now, for small ϵ it is obvious that r and ϕ are slowly varying with a slow time of the order of ϵt. We now average (24)-(25) by integrating over one period of the unperturbed system (2π) *with the approximation of keeping r and ϕ constant over such a period.* Thus

$$\dot{r} \approx \epsilon \int_0^{2\pi} dt h(r\cos(t + \phi), -r(t)\sin(t + \phi), t)\sin(t + \phi) \tag{26}$$
$$r\dot{\phi} \approx \epsilon \int_0^{2\pi} dt h(r\cos(t + \phi), -r(t)\sin(t + \phi), t)\cos(t + \phi) \tag{27}$$

For a non-linear pendulum oscillating close to $\theta = 0$ we can expand as $\sin\theta \approx \theta - \frac{1}{6}\theta^3$ and take our equation as

$$\ddot{x} + x + \epsilon h(x, \dot{x}, t) = 0 \tag{28}$$

with $\epsilon h(x, \dot{x}, t) = \kappa x + \gamma\dot{x} - \beta x^3 - F\cos t$ Here time has been scaled such that the frequency of the external forcing is $\omega = 1$ and the *detuning* κ describes the deviation between this frequency and the natural frequency of the oscillator as $\omega_0 = 1 + \kappa$.

We now insert the expression for h into the averaged equations (26)-(27). For the forcing term we use the addition formulas for $\sin(t + \phi)$ and $\cos(t + \phi)$ and we get

$$\dot{r} = -\frac{1}{2}F\sin\phi - \frac{1}{2}\gamma r \tag{29}$$
$$r\dot{\phi} = -\frac{1}{2}F\cos\phi + \frac{1}{2}\kappa r - \frac{3}{8}\beta r^3 \tag{30}$$

2.1 The undamped case

Let us first study the case $\gamma = 0$. The fixed points must satisfy: $\sin\phi = 0$, which means that $\phi = 0$ or $\phi = \pi$ and $\cos\phi = \pm 1$. Thus (with the notation $a = 3\beta/4$)

$$r(\kappa - ar^2) = \pm F \tag{31}$$

which gives the response $r(\kappa)$ for a given $F > 0$ shown in Fig. 1 (top).

The parenthesis on the left hand side is zero for $r = r_0(\kappa) = \sqrt{\frac{\kappa}{a}}$. This line corresponds to the limit $F = 0$. To the left of (or above) $r = r_0(\kappa)$ the negative sign must be chosen in (31), corresponding to $\phi = \pi$. In this case (31) has a unique solution $r_F(\kappa)$. To the right of (or below) $r = r_0(\kappa)$ the positive sign must be chosen in (31), corresponding to $\phi = 0$. In this case (31) has two solutions $r_F(\kappa)$ as long as F is smaller than the maximum value of $\kappa r - ar^3$, which is $2a^{-1/2}(\kappa/3)^{3/2}$. The double-valuedness of the response in this regime, corresponds to the existence of *turning points*, where $d\kappa/dr = 0$. Differentiating (31) with r gives

$$\kappa - ar^2 - 2ar^2 + \frac{d\kappa}{dr} = 0 \tag{32}$$

and thus the turning points occur on the line

$$\kappa - 3ar^2 = 0 \tag{33}$$

or $r = r_1(\kappa) = \sqrt{\frac{\kappa}{3a}}$.

The branch between the two curves r_0 and r_1 is actually *unstable*. We see this by calculating the Jacobian:

$$J = \begin{pmatrix} \frac{\partial \dot r}{\partial r} & \frac{\partial \dot r}{\partial \phi} \\ \frac{\partial \dot\phi}{\partial r} & \frac{\partial \dot\phi}{\partial \phi} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}F\cos\phi \\ \frac{F\cos\phi}{2r^2} - 2ar & \frac{F\sin\phi}{2r} \end{pmatrix} \tag{34}$$

At the fixed points

$$J = \begin{pmatrix} 0 & -\frac{1}{2}r(\kappa - ar^2) \\ \frac{1}{2r}(\kappa - ar^2) - ar & 0 \end{pmatrix} \tag{35}$$

so that $\tau = \mathrm{Tr}J = 0$ and $\Delta = \mathrm{Det}J = (\kappa - ar^2)(\kappa - 3ar^3)/4$. This is negative precisely between the two curves r_0 and r_1. Outside of them the solutions are marginally stable (centers) and the system is in fact Hamiltonian, just like the unaveraged system.

