5 Buoyancy

Fishes, whales, submarines, balloons and airships all owe their ability to float to buoyancy, the lifting power of water and air. The understanding of the physics behind buoyancy goes back as far as antiquity and has probably sprung from the interest in ships and shipbuilding in classic Greece.

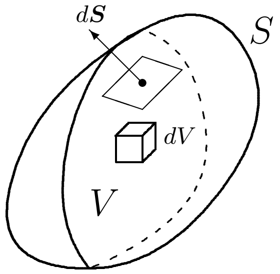
The basic principle is due to Archimedes. His famous Law states that the buoyancy force on a body is equal and oppositely directed to the weight of the fluid that the body replaces. Actually the Law was not just one law, but a set of four propositions dealing with different configurations of body and liquid [6]. Before his time one had thought that the shape of a body determined whether it would sink or float.

Archimedes, Greek scientist (287–212 BC). Often cited as the father of fluid mechanics.

Stability of floating bodies is not only of importance to shipbuilding, but also to anyone who has ever tried to stand up in a small rowboat. Newtonian mechanics may not only be used to derive Archimedes' Principle for equilibrium of floating bodies, but also to characterize the deviations from equilibrium and the restoring forces. Although being in hydrostatic equilibrium, a body floating in or on water is not in complete mechanical balance in every orientation, because the center of mass of the body and the center of mass of the water, it has displaced, do not in general coincide. The latter is also called the center of buoyancy. For a body of the fluid itself, the center of mass always coincides with the center of buoyancy.

In general, the mismatch between the centers of mass and buoyancy for a floating body creates a moment of force, or torque, which tends to rotate the body towards a stable equilibrium. For submerged bodies the stable equilibrium will always be with the center of gravity directly below the center of buoyancy. For bodies floating stably on the surface, a ship, the center of gravity is normally found directly above the center of buoyancy.

5.1 Global hydrostatic equilibrium



A control volume V with its enclosing surface S, a volume element dV and a surface element $d\boldsymbol{S}$.

An arbitrary macroscopic volume of fluid, often called a *control volume*, is in general subject to two kinds of forces: volume (or body) forces described by a force density field $\boldsymbol{f}$ (usually gravity $\boldsymbol{f} = \rho \boldsymbol{g}$), and contact or surface forces that for a fluid at rest are described entirely by the pressure field p. The total force on the control volume V with surface S is consequently the sum of two contributions

$$\boldsymbol{\mathcal{F}} = \int_V \boldsymbol{f}\, dV - \oint_S p\, d\boldsymbol{S} \ . \tag{5-1}$$

The first term is for the case of gravity just the weight of the fluid in the volume and the second is the *buoyancy* force. The circle in the symbol for the surface integral is in fact redundant, but is there only to remind us that the surface is closed.

In hydrostatic equilibrium, the total force must vanish for any volume of fluid,

$$\boxed{\int_V \boldsymbol{f}\, dV - \oint_S p\, d\boldsymbol{S} = \boldsymbol{0}} \ . \tag{5-2}$$

In other words, buoyancy must exactly cancel the weight. It was this rule we used intuitively on page 55 to calculate the pressure in the incompressible sea.

If the cancellation is not exact, as for example when a volume of water is heated or cooled relative to its surroundings, the fluid *must* start to move, either upwards if it weighs less than the ambient fluid or downwards if it weighs more.

Gauss' theorem

In the preceding chapter we established the local equation of hydrostatic equilibrium (4-8). Using it to eliminate the body force in the global equation (5-2), we get

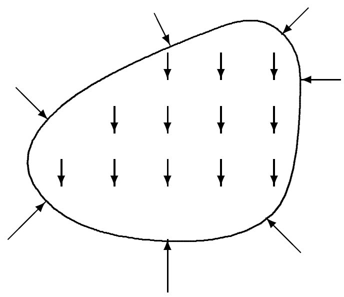
$$\boxed{\oint_S p\, d\boldsymbol{S} = \int_V \boldsymbol{\nabla} p\, dV} \ . \tag{5-3}$$

Normally we think of the body force as given and the equilibrium pressure as obtained by solving $\boldsymbol{\nabla} p = \boldsymbol{f}$. But we may equally well view the pressure as given and use the equilibrium equation to calculate the necessary force $\boldsymbol{f} = \boldsymbol{\nabla} p$. The above equation must for this reason be a purely mathematical relationship between a closed surface integral over an arbitrary function $p(\boldsymbol{x})$ and the integral of its gradient $\boldsymbol{\nabla} p(\boldsymbol{x})$ over the enclosed volume. The equation is one version of *Gauss' theorem* which we shall meet in several different guises in the course of this book. The theorem is a generalization of the familiar integration rule $f(b) - f(a) = \int_a^b f'(x)\, dx$, and a formal proof may also be derived along these lines (see problem 5.1).

