26 Aerodynamics

WARNING: unfinished

The take-off of a large airplane never ceases to wonder passengers and bystanders alike. After building up speed during a brief half-minute run, gravity lets go and the plane marvellously lifts off. For this to happen, it is obvious that the engines and the air flow must generate a vertical force, a lift, which is larger than the weight of the aircraft. After becoming airborne the airplane accelerates further while climbing with a fairly large angle of attack. During this phase, part of the vertical lift must be generated directly by the engines, even though the major part of the engine thrust goes to accelerate the aircraft and to counteract the increasing drag on the airframe.

Aloft in level flight at constant speed, the aerodynamic lift must almost exactly balance the weight, whereas the engine thrust goes to oppose the drag. What is not obvious to most passengers is how the lift depends on the forward speed and the shape of the airframe, especially the wings.

The general explanation of aerodynamic lift is quite simple, even if it was only in the beginning of the twentieth century — about the same time as the first airplanes were built — that the details became understood. In a nearly ideal flow, pressure is the dominant stress acting on any surface. Consequently, for a lift to exist, the pressure must be higher underneath the wing than above. Bernoulli's theorem then implies that the airspeed must be higher above the wing than below, effectively creating a circulation around the wing, a kind of bound vortex superimposed on the general airflow. Without this circulation, caused by the shape and flying attitude of the wing, there can be no lift.

Kutta (1902) and Zhukovskii (1906) were the first to determine the relation between circulation and lift.

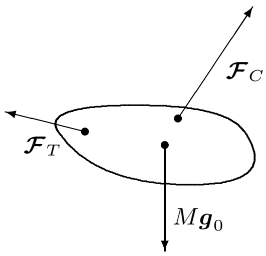
Aerodynamics is a huge subject of importance for all objects moving through the air, such as rifle bullets, rockets, airplanes, cars, and sailing ships, and even for submarines moving through water. In this chapter only the most basic aerodynamic theory will be presented.

26.1 Aerodynamic forces

The only way a fluid can act on a body immersed in it, is through contact forces, described by the stress tensor $\boldsymbol{\sigma}$. In total the hydrodynamic contact forces add up to

$$\boxed{\boldsymbol{\mathcal{F}}_C = \oint_{\mathsf{body}} \boldsymbol{\sigma} \cdot d\boldsymbol{S}} \ , \tag{26-1}$$

integrated over the body surface. In principle this includes hydrostatic buoyancy forces, but as they only serve to diminish the effective gravitational mass of a body, they will be disregarded here. For the purposes of aerodynamics even if the 'air' is another fluid such as water, the gravitational mass of a body is simply the inertial mass with buoyancy subtracted. This may of course lead to objects with negative gravitational mass, for example balloons.



Sketch of force balance for a non-rotating body moving with constant velocity through a fluid. The total force and the total couple both vanish.

The total force $\boldsymbol{\mathcal{F}}$ acting on a body determines the acceleration of its center of mass. Including standard gravity $M\boldsymbol{g}_0$ and engine thrust $\boldsymbol{\mathcal{F}}_T$ together with the hydrodynamic contact forces, we have

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_C + \boldsymbol{\mathcal{F}}_T + M\boldsymbol{g}_0 \ . \tag{26-2}$$

For a non-accelerated object, the total force must of course vanish.

The moment of force

The three types of forces do not attack in the same point. The point of attack of a force may generally be defined to be the point for which the total moment of force, or couple, is the same as the moment of the total force. For the gravitational force this is the center of mass, and for the engine thrust it must be close to the position of the engine itself. The hydrodynamic contact forces have a total moment

$$\boldsymbol{\mathcal{M}}_C = \oint_{\mathsf{body}} \boldsymbol{x} \times \boldsymbol{\sigma} \cdot d\boldsymbol{S} = \boldsymbol{x}_C \times \boldsymbol{\mathcal{F}}_C \ , \tag{26-3}$$

where the point of attack $\boldsymbol{x}_C$ has been defined so that the moment around this point vanishes.

For a steadily moving, non-rotating body, the total moment of all forces must also vanish. In the same way as floating bodies, ships and icebergs, are normally in stable hydrostatic equilibrium, aircraft should also generally be hydrodynamically stable. This means that a small perturbation of the body's attitude should generate a couple counteracting the perturbation. Most modern aircraft are dynamically stable, and that is very good for amateur pilots, but in military fighter planes, stability is sometimes traded for maneuverability. Certain modern fighter planes can in fact only maintain a stable attitude through corrections continually applied to the control surfaces by a fast computer.