2.2 The damped case

Now we include damping, i.e. $\gamma > 0$ in (29)-(30). The fixed points now satisfy:

$$F\sin\phi = -\gamma r \tag{36}$$
$$F\cos\phi = r(\kappa - ar^2) \tag{37}$$

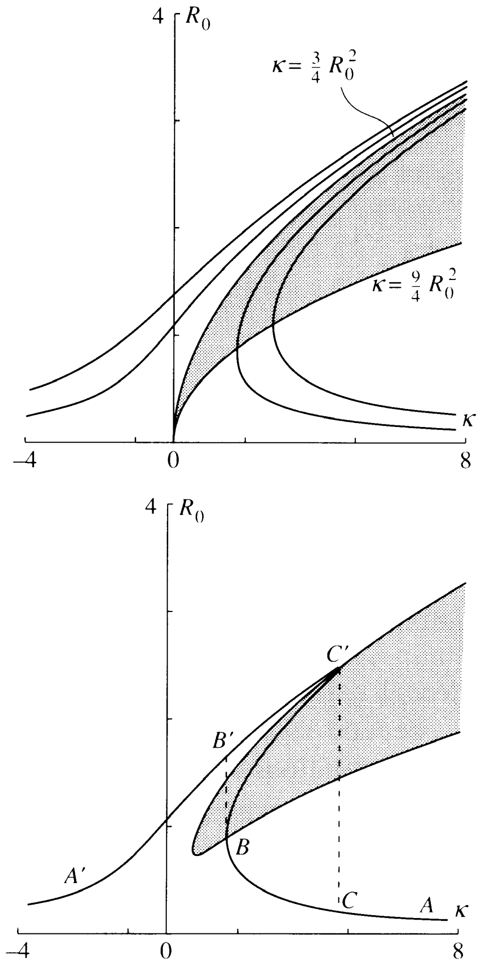

Figure 1: Typical response diagram of a nonlinear oscillator near resonance. Top: Undamped case. The two curves $r = R_0(\kappa)$ are computed from (31) with $\beta = 1$ and $F = 1$ and $F = 2$. Bottom: Damped case. Here $r = R_0(\kappa)$ is computed from (38) with $\beta = 1$, $\gamma = 0.4$ and $F = 1$.

and this leads to

$$r^2(\gamma^2 + (\kappa - ar^2)^2) = F^2 \tag{38}$$

for the response $r(\kappa)$ for a given $F > 0$ shown in Fig. 2 (bottom). Here we again use the abbreviation $a = 3\beta/4$.

Again, we can study the turning points seen in Fig. 2 by differentiating (38) with r. Thus

$$r(\kappa - ar^2)\frac{d\kappa}{dr} = -(\gamma^2 + (\kappa - ar^2)(\kappa - 3ar^2)) \tag{39}$$

and thus the turning points occur when the right-hand side is zero as shown in Fig. 2. Now, in addition, we can have $d\kappa/dr = \infty$ or $dr/d\kappa = 0$ and this occurs on the line $r_0(\kappa)$.

The stability of the fixed points can again be found from the Jacobian:

$$J = \begin{pmatrix} -\frac{1}{2}\gamma & -\frac{1}{2}F\cos\phi \\ \frac{F\cos\phi}{2r^2} - 2ar & \frac{F\sin\phi}{2r} \end{pmatrix} \tag{40}$$

and for the fixed points

$$J = \begin{pmatrix} -\frac{1}{2}\gamma & -\frac{1}{2}r(\kappa - ar^2) \\ \frac{1}{2r}(\kappa - ar^2) - ar & -\frac{1}{2}\gamma \end{pmatrix} \tag{41}$$

so that $\tau = \mathrm{Tr}J = -\gamma$ and $\Delta = \mathrm{Det}J = (\gamma^2 + (\kappa - ar^2)(\kappa - 3ar^3))/4$. Again the instability region corresponds to the region between the turning points. This means that the system will show *hysteresis*: if we start from small κ (large ω) the response will follow the upper branch until the turning point at C'. There it becomes unstable and jumps down to the lower branch at C. Coming from the other side (small ω) the solution follows the lower branch further - all the way down to B, where it jumps up to B'.