5.2 Archimedes' principle

Mechanical equilibrium takes a slightly different form when a body of another material is immersed in the fluid. If its material is incompressible, the body retains its shape and displaces an amount of fluid with exactly the same volume. If the body is compressible, as a rubber ball, the volume of displaced fluid will be smaller. The body may even absorb fluid, like the piece of bread you dunk into your coffee, but then the physics becomes more complicated, and we shall disregard this possibility in the following. A body which is only partly immersed in the fluid may formally be viewed as a body that is fully immersed in fluid with properties, such as mass density and equation of state, that vary from place to place. This also covers the case where part of the body is in vacuum which may be thought of as a fluid with the extreme properties, $\rho = p = 0$.

Weight and buoyancy



Gravity attacks a body all over its volume, while pressure only attacks at the surface.

Let the actual, perhaps compressed, volume of the immersed body be V with surface S. In the field of gravity the body is subject to two forces: its weight

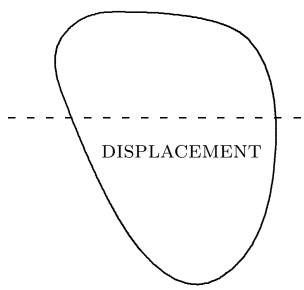
$$\boldsymbol{\mathcal{F}}_G = \int_V \rho_{\text{body}}\, \boldsymbol{g}\, dV \ , \tag{5-4}$$

and the buoyancy due to pressure acting at its surface,

$$\boldsymbol{\mathcal{F}}_B = -\oint_S p\, d\boldsymbol{S} \ . \tag{5-5}$$

In general these two forces do not have to be in balance. If the resultant $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B$ is directed upwards (against the direction of gravity) the body will rise to the surface, otherwise if directed downwards the body will sink to the bottom. In mechanical equilibrium the two forces must exactly cancel each other so that the body can remain in place.

Assuming that the body itself does not contribute to the field of gravity, the local balance of forces in the fluid (4-8) will be the same as before. In particular the pressure in the fluid cannot depend on whether the volume V contains material that is different from the fluid itself. The pressure on the surface of the immersed body must for this reason be identical to the pressure on a body of fluid of the same shape. But then the global equilibrium condition (5-2) tells us that the buoyancy force exactly balances the weight of the displaced fluid, so that



For a partially submerged body the displacement is the amount of water that has been displaced by the volume of the body below the waterline.

$$\boldsymbol{\mathcal{F}}_B = -\oint_S p\, d\boldsymbol{S} = -\int_V \rho_{\text{fluid}}\, \boldsymbol{g}\, dV \ . \tag{5-6}$$

This is indeed Archimedes' principle: *the force of buoyancy equals (minus) the weight of the displaced fluid.*

The total force on the body may then be written

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B = \int_V (\rho_{\text{body}} - \rho_{\text{fluid}})\boldsymbol{g}\, dV \ , \tag{5-7}$$

explicitly confirming that when the body is made from the same fluid as its surroundings, so that $\rho_{\text{body}} = \rho_{\text{fluid}}$, the resultant force vanishes automatically. In general, however, the distribution of mass in the body and in the displaced fluid will be different.

Example 5.2.1: How much can a helium-filled balloon lift? Assuming that the helium and the surrounding air are at the same temperature and pressure and using the ideal gas law, it follows that the ratio of densities of helium to air is equal to the ratio of the molecular weights. The lift (in kilograms per unit of volume at sea level) then becomes $\rho_{\text{atm}} - \rho_{\text{He}} = (1 - M_{\text{He}}/M_{\text{atm}})\rho_{\text{atm}} \approx 0.86 \times 1.2\ \mathsf{kg/m^3} \approx 1\ \mathsf{kg/m^3}$. A spherical balloon with a diameter of 10 m can thus lift more than 500 kg, including of course its own mass.