Steady flight

We shall mostly be interested in non-accelerated objects in steady flight, *i.e.* moving with constant center of mass velocity through a fluid which would have been at rest, were it not for the presence of the moving body.

Asymptotically, sufficiently far from the body, the fluid will always be at rest. As we have discussed before, it is more convenient to avoid the explicit time-dependent boundary conditions on the body surface in the rest frame of the fluid by going to the rest frame of the body, where the fluid asymptotically moves with steady uniform velocity. Since forces in Newtonian mechanics are the same in all inertial systems, it does not matter for the discussion whether we choose the one or the other frame.

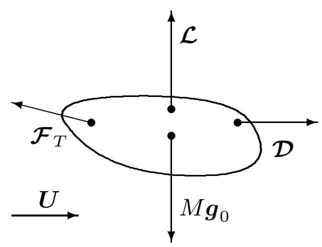
The choice of the rest frame of the body does not guarantee that the flow induced by the body will be steady, but at least there is no explicit hindrance for it, and we shall mostly assume that it is. The actual flow pattern in the rest frame of the body, steady or not, depends on the shape of the body, the speed of the flow and the viscosity of the fluid. As we have discussed so often before, a natural controlling parameter is the Reynolds number, $\mathsf{Re} = UL/\nu$ where U is the asymptotic flow speed, L the body dimension and ν the kinematic viscosity.

Lift and drag

It is convenient to resolve the total contact force into two components, one which is orthogonal to the asymptotic flow velocity $\boldsymbol{U}$, and one which is parallel with it,

$$\boldsymbol{\mathcal{F}}_C = \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{D}} \; . \tag{26-4}$$

These components are called *lift* and *drag*, respectively, and formally we have $\boldsymbol{\mathcal{L}} = \boldsymbol{e}_U \times (\boldsymbol{\mathcal{F}}_C \times \boldsymbol{e}_U)$ and $\boldsymbol{\mathcal{D}} = (\boldsymbol{\mathcal{F}}_C \cdot \boldsymbol{e}_U)\boldsymbol{e}_U$, where $\boldsymbol{e}_U = \boldsymbol{U}/U$ is a unit vector in the direction of the asymptotic flow velocity. Whereas we expect the drag always to be in the same direction as the velocity, so that sign of $\mathcal{D} = \boldsymbol{\mathcal{F}}_C \cdot \boldsymbol{e}_U$ is always positive, the lift may take any direction, as long as it is orthogonal to the velocity.



Sketch of forces in steady horizontal powered flight (towards the left). Lift ($\boldsymbol{\mathcal{L}}$) almost balances the weight ($M\boldsymbol{g}_0$), and drag ($\boldsymbol{\mathcal{D}}$) almost balances the thrust ($\boldsymbol{\mathcal{F}}_T$).

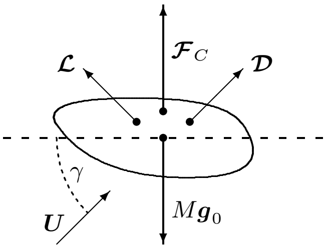
It is, for example, dangerous to fly inverted on the back close to the ground, because the gut reaction of pulling the stick towards you to get away from the ground will generate an extra lift which sends you directly into the ground. During turn and bank (see below), an airplane also generates lift away from the vertical, which creates the force necessary to change its direction.

In powered steady flight the sum of all forces must vanish

$$\boldsymbol{\mathcal{F}}_T + \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{D}} + M\boldsymbol{g}_0 = 0 \; . \tag{26-5}$$

For ordinary aircraft in powered level flight, almost all of the lift is aerodynamic, whereas the drag is almost entirely balanced by engine thrust. The engine thrust does, however, not have to be directly opposite to the wind, and part of it may go to produce lift, as for example during takeoff or a steep climb. The extreme case is a hovering helicopter that uses all the engine thrust for lift.

Unpowered descent



Sketch of forces in unpowered steady descent through fluid at approximately 45° glide angle. Lift ($\mathcal{L}$) and drag ($\mathcal{D}$) collaborate to balance the weight ($M\boldsymbol{g}_0$).