3. Properties of the Poincaré map

Let $P_{t_0}^t$ be the map that propagates the solution forward from time t_0 to time t. Thus $(x(t), y(t)) = P_{t_0}^t(x_0, y_0)$ is the orbit point with initial condition $(x(t), y(t)) = (x_0, y_0)$. The area contraction of this discrete map in the (x, y) plane is given by the determinant

$$J = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} \end{bmatrix} \tag{42}$$

thus

$$\dot{J} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x_0} & \frac{\partial \dot{x}}{\partial y_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} \end{bmatrix} + \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} \\ \frac{\partial \dot{y}}{\partial x_0} & \frac{\partial \dot{y}}{\partial y_0} \end{bmatrix} \tag{43}$$

using e.g.

$$\frac{\partial \dot{x}}{\partial x_0} = \frac{\partial \dot{x}}{\partial x}\frac{\partial x}{\partial x_0} + \frac{\partial \dot{x}}{\partial y}\frac{\partial y}{\partial x_0} = \frac{\partial f_1}{\partial x}\frac{\partial x}{\partial x_0} + \frac{\partial f_2}{\partial y}\frac{\partial y}{\partial x_0} \tag{44}$$

and

$$\dot{J} = (\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y})J = -\gamma \tag{45}$$

which leads to

$$J(t) = J(0)e^{\int_0^t (\partial f_1/\partial x + \partial f_2/\partial y)dt} = e^{-\gamma t} \tag{46}$$

since $J(0) = 1$. This corresponds well to (9) since the z-direction does not contribute to the contraction.

4. Period doubling and symmetry breaking.

The equations (4) are invariant under the transformation: $x \to -x$, $y \to -y$ and $t \to t + T/2$, where $y = \dot{x}$ and $T = 2\pi/\omega$. Thus there exists symmetric solutions $x(t)$ for which $x(t + T/2) = -x(t)$ and $y(t + T/2) = -y(t)$, but it is important to note that the solutions are not necessarily symmetric. If they are not symmetric we can find another solution by performing the symmetry transformation $x \to -x(t+T/2)$, $y \to -y(t+T/2)$. Let again $(x(t_1), y(t_1)) = P_{t_0}^{t_1}(x_0, y_0)$ be the solution of (1) at time t_1 with initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$. The Poincaré map $\mathbf{P} = P_0^T = P_{T/2}^T P_0^{T/2}$. The symmetry of (1) means that

$$(x, y) = P_0^{T/2}(x_0, y_0) \tag{47}$$

implies that

$$(-x, -y) = P_{T/2}^T(-x_0, -y_0) \tag{48}$$

and sice this is true for all (x_0, y_0) the half-period mappings must satisfy

$$P_{T/2}^T = (-I) \circ P_0^{T/2} \circ (-I) \tag{49}$$

where I is the unit matrix (that is $(-I)(x, y) = (-x, -y)$). Thus the Poincaré map can be written as the *square* of another operator (i.e. the composition of another operator with itself): $\mathbf{P} = ((-I) \circ P_0^{T/2}) \circ ((-I) \circ P_0^{T/2}) = ((-I) \circ P_0^{T/2})^2$.

A *symmetric* orbit is not only a fixed point of $\mathbf{P}$ but also of $(-I) \circ P_0^{T/2}$. Non-symmetric orbits are two cycles of this map. The eigenvalues of $\mathbf{P}$ for the symmetric orbits are the square of the eigenvalues of $(-I)P_0^{T/2}$. Period doubling means that one of the eigenvalues of $\mathbf{P}$ must go through -1. Thus $(-I)P_0^{T/2}$ must have eigenvalues $\pm i$ and this means that the other eigenvalue of $\mathbf{P}$ must also be -1. This is impossible since (1) is damped. In fact the product of the two eigenvalues must be $e^{-\gamma T}$. Thus period doubling must be preceeded by symmetry breaking.