Notice that Archimedes' principle is valid even if the gravitational field varies appreciably across the body. Archimedes principle fails, if the body is so large that its own gravitational field cannot be neglected, such as would be the case if an Earth-sized body fell into Jupiter's atmosphere. The extra compression of the fluid and the associated change in buoyancy caused by the body's own gravitational field is calculated section 7.1.

Constant field of gravity

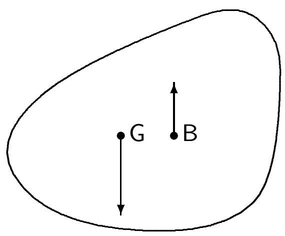
A major simplification happens in a constant gravitational field, $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$. Then the weight of the body becomes

$$\boldsymbol{\mathcal{F}}_G = M_{\text{body}}\, \boldsymbol{g}_0 \ , \tag{5-8}$$

and the buoyancy force becomes

$$\boldsymbol{\mathcal{F}}_B = -M_{\text{fluid}}\, \boldsymbol{g}_0 \ . \tag{5-9}$$

Since the total force is the sum of these contributions, one might say that buoyancy acts as if the displacement were filled with fluid of negative mass $-M_{\text{fluid}}$. Alternatively, one may view the buoyancy force as a kind of antigravity.



Center of gravity (G) and of buoyancy (B) for a body not in mechanical equilibrium (this is indicated by the different size of the force vectors). In calculating the moments, the forces may be assumed to attack in these centers.

Centers of gravity and buoyancy

The center of mass of the body is in this chapter called the *center of gravity* and denoted $\boldsymbol{x}_G$, in order to distinguish it from the center of mass of the displaced fluid which is called the *center of buoyancy* and denoted $\boldsymbol{x}_B$. We have previously seen that knowing the center of mass, the moment of gravity (3-12) is calculated as

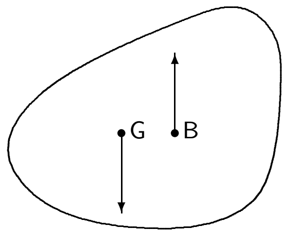
$$\boldsymbol{\mathcal{M}}_G = \boldsymbol{x}_G \times M_{\text{body}}\boldsymbol{g}_0 \ . \tag{5-10}$$

Similarly, the moment of buoyancy becomes

$$\boldsymbol{\mathcal{M}}_B = -\boldsymbol{x}_B \times M_{\text{fluid}}\boldsymbol{g}_0 \ , \tag{5-11}$$

and the total moment of force acting on the body is the sum of these two contributions.

In calculating the moment of buoyancy, we have tacitly used that the total moment automatically vanishes for any control volume of fluid in hydrostatic equilibrium (see problem 5.10 for a proof). This result may be formulated as a corollary to Archimedes' theorem: *the moment of buoyancy is equal to (minus) the moment of gravity of the displaced fluid.*



Body with gravity in equilibrium with buoyancy. In this case, the moment of force sticks out of the paper. The moment will for a submerged body tend to rotate it in the anticlockwise direction and thus bring the center of gravity below the center of buoyancy.

5.3 Hydrostatic stability

In mechanical equilibrium the mass of the displaced fluid must be exactly equal to the mass of the body in order for the total force to vanish,

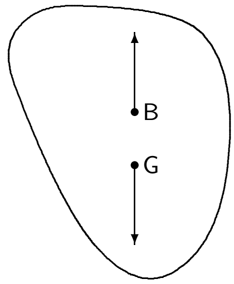
$$\boxed{M_{\text{fluid}} = M_{\text{body}}} \; . \tag{5-12}$$

In the following we assume this to be fulfilled and denote the common mass M. The total moment of force (couple or torque) then becomes independent of the origin of the coordinate system, and is given by

$$\boldsymbol{\mathcal{M}} = (\boldsymbol{x}_G - \boldsymbol{x}_B) \times M\boldsymbol{g}_0 \; . \tag{5-13}$$

The equilibrium orientations are as mentioned determined by the requirement that the total moment vanishes. Except for the trivial case where the centers coincide, the above equation tells us that the moment can only vanish if the centers of gravity and buoyancy lie on a vertical line, so that $\boldsymbol{x}_G - \boldsymbol{x}_B \sim \boldsymbol{g}_0$.

Submerged body



A fully submerged body in stable equilibrium must have the center of gravity directly below the center of buoyancy.