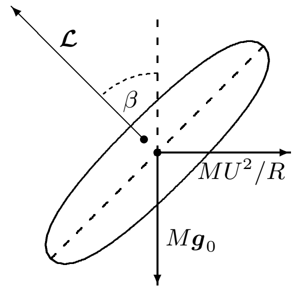
If the thrust vanishes, $\mathcal{F}_T = 0$, lift and drag must alone balance the object's weight, *i.e.* $\boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{D}} + M\boldsymbol{g}_0 = 0$. Most freely falling objects quickly reach a constant terminal velocity. Stones usually fall vertically, whereas paper gliders and parachutists in free fall in addition will attain sometimes large horizontal speeds.

An aircraft in unpowered flight is able to glide towards the ground with constant velocity and constant rate of descent. It is part of early training of pilots to learn how to handle their craft in unpowered steady descent, and usually the aircraft is so dynamically stable, that it by itself ends up in a steady glide, if the engine power is cut. Paper gliders on the other hand often go through a series of swooping dives broken by stalls.

During a steady unpowered descent the air hits the aircraft from below at an angle γ with the horizontal, called the *glide angle*. The maximal ratio between horizontal and vertical speed is called the *glide ratio* and is given by $\cot\gamma_{\rm min} \approx 1/\gamma_{\rm min}$ when it is small. Typical commercial aircraft have glide ratios of 10, so if the engines set out in a commercial jet in 10 km altitude, the pilot has to look for a place to land inside 100 km. Real glider planes may reach glide ratios around 35–45 and in extreme cases 75.

In a steady unpowered glide the total hydrodynamic contact force must be equal and opposite to the weight of the aircraft. Its point of attack must be vertically above the center of gravity in order not to generate a turning couple and for the descent to be stable. Since the contact force is sum of lift and drag, we have $\mathcal{L}/\cos\gamma = \mathcal{D}/\sin\gamma = Mg_0$. From the glide angle and the weight of an aircraft, we may thus determine both lift and drag.

Banked turn



Sketch of forces in the transverse plane during a 45° banked turn.

Consider an aircraft turning steadily under power in a horizontal circle of radius R. From the rest frame of the aircraft, the air again flows steadily past with velocity U, but now there is also a centrifugal force MU^2/R directed away from the center of the turn. The engine thrust is assumed to balance the drag, and the lift must therefore balance the vector sum of the weight and the centrifugal force. Let β be the angle between the vertical and the lift vector. Since $\mathcal{L}\sin\beta = MU^2/R$ and $\mathcal{L}\cos\beta = Mg_0$ we get $\cot\beta = Rg_0/U^2$.

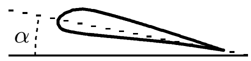
Usually airplanes are banked (tilted) through the angle β during turns, such that the floor remains orthogonal to the lift. The effective gravity felt inside the airplane during such a clean turn is (in units of the standard gravity) $g_{\rm eff}/g_0 = \mathcal{L}/Mg_0 = 1/\cos\beta$, also called the g-factor. In a clean 60° banked turn, one pulls a g-factor of 2. In order to avoid passenger discomfort, most commercial aircraft rarely bank beyond 15°, which at a speed of 900 kilometers per hour yields a clean turn diameter $2R = 2U^2\cot\beta/g_0 \approx 50$ kilometers, and a full turn takes about 10 minutes.

Aircraft controls

The majority of all aircraft that have ever been built are symmetric under reflection in a midplane. The wings are more or less placed in a plane orthogonal to the midplane, but often swept somewhat backwards and a bit upwards. The symmetry plane and the wing plane define the three orthogonal axes of the aircraft.

The first axis is the longitudinal axis which runs along the body of the aircraft in the intersection of the midplane and the wing plane. Rotation around this axis is called *roll*, and is normally controlled by the *ailerons* on the wings. When the pilot moves the stick from side to side the ailerons move oppositely to each other and create a rolling moment around the this axis. The second axis lies in the wing plane orthogonally to the midplane. Rotation around this axis is called *pitch* and normally controlled by the *elevator*. The pilot moves the stick forward and backward to create a pitching moment around this axis. The third axis is orthogonal to the other two and therefore to both the wings and the body. Rotation around this axis is called *yaw* and normally controlled by the *rudder*. The pilot presses pedals in order to move the rudder and create a yawing moment around this axis.

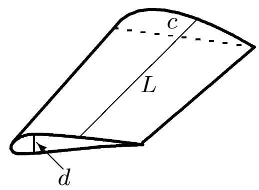
An aircraft can in principle move through the air in any attitude — and some pilots enjoy doing just that — but there is an intended normal flying attitude with the wings horizontal and orthogonal to the airflow. The main controlling parameter for lift is then the angle of attack α, also called the angle of incidence, between the plane of the aircraft and the airflow. It is usually defined such that the lift vanishes for $\alpha = 0$, and it is controlled by the elevator.



The angle of attack α. A wing may also have aerodynamic twist, such that the angle of attack varies along the wing.

Wing dimensions

An airplane wing may be characterized by three different length scales: the length or *span* of the wing L, the transverse width or *chord* c, and the thickness d. A wing can only in the coarsest of approximations be viewed as a rectangular box of these dimensions. Normally wings or *airfoils* are designed with curved leading and sharp trailing edges for maximal lift and minimal drag, and they are usually thin ($d \ll c$) with small aspect ratio ($c \ll L$), but other shapes are also used, for example delta wings.



The dimensions of a wing are characterized by the span L, the chord c, and the thickness d. The wing profile depicted here also carries aerodynamic twist.

Close to the wing surfaces there are thin boundary layers, in which the flow velocity changes dramatically from zero at the skin of the wing to a speed nearly equal to the asymptotic airspeed just outside.

> Consider a small airplane with a wing span of 10 meters, a chord of 1 meter, and thickness 10 centimeters, moving at a speed of 100 meters per second. The Reynolds number for the flow across the wing becomes $\mathsf{Re} = Uc/\nu \approx 10^7$. For smooth wing surfaces, the effect of viscosity is limited to thin laminar boundary layers with an estimated thickness of about $3c/\sqrt{\mathsf{Re}} \approx 1$ millimeter.

26.2 Thin wing estimates

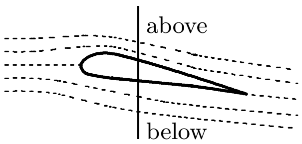
As mentioned in the introduction to this chapter, lift in powered horizontal flight at constant velocity is primarily due to a pressure difference between the upper and lower wing surfaces, provided the flow is nearly ideal and the wings are thin. Drag is on the other hand due to several effects. First, there is skin friction due to the boundary layer. Second, there is form drag due to the wing obstructing the flow, and finally for a wing of finite span, there is induced drag due to vortices trailing the wing tips. We shall now estimate all these effects from simple physical arguments.

Lift

For a thin, long wing with small angle of incidence the pressure difference between the upper and lower surfaces must approximately be

$$\Delta p \approx -\frac{\mathcal{L}}{Lc} \;, \tag{26-6}$$

where Lc is a measure of the wing area.



Sketch of streamlines around a wing with positive lift. The wing profile accelerates the flow across the top of the wing and retards it below, and thus creates a larger pressure below than above. Notice the front and rear stagnation points.

For the small airplane of the preceding section, weighing perhaps 1000 kilograms, with a wing area of 10 square meters we find in steady level flight a pressure difference of 10^3 Pascal or 10 millibar. Not very impressive, but it nevertheless carries the plane!

Consider now two streamlines coming in from infinity and passing on each side of the wing. Since they start out with the same velocity and pressure, Bernoulli's theorem may be used to relate the pressures and velocities in nearby points on a vertical line above and below the wing, just outside the boundary layers,

$$p_{\text{above}} + \frac{1}{2}\rho_0 \boldsymbol{v}_{\text{above}}^2 = p_{\text{below}} + \frac{1}{2}\rho_0 \boldsymbol{v}_{\text{below}}^2 \;. \tag{26-7}$$

From this we get

$$\begin{aligned}
\Delta p = p_{\text{above}} - p_{\text{below}} &= -\frac{1}{2}\rho_0(\boldsymbol{v}_{\text{above}}^2 - \boldsymbol{v}_{\text{below}}^2) \;, \\
&\approx -\frac{1}{2}\rho_0(\boldsymbol{v}_{\text{above}} + \boldsymbol{v}_{\text{below}}) \cdot (\boldsymbol{v}_{\text{above}} - \boldsymbol{v}_{\text{below}}) \;, \\
&\approx -\rho_0 U \Delta v \;.
\end{aligned}$$

In the last line we have used that the flow velocity over most of the wing surface must be close to the asymptotic velocity U, an assumption which must break down near the leading and trailing ends of the wing, where there are stagnation points with vanishing flow speed. For a sufficiently thin planar wing, these end effects may hopefully be disregarded.