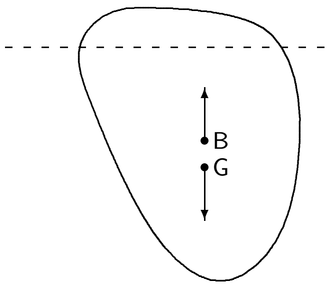
For a fully submerged body, like a fish or a submarine, both centers are always in the same place relative to the body. If the body is rotated so that the center of gravity is displaced horizontally with respect to the center of buoyancy, the direction of the moment will always tend to turn the body so that the center of gravity is lowered with respect to the center of buoyancy.

The situation is therefore completely stable when the center of gravity lies vertically below the center of buoyancy. A small perturbation away from this orientation will soon be corrected and the body brought back to the equilibrium orientation. The direction of the moment shows that the other equilibrium orientation with the center of gravity above the center of buoyancy is unstable.

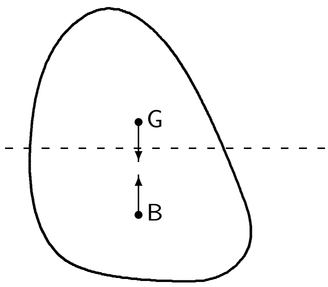
This is why the gondola hangs below an airship or balloon, and why fish go belly-up when they die, because they lose control of the swim bladder which enlarges into the belly and reverses the positions of the center of gravity and buoyancy.

Floating body

At the surface of a liquid, a body such as a ship or an iceberg will according to Archimedes' principle always arrange itself so that the mass of displaced liquid



A partially submerged body may have a stable equilibrium with the center of gravity directly below the center of buoyancy.



A partially submerged body may have a stable equilibrium with the center of gravity directly above *the center of buoyancy.*

exactly equals the mass of the body. Here we assume that there is vacuum or a very light fluid such as air above the liquid. The center of gravity is as always in the same place relative to the body, but the center of buoyancy depends now on the orientation of the body, because the volume of displaced fluid changes place and shape (while keeping the mass constant) when the body orientation changes.

Stability can again only occur when the two centers lie on the same vertical line, but there may be more than one point of stability. A sphere made of homogeneous wood floating on water, is stable in all orientations . None of them are in fact truly stable, because it takes no force to move from one to the other. This is however a marginal case.

A floating body may like a submerged body also possess a stable orientation with the center of gravity directly below the center of buoyancy. A heavy keel may, for example, used to lower the center of gravity of a sailing ship so much that this orientation becomes the only stable equilibrium. In that case it becomes virtually impossible to capsize the ship, even if the wind is very strong.

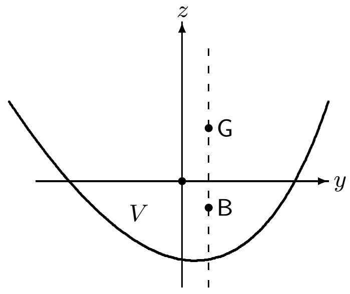
The stable orientation for most floating objects, such as ships, will in general have the center of buoyancy situated *below* the center of gravity. This happens always when an object of constant mass density floats on top of a liquid of constant mass density, for example an iceberg on water. The part of the iceberg that lies below the waterline must have its the center of buoyancy in the same place as its the center of gravity. The part of the iceberg lying above the water cannot influence the center of buoyancy whereas it always will always shift the real center of gravity upwards.

How can that situation ever be stable? Will the restoring couple not be of the wrong sign? Why don't ducks and tall ships capsize spontaneously? The qualitative answer is that when the body is rotated away from an equilibrium orientation, the volume of displaced water will change position in such a way that the center of buoyancy is shifted back to the other side of the center of gravity and the direction of the restoring moment reversed.

5.4 Ship stability analysis

Sitting comfortably in a small rowboat, it is fairly obvious that the center of gravity lies above the center of buoyancy, and that the situation is stable with respect to small movements of the body. But many a fisherman has learnt that suddenly standing up compromises the stability, sending him out among the fishes. There are, as we shall see, strict conditions on the distance between the centers of buoyancy and gravity.

Most ships are mirror symmetric in a plane, but we shall be more general and consider a "ship" of an arbitrary shape. We shall assume that the ship is in an equilibrium orientation and calculate the moment that arises when it is brought slightly out of equilibrium. To lowest order of approximation, the stability turns out to be essentially a two-dimensional problem, depending mainly on the outline of the ship's hull in the waterline and the displaced volume.