Combining this result with (26-6) we obtain an estimate of the average flow velocity difference in units of the general flow velocity

$$\frac{\Delta v}{U} = \frac{\mathcal{L}}{\rho_0 L c U^2} \ . \tag{26-8}$$

For the example of the small airplane in level flight from the preceding section we find $\Delta v/U \approx 0.1$. The lift necessary to keep the plane aloft requires a velocity difference of 10 m/s on the two sides of the wing.

Since the lift vanishes for zero angle of attack , it is reasonable to expect that for small values of α we have $\Delta v/U \approx k\alpha$ with a numerical $\mathcal{O}(1)$ coefficient k, depending on the actual shape of the wing. A typical value is $k = \pi$, valid for a general class of wing shapes, called Zhukovskii profiles, including among them a flat plate. In the example above, this leads to $\alpha \approx 2°$.

It is customary to define the dimensionless quantity

$$\boxed{C_{\rm lift} = \frac{\mathcal{L}}{\frac{1}{2}\rho_0 L c U^2} \approx \frac{2\Delta v}{U} \approx 2\pi\alpha} \ , \tag{26-9}$$

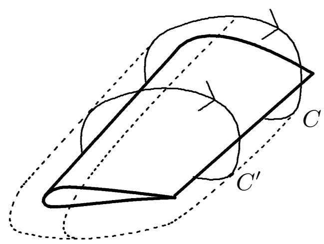
called the *lift coefficient*. The denominator is the stagnation pressure $\frac{1}{2}\rho_0 U^2$ times the area of the wing. In the example we have $C_{\rm lift} \approx 0.2$.

Circulation

The difference in flow velocity between the wing surfaces (outside the boundary layers) not only gives rise to a pressure difference and thereby lift, but also implies that there is circulation $\Gamma \approx -c\Delta v$ around the wing in the clockwise direction, as indicated by the negative sign. Using (26-8) we may estimate the lift as

$$\boxed{\mathcal{L} \approx -\rho_0 U L \Gamma} \ . \tag{26-10}$$

In section 26.3 we shall see that there is an exact relation of this form in ideal irrotational flow. The realization that lift and circulation are two sides of the same coin, is probably *the single most important insight into the mechanics of flight.*



Sketch of vortex bound to the (left) wing of an aircraft. The circulation is the same for the two curves C and C', provided there is no vorticity flux through the surface bounded by these curves.

The bound vortex

In nearly ideal irrotational flow, the circulation is the same around any curve circling the wing, because Stokes' theorem relates the difference in circulation between two such curves to the flux of vorticity (which is assumed to vanish) through the surface bounded by the two curves. For an infinitely long wing this may be true everywhere, but for a finite wing, the assumption has to break down, because one of the curves may be slid over the tip of the wing and shrunk to a point with no circulation. The conclusion is that since lift requires non-zero circulation, vorticity must come off somewhere along the wing.

The shedding of vorticity from a wing of finite span depends strongly on its shape. A wing which tapers towards the tip will shed vorticity everywhere along its trailing edge, but most near the tip. For a rectangular wing with constant chord, the vorticity will tend to appear very close to the tip. In any case, the vorticity coming off the tip is turned backwards, forming a trailing vortex in continuation of the bound vortex. Alternatively one may see the trailing vortex as a flow from below to above the wing, seeking to equalize the higher pressure underneath the wing with the lower pressure on top.

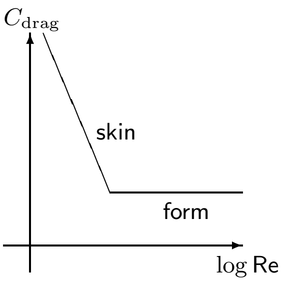
Skin friction

The shear stress acting on the wing may be estimated to be $\sigma \approx \eta U/\delta$ where δ is the width of the boundary layer. The layer is, however, not of the same thickness across the wing, but is thinnest at the leading edge and becomes thicker towards the rear and may in special cases even separate before the trailing edge. For a long wing of large aspect ratio L/c we find approximatively, by integrating over the chord, the drag due to skin friction

$$\mathcal{D}_{\text{skin}} \approx 2\eta L U \int_0^c \frac{dx}{\delta(x)}$$

with a factor of 2 due to the wing having two sides. Had the wing been completely flat, Blasius' model of the boundary layer would have yielded the result (**??**), *i.e.* $\delta(x) = 3\sqrt{\nu x/U}$. Using this result anyway, we obtain the skin drag coefficient

$$\boxed{C_{\text{skin}} = \frac{\mathcal{D}_{\text{skin}}}{\frac{1}{2}\rho_0 L c U^2} \approx \frac{8}{3}\frac{1}{\sqrt{\mathsf{Re}}}}\,, \tag{26-11}$$