The ship in an equilibrium orientation, stable or unstable, shown in a vertical plane containing the aligned centers of gravity and buoyancy.

Equilibrium configuration

Water is assumed to have constant density ρ_0, so that the mass of the displaced water (and the ship!) is $M = \rho_0 V$ where V is the volume of the ship's displacement . The center of buoyancy is simply the average of $\boldsymbol{x}$ over V,

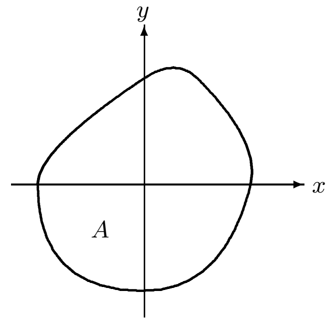
$$\boldsymbol{x}_B = \frac{1}{V}\int_V \boldsymbol{x}\, dV \ , \tag{5-14}$$

whereas the position of the center of gravity depends on the actual mass distribution in the ship.

Let A be the area that the ship's hull traces out in the waterline. We choose a flat-earth coordinate system with origin at the center of this area, defined by the vanishing of the integrals,

$$\int_A x\, dA = \int_A y\, dA = 0 \ . \tag{5-15}$$

This choice is not really necessary but just convenient for the following calculation.



The area A of the ship in the waterline may be of quite arbitrary shape. The z-axis is vertical, sticking out of the paper.

Tilting the ship

The ship is now heeled over or tilted slightly by a small positive angle α around the x-axis, so that the previous waterline area A becomes the plane $z = \alpha y$. To first order in α, this tilt causes the center of gravity to be shifted to the left by the amount

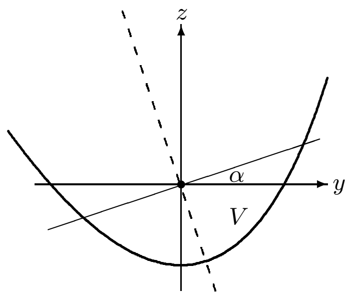
$$\delta y_G = -\alpha z_G \ . \tag{5-16}$$

There will also be a small shift in the z-direction of second order in α, which we shall disregard.

The center of buoyancy is shifted in the same way by the tilt, but because the displacement changes, there will also be a contribution Δy_B from the change in displacement, so that we may write

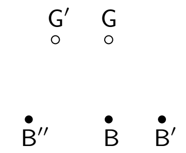
$$\delta y_B = -\alpha z_B + \Delta y_B \tag{5-17}$$

for the full shift. The change in displacement consists of two roughly wedge-shaped regions between the old and the new waterline. The tilt causes displaced water to be "moved" from the wedge with $y > 0$ to the wedge with $y < 0$. We expect that this will shift the center of buoyancy to the left, *i.e.* $\Delta y_B < 0$. For the ship to be stable, the shift must be so large that the new center of buoyancy comes to lie to the left of the center of gravity.

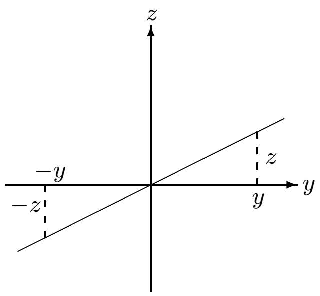


Ship tilted an angle α around the x-axis. The displaced water is moved from the wedge-shaped region on the right to the region on the left (for $\alpha > 0$). They must be equal in volume to first order in α.

In order for the tilted ship remain in buoyant equilibrium, the total displacement must be the same after the tilt, so that the volumes of these two regions must be equal. This is in fact the case with our choice of origin of coordinates



The tilt moves the center of gravity from G to G′, and the center of buoyancy from B to B′. In addition, the shift in displaced water moves the center of buoyancy back to B″. In stable equilibrium this point must lie to the left of the new center of gravity.



in the center of the ship. With $z = \alpha y$ we find the difference between the two volumes

$$\int_{A(y>0)} z\, dA - \int_{A(y<0)} (-z)\, dA = \int_A z\, dA = \alpha \int_A y\, dA = 0\ .$$

Here we have also disregarded the small corrections of order α^2 stemming from the changes of shape of the hull in the waterline that occur because of the tilt.