Sketch of behavior of drag coefficient C_{drag} as a function of $\log \mathsf{Re}$ for a bluff body. For sufficiently high Reynolds number the form drag will dominate over skin friction.

where $\mathsf{Re} = Uc/\nu$ is the Reynolds number appropriate for flow across the wing. In the example we get $C_{\text{skin}} \approx 10^{-3}$.

Form drag

It is a common experience that the mere shape of a body gives rise to a drag. Sticking a hand out the window in a car at high speed one feels a drag which is obviously much greater when the hand is orthogonal to the wind than parallel with it (for intermediate angles there is also a strong lift acting on the hand).

A naive guess would be that the moving object creates a 'hole' in the air behind the body and absorbs all the momentum of the air hitting it. This picture would be right if 'air' consisted of globs of sticky glue and leads to an estimate for the form drag of $\mathcal{D}_{\text{form}} \approx \rho_0 U^2 A$ where A is the cross-sectional area of the body. This is not unreasonable for bluff bodies of roughly spherical shape, because they tend to leave a turbulent wake with a cross section a sizable fraction of that of the body, and a considerably lower than asymptotic flow velocity. Furthermore, since for such bodies form drag is independent of viscosity, form drag will always win over skin friction for sufficiently large Reynolds numbers.

For a thin 'streamlined' wing, narrowing down into a sharp trailing edge, the flow will be laminar and ideal everywhere, except in the boundary layers. We have earlier mentioned (and shall prove it in the following section) that a body in an ideal fluid does not experience any drag at all, independently of its shape. Both skin friction and form drag therefore owe their existence to viscosity, but where skin friction is due to shear stresses in the boundary layer, form drag arises from the tiny changes in pressure distribution over the body caused by the boundary layers.

In eq. (**??**) we estimated the small pressure changes in the boundary layer to be of magnitude $q \approx \rho_0 U^2 (\delta/c)^2$, expressed in terms of the parameters of the wing. Multiplying with the wing area Lc we obtain an estimate, $\mathcal{D}_{\text{form}} \approx \rho_0 U^2 Lc(\delta/c)^2$, for the influence of this tiny pressure. The corresponding drag coefficient becomes

$$\boxed{C_{\text{form}} = \frac{\mathcal{D}_{\text{form}}}{\frac{1}{2}\rho_0 U^2 Lc} \approx \frac{1}{\mathsf{Re}}} \,. \tag{26-12}$$

For a streamlined wing in normal flight with small angle of attack, form drag is completely negligible compared to skin friction. In the example we have $C_{\text{form}} \approx 10^{-7}$.

If the boundary layers are not so well-behaved, but separate from the wing surface before the trailing edge, or develop turbulence, the ideal flow pattern may change, so that form drag grows beyond the above estimate. This may happen for larger angles of attack, giving rise to much larger form drag. The point of separation for the boundary layer on top of the wing may suddenly and dramatically shift forward at a certain angle of attack, losing almost all of the lift. The wing and the aircraft are then said to have *stalled*. Most aircraft come fairly close to the stall limit when during landing just before touchdown.

Induced drag

The two vortices trailing from the wing tips of an aircraft rotate in opposite directions from below the wing to above, and must by Stokes theorem carry the same circulation Γ as the vortices bound by the wings. They are created at a rate given by the speed U of the airplane and persist indefinitely in a truly ideal fluid. In a real fluid they dissolve after a certain time. The drag caused by 'feeding out' the trailing vortices from the aircraft may be estimated from their kinetic energy per unit of length. For a vortex with core radius of a, the typical flow speed is $\Gamma/2\pi a$ so that $\mathcal{D}_{\text{induced}} \approx \rho_0 \pi a^2 (\Gamma/2\pi a)^2 \approx \rho_0 \Gamma^2/4\pi \approx \rho_0 U^2 c^2 \pi \alpha^2/4$. In other words

$$\boxed{C_{\text{induced}} = \frac{\mathcal{D}_{\text{induced}}}{\frac{1}{2}\rho_0 U^2 Lc} \approx \frac{c}{L}\alpha^2} \,, \tag{26-13}$$

apart from a numerical constant. For the example above we find $C_{\text{induced}} \approx 10^{-4}$, in roughly the same ballpark as the drag due to skin friction.

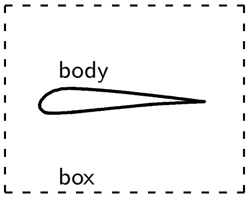
∗ 26.3 Lift in an ideal fluid

In an ideal fluid there are no shear forces, and the total contact force (26-1) acting on the body is entirely due to the pressure, $\boldsymbol{\sigma} = -p\mathbf{1}$. From Euler's equations (15-10) it follows that the divergence of the tensor $\rho_0 \boldsymbol{v}\boldsymbol{v} + p\mathbf{1}$ vanishes. Explicitly we have

$$\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{v}\boldsymbol{v} + p\mathbf{1}) = \rho_0(\mathsf{div}\,\boldsymbol{v})\boldsymbol{v} + \rho_0(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla} p = 0 \ .$$

Integrating this divergence over a volume of fluid with surface S, and using Gauss' theorem we find

$$\oint_S (\rho_0 \boldsymbol{v}\boldsymbol{v} + p\mathbf{1}) \cdot d\boldsymbol{S} = 0 \ .$$



Body and enclosing box.

Finally, choosing the volume of fluid to lie between the body and an arbitrary surface containing the body, a "box", we obtain

$$\boxed{\boldsymbol{\mathcal{F}}_C = -\oint_{\mathsf{body}} p\, d\boldsymbol{S} = -\oint_{\mathsf{box}} (\rho_0 \boldsymbol{v}\boldsymbol{v} + p\mathbf{1}) \cdot d\boldsymbol{S}} \ . \tag{26-14}$$

because the normal component of the velocity $\boldsymbol{v} \cdot d\boldsymbol{S}$ vanishes at the body surface in an ideal fluid. This equation simply expresses that all the momentum transferred to the body by the pressure acting at its surface, must also pass through any other surface completely enclosing the body.

The pressure on the boundary of the box may be determined from Bernoulli's theorem. For all points along streamlines coming in from infinity, where the pressure is assumed to vanish, we have

$$p + \frac{1}{2}\rho_0 \boldsymbol{v}^2 = \frac{1}{2}\rho_0 \boldsymbol{U}^2 \ . \tag{26-15}$$

Assuming that all streamlines crossing the box indeed come in from infinity, then this is valid everywhere on the box. This would be trivially true if the flow were everywhere irrotational, but as we have seen, this cannot always be assumed.

Let now the box be expanded so much that the velocity field everywhere on its boundary is only perturbed infinitesimally away from its asymptotic value, $\boldsymbol{v} \approx \boldsymbol{U}$. To leading order in the perturbation $\boldsymbol{v} - \boldsymbol{U}$ we find the pressure

$$p = \frac{1}{2}\rho_0(\boldsymbol{U}^2 - \boldsymbol{v}^2) \approx -\rho_0(\boldsymbol{v} - \boldsymbol{U}) \cdot \boldsymbol{U} \ . \tag{26-16}$$

Similarly, because $\oint_{\mathsf{box}} \boldsymbol{v} \cdot d\boldsymbol{S} = 0$ due to mass conservation, we have

$$\oint_{\mathsf{box}} \boldsymbol{v}\,\boldsymbol{v} \cdot d\boldsymbol{S} = \oint_{\mathsf{box}} \boldsymbol{U}\,\boldsymbol{v} \cdot d\boldsymbol{S} + \oint_{\mathsf{box}} (\boldsymbol{v} - \boldsymbol{U})\boldsymbol{v} \cdot d\boldsymbol{S} \approx \oint_{\mathsf{box}} (\boldsymbol{v} - \boldsymbol{U})\boldsymbol{U} \cdot d\boldsymbol{S} \ .$$

Putting it all together we get

$$\boxed{\boldsymbol{\mathcal{F}}_C = \rho_0 \oint_{\mathsf{box}} [(\boldsymbol{v} - \boldsymbol{U}) \cdot \boldsymbol{U}\,\mathbf{1} - (\boldsymbol{v} - \boldsymbol{U})\boldsymbol{U}] \cdot d\boldsymbol{S}} \tag{26-17}$$

to lowest order in the perturbation.