The backshift of the y-coordinate of the center of buoyancy due to the change in displacement may be calculated in the same way. Since it is the y-coordinate of the center-of-mass of the displaced water, we must subtract the contribution from the right-hand wedge and add in the contribution from the left-hand wedge. The change in the center-of-mass integral then becomes

$$V \Delta y_B = -\int_{A(y>0)} yz\, dA + \int_{A(y<0)} y(-z)\, dA = -\alpha \int_A y^2\, dA\ .$$

The integral

$$\boxed{I = \int_A y^2\, dA} \tag{5-18}$$

is the second moment of the area along the the tilt axis, and we may finally write

$$\Delta y_B = -\alpha \frac{I}{V}\ . \tag{5-19}$$

In agreement with expectation, the shift Δy_B is negative.

Stability condition

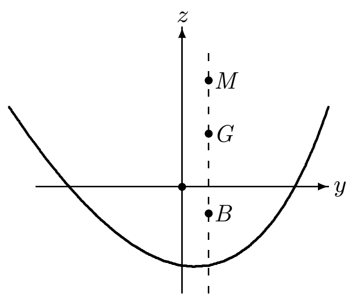
In order for the ship to be stable, the new center of buoyancy must as mentioned fall to the left of the center of gravity, so that a negative moment arises along the x-axis, tending to counteract the tilt. In other words we must require $\delta y_B < \delta y_G$, or

$$\boxed{z_G < z_B + \frac{I}{V}}\ . \tag{5-20}$$

This is the condition which limits the vertical distance between the centers of gravity and buoyancy.

Example 5.4.1: It is this condition is that you violate when you stand up in a small rowboat. Assuming that the boat is roughly 1 m by 2 m, the moment of the area becomes $I \approx 2\ \mathrm{m} \times \frac{2}{3}(1/2\ \mathrm{m})^3 \approx 1/6\ \mathrm{m}^4$. If you weigh 75 kg and the boat weighs 50 kg, the displacement will be $V \approx 0.125\ \mathrm{m}^3$, leading to $z_G - z_B < 1.3$ m. A person getting up from his seat may indeed raise the center of gravity so much that the boat becomes unstable.

The orientation of the coordinate system with respect to the ship's hull was not specified in the analysis and is therefore valid for a tilt in any direction. For a ship to be fully stable, the stability condition must be fulfilled for all possible tilt directions, so the second moment of the area on the right hand side of (5-20) should in fact be replaced by the smallest possible one. Often it is quite obvious which moment is the smallest. Many modern ships are extremely long with the same cross section for most of their length and a mirror symmetry through a vertical plane. For such ships the smallest moment is clearly obtained with the roll axis parallel to the longitudinal axis of the ship.



Stable ship. The metacenter lies above the center of gravity.

The metacenter

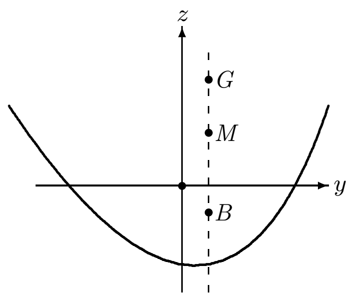
The quantity on the right hand side of (5-20)

$$z_M = z_B + \frac{I}{V} \tag{5-21}$$

defines the z-coordinate of a point situated vertically above the center of buoyancy in equilibrium, called the *metacenter*. Geometrically, the center of gravity must lie below the metacenter, $z_G < z_M$, for the ship to be stable. A good captain always knows the position of his ship's metacenter.

The restoring moment becomes

$$\mathcal{M}_x = (y_G - y_B)M(-g_0) = -\alpha\rho_0 V g_0(z_M - z_G) \ , \tag{5-22}$$

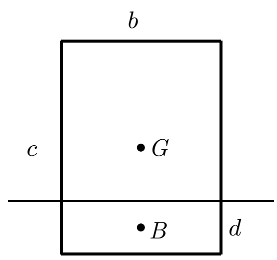


Unstable ship. The metacenter lies below the center of gravity.

and is proportional to the signed distance between the metacenter and the center of gravity. When the ship is stable, the moment is evidently negative and will tend to diminish a positive tilt angle α. The closer the center of gravity comes to the metacenter, the smaller will the restoring moment be, and the longer will the period of rolling oscillations. The actual roll period depends also on the (real) moment of inertia of the ship around the roll axis (see problem 5.6).

Floating board

The simplest example to which we may apply the stability criterion is that of a rectangular board or box of dimensions a, b and c in the three coordinate directions. The board is assumed to be made from a uniform material with constant density ρ_1 and floats in water of constant density ρ_0. The draft (depth below the waterline) is d, and in hydrostatic equilibrium we must have $dab\rho_0 = abc\rho_1$ or $\rho_1/\rho_0 = d/c$.



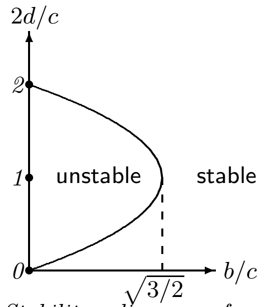
Board floating on water. The board has length a into the paper.

The distance between the center of gravity and the center of buoyancy is $(c-d)/2$, and the second moment of the waterline area becomes (for tilts around the x-axis)

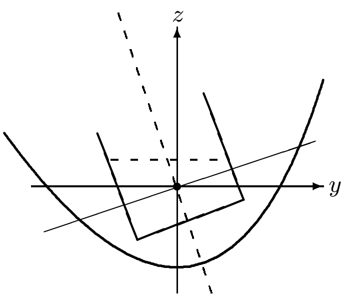
$$I = \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, y^2 = \frac{1}{12}ab^3 \ . \tag{5-23}$$

Since $V = abd$, the stability condition (5-20) for tilts around the y-axis then takes the form

$$c < d + \frac{b^2}{6d} \ . \tag{5-24}$$



Stability diagram for the floating board.



Tilted ship with an open container filled with liquid.

Rearranging this may be written

$$\left(\frac{2d}{c} - 1\right)^2 > 1 - \frac{2}{3}\left(\frac{b}{c}\right)^2 . \tag{5-25}$$

When $b/c > \sqrt{3/2} = 1.2247\ldots$, the right hand side becomes negative and the inequality is always fulfilled. On the other hand, when $b/c < \sqrt{3/2}$, *i.e.* if the width of the board is less than 122% of the heigth, there is a range of draft values around $d = c/2$ (corresponding to density values around $\rho_1 = \rho_0/2$), for which the board is unstable and will keel over and come to rest in another orientation (see problem 5.7).

This stability analysis is valid for other configurations of mass inside the board, as long as the total mass and the center of gravity are unchanged, as is for example the case for a box-shaped "ship" with a hull constructed from plates of uniform thickness.

Ship with liquid cargo

Many ships carry liquid cargos, oil, water, etc. When the tanks are not completely filled this kind of cargo may strongly influence the stability of the ship. In heavy weather or due to accidents, car ferries may inadvertently also get a layer of water on the car deck.

The main effect of an open liquid surface inside the ship is that the center of mass is shifted in the same direction by the redistribution of real liquid as the shift in the center of buoyancy due to the change in displaced water, *i.e.* towards negative y-values. This disturbs the stability and creates a competition between the liquid carried by the ship and the water displaced by the ship.

For the case of a single open tank, let as before the displaced volume (including the weight of the liquid) be V and the second moment I. A similar calculation then shows that the change in center of gravity due to the movement of a wedge of liquid of density ρ_1 becomes

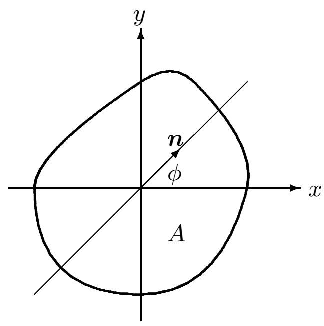
$$\Delta y_G = -\alpha \frac{\rho_1 I_1}{M} = -\alpha \frac{\rho_1}{\rho_0} \frac{I_1}{V} \tag{5-26}$$

where I_1 is the second moment of the open liquid surface. The metacentric height now becomes

$$\boxed{z_M = z_B + \frac{I}{V} - \frac{\rho_1}{\rho_0}\frac{I_1}{V}} \tag{5-27}$$

The effect of the moving liquid is to lower the metacentric height with possible destabilization as result (see problem 5.9). The unavoidable sloshing of the liquid may further compromise the stability.

The destabilizating effect of a liquid cargo may be counteracted by having a number of smaller containers or compartments, instead of a single container with an open surface. In car ferries this option is not open because it would hamper efficient loading of the cars, and such ships remain susceptible to water on the car deck.



Tilt axis $\boldsymbol{n}$ forming an angle ϕ with the x-axis.