No drag in ideal fluids

In standard coordinates with the x-axis along the asymptotic velocity, so that $\boldsymbol{U} = (U, 0, 0)$, it follows immediately that the drag vanishes $\mathcal{D} = (\boldsymbol{\mathcal{F}}_C)_x = 0$. This is finally the proof of d'Alemberts paradox, that there can be no drag on a body of arbitrary shape in an ideal fluid.

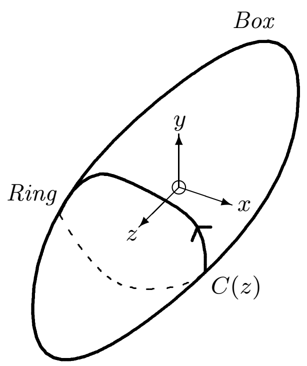
Actually we have only proven it to be true to first order in the perturbation $\boldsymbol{v} - \boldsymbol{U}$. There will be corrections of order $|\boldsymbol{v} - \boldsymbol{U}|^2$ to the total contact force. It follows however from the equations of motion that at great distances r from the body, the perturbation will fall off as $1/r^2$. This agrees with (26-17), because the surface of a large 'box' grows as r^2, so that the first order corrections become independent of r. By the same argument the second order corrections will (26-17) will fall off as $1/r^2$, and can be ignored.

Lift and circulation

Choosing the y-axis along the lift, so that $\mathcal{L} = (\boldsymbol{\mathcal{F}}_C)_y$ we find from (26-17)

$$\mathcal{L} = \rho_0 U \oint_{\mathsf{box}} ((v_x - U)dS_y - v_y dS_x) \ . \tag{26-18}$$

The lift only depends on quantities relating to the xy-plane. It is therefore convenient to view the surface of the large box as consisting of a sequence of curves or 'rings' $C(z)$ lying entirely in the xy-plane for every z that is covered by the box, say $z_1 < z < z_2$. We shall assume that the box is everywhere convex, or 'cigar-shaped', so that there will be no problems with infolding and crossing. If the line element of a ring is $d\boldsymbol{\ell} = (dx, dy, 0)$, then the surface element is becomes $d\boldsymbol{s} = d\boldsymbol{\ell} \times \boldsymbol{e}_z dz = (dydz, -dxdz, 0)$. Hence



Cigar-shaped 'box' surrounding the body at great distances. It may be viewed as a sequence of oriented closed curves or 'rings' $C(z)$, for each z parallel with the xy-plane.

$$\mathcal{L} = \rho_0 U \int_{z_1}^{z_2} dz \oint_{C(z)} (-(v_x - U)\, dx - v_y\, dy) \ . \tag{26-19}$$

But the U term cannot contribute to the contour integral, so that we get

$$\mathcal{L} = \rho_0 U \int_{z_1}^{z_2} dz \oint_{C(z)} (-\boldsymbol{v} \cdot d\boldsymbol{\ell}) \ . \tag{26-20}$$

Defining the circulation around $C(z)$

$$\Gamma(z) = \oint_{C(z)} \boldsymbol{v} \cdot d\boldsymbol{\ell} \ , \tag{26-21}$$

we finally obtain

$$\boxed{\mathcal{L} = -\rho_0 U \int_{z_1}^{z_2} \Gamma(z) dz} \ . \tag{26-22}$$

The Kutta-Zhukovskii theorem

This is the Kutta-Zhukovskii theorem, relating lift to circulation. As long as there is no vorticity component in the z-direction, we may use Stokes' theorem and contract the rings all the way down to the surface of the body, or at least to the boundary layer. For a long wing of constant profile with trailing wing tip vortices coming off in the x-direction, this condition is well fulfilled, and we arrive at the result

$$\mathcal{L} = -\rho_0 U L \Gamma \tag{26-23}$$

which we previously (in (26-10)) had only estimated.

The Kutta-Zhukovskii theorem is not everything. One still has to calculate the flow around a wing shape and the circulation associated with it. There is as mentioned before a family of airfoils due to Zhukovskii, derived from a cylinder by affine transformation, for which it is possible to evaluate the circulation, and thereby the lift, exactly. We shall not go further into these questions here.

Problems

26.1 Show that the induced drag is smaller if the single trailing vortex is divided into a number of smaller vortices coming off the wing.