∗ Principal roll axis

It has already been remarked that the metacenter for absolute stability is determined by the smallest second moment of the waterline area. If instead of tilting the ship around the x-axis, it is tilted around an axis $\boldsymbol{n} = (\cos\phi, \sin\phi, 0)$ forming an angle ϕ with the x-axis. Since this configuration is obtained by a simple rotation through ϕ around the z-axis, the transverse coordinate to be used in calculating the second moment becomes $y' = y\cos\phi - x\sin\phi$, and we find

$$I = \int_A (y')^2\, ds = I_{xx}\cos^2\phi + I_{yy}\sin^2\phi + 2I_{xy}\sin\phi\cos\phi = \boldsymbol{n}\cdot\mathbf{I}\cdot\boldsymbol{n} \qquad (5\text{-}28)$$

where I_{xx}, I_{yy} and I_{xy} are the elements of the matrix

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{pmatrix} = \int_A \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} ds \qquad (5\text{-}29)$$

The extrema of the positive definite quadratic form $\boldsymbol{n}\cdot\mathbf{I}\cdot\boldsymbol{n}$ are found from the eigenvalue equation $\mathbf{I}\cdot\boldsymbol{n} = \lambda\boldsymbol{n}$ (see problem 5.5). The eigenvector corresponding to the smallest eigenvalue is called the *principal roll axis* of the ship and its eigenvalue determines the metacenter for absolute stability.

Problems

5.1 Derive Gauss' theorem (5-3) for a convex volume using the partial integration rule $\int_a^b f'(x)dx = f(b) - f(a)$.

5.2 Show that when the total force on a body vanishes, the moment of force becomes independent of the origin of the coordinate system.

5.3 A stone weighs 1000 N in air and 600 N when submerged in water. Calculate the volume and average density of the stone.

$*$ **5.4** A hydrometer with mass $M = 4\ g$ consists of a spherical glass container filled with air and a long thin cylindrical stem of radius $a = 2$ mm. The sphere is weighed down so that the apparatus will float stably with the stem pointing vertically upwards and crossing the fluid surface at at some point. How much deeper (h) will it float in alcohol (mass density $\rho_1 = 0.78$ $\mathsf{g/cm^3}$) than in oil (mass density $\rho_0 = 0.82$ $\mathsf{g/cm^3}$)?

5.5 Show that the extrema of a 2×2 quadratic form $\boldsymbol{n} \cdot \mathbf{I} \cdot \boldsymbol{n}$, where $\boldsymbol{n} = (\cos\phi, \sin\phi)$ is a unit vector, are determined by the eigenvectors of $\mathbf{I}$ satisfying $\mathbf{I} \cdot \boldsymbol{n} = \lambda \boldsymbol{n}$.

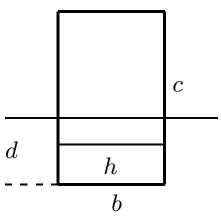
$*$ **5.6** Show that in a stable orientation the angular frequency of small oscillations around around a major axis of a ship is

$$\omega = \sqrt{\frac{Mg_0}{J}(z_M - z_G)}$$

where J is the moment of inertia of the ship around this axis.

5.7 Find the stable orientations of a cubic wooden block with density equal to half that of water.

5.8 A cylindrical wooden stick (density $\rho_1 = 0.65$ $\mathsf{g/cm^3}$) floats in water (density $\rho_0 = 1$ $\mathsf{g/cm^3}$). The stick is loaded down with a lead weight (density $\rho_2 = 11$ $\mathsf{g/cm^3}$) at one end such that it floats in vertical position with a fraction $f = 1/10$ of its length out of the water. a) What is the ratio (M_1/M_2) between the masses of the wooden stick and the lead weight? b) How large a fraction can stick out of the water (disregarding questions of stability)?



A 'car ferry' with water on the deck

5.9 A car ferry is extremely sensitive to water on the car deck. Consider a ferry in the shape of a long box-shaped hull of mass M and essentially no thickness. The length of the ferry is a, the width b, and the height c. The density of water is ρ_0. a) Calculate the draft d_0 when there is no water in the hull, b) Calculate the draft (d) when there is water of depth h inside the hull. c) Determine the center of mass height in equilibrium. d) Evaluate the stability condition.

$*$ **5.10** Prove without assuming constant gravity that the moment of buoyancy equals (minus) the moment of gravity of the displaced fluid (corollary to Archimedes' law).