# 10 Linear elasticity

When you bend a stick the reaction grows noticeably stronger the further you go — until it perhaps breaks with a snap. If you release the bending force before it breaks, the stick straightens out again and you can bend it again and again without it changing its reaction or its shape. That is elasticity.

In elementary mechanics the elasticity of a spring is expressed by Hooke's law which says that the force necessary to stretch or compress a spring is proportional to how much it is stretched or compressed. In continuous elastic materials Hooke's law implies that stress is proportional to strain. Some materials that we usually think of as highly elastic, for example rubber, do not obey Hooke's law except for very small deformation. When stresses grow large, most materials deform more than predicted by Hooke's law. The proper treatment of non-linear elasticity goes far beyond the simple linear elasticity which we shall discuss in this book.

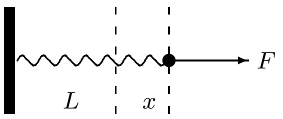
*Robert Hooke (1635–1703), English physicist. Worked on elasticity, built telescopes, and the discovered diffraction of light. The famous law which bears his name is from 1660. He stated already in 1678 the inverse square law for gravity, over which he got involved in a bitter controversy with Newton.*

The elastic properties of continuous materials are determined by the underlying molecular structure, but the relation between material properties and the molecular structure and arrangement in solids is complicated, to say the least. Luckily, there are broad classes of materials that may be described by a few material constants which can be determined by macroscopic experiments. The number of such constants depends on the complexity of the crystalline structure of the material, but we shall almost exclusively concentrate on the properties of structureless, isotropic elastic materials, described by just two material constants, Young's modulus and Poisson's ratio.

*Thomas Young (1773–1829). English physician, physicist, and egyptologist. He observed the interference of light and was the first to propose that light waves are transverse vibrations, explaining thereby the origin of polarization. He contributed much to the translation of the Rosetta stone.*

In this chapter, the emphasis will be on matters of principle. We shall derive the basic equations of linear elasticity, but only solve them in the simplest possible cases. In the next chapter we shall solve these equations in a number of generic situations of more practical interest.

## 10.1 Hooke's law



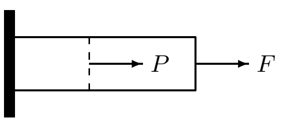

*A spring anchored at the left and pulled towards the right by a force $F$ will be stretched by the amount $x = F/k$.*

Ideal massless elastic springs obeying Hooke's law are a mainstay of elementary mechanics. If a spring of length $L$ is anchored in one end and pulled at the other with a force $F$, its length is increased to $L + x$. Hooke's law states that there is proportionality between force and change in length, $F = kx$, with a constant of proportionality, $k$, called the *spring constant*.

Real springs are physical bodies with mass, shape and internal molecular structure. Almost any solid body, anchored in one end and pulled in the other, will react like a spring, when the pull is not too strong. Basically, this reflects that interatomic forces are approximately elastic, when the atoms are only displaced slightly away from their positions (problem 10.1).

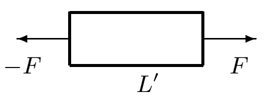
### Young's modulus

Many elastic bodies that we handle daily, for example rubber bands, piano wire, sticks, or water hoses, are long string-like objects with constant cross section, typically made from homogeneous and *isotropic* material without any particular internal structure. Their uniform composition and simple form make material strings convenient models for real springs.




*The same tension must act on any cross section in the string.*

The force, $F$, necessary to extend the length of a string by $x$ must be proportional to the area, $A$, of the string cross section, because if we bundle $N$ such such strings loosely together to make a thicker string of area $NA$, the total force will have to be $NF$ in order to get the same change of length. This shows that the relevant quantity to speak about is not the force $F$ itself, but rather the (average) normal stress, or tension, $P = NF/NA = F/A$, which is independent of the size of the cross section.




*The force acts in opposite directions at the terminal cross sections of a smaller slice of the string. The extension is proportionally smaller.*

Since the same force, $F$, acts on any cross section of the string, the tension, $P = F/A$, must also be the same everywhere along the string. For a smaller slice of the string of length $L' < L$, the uniformity implies that such a part will be stretched proportionally less, *i.e.* $x'/L' = x/L$. This shows that the relevant parameter is not the absolute change of length, $x$, but rather the relative longitudinal extension, or strain, $u_{xx} = x/L$, which must be independent of the length of the string.

Putting everything together, we conclude that the quantity

$$E = \frac{P}{u_{xx}} = \frac{F/A}{x/L} = k\frac{L}{A} \tag{10-1}$$

must be independent of both the length of the spring and the area of its cross section. It is a material constant, called the *modulus of extension* or *Young's modulus* (1807), and given this quantity we may calculate the actual spring constant,

$$k = E\frac{A}{L} , \tag{10-2}$$

for a string of length $L$ and cross section $A$ made from this material.

Young's modulus characterizes the behavior of the material of the spring, when stretched in one direction. The relation (10-1) also tells us that a unidirectional tension $\sigma_{xx} = P$ creates a relative extension,

$$u_{xx} = \frac{P}{E} \,, \tag{10-3}$$

in the material. Evidently, Hooke's law leads to a *linear* relation between stress, $P$, and strain, $u_{xx}$, and materials with this property are generally called linear.

Young's modulus is by way of its definition (10-1) measured in units of pressure, and typical values for metals are, like the bulk modulus, of the order of $10^{11}$ Pa $=$ $10^6$ bar. In the same way as the bulk modulus is a measure of the in-compressibility of a material, Young's modulus is a measure of the *in-stretchability*. The larger it is, the harder it is to stretch the material. In order to obtain a large strain $u_{xx} \approx 100$ %, one would have to apply stresses of magnitude $P \approx E$, as shown by (10-3). Such strains are of course not permitted in the theory of small deformations, but Young's modulus nevertheless sets the scale.

| *Material* | $E$ *GPa* | $\nu$ *%* |
|---|---|---|
| *Wolfram* | *411* | *28* |
| *Plain steel* | *205* | *29* |
| *Nickel* | *200* | *31* |
| *Cast iron* | *152* | *27* |
| *Copper* | *130* | *34* |
| *Titanium* | *116* | *32* |
| *Brass* | *100* | *35* |
| *Glass* | *75* | *17* |
| *Aluminium* | *70* | *35* |
| *Magnesium* | *45* | *29* |
| *Lead* | *16* | *44* |

*Young's modulus and Poisson's ratio for various materials (from [1])*

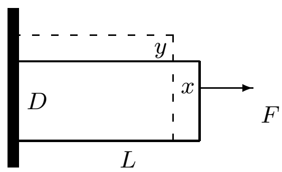
**Example 10.1.1:** At company outings, employees often play the game of pulling in teams at each end of a rope. Before the inevitable terminal instability sets in, there is often a prolonged period where the two teams pull with almost equal force $F$. If the teams each consist of 10 persons, all pulling with about their average weight of 70 kg, the total force becomes $F = 7{,}000$ N. For a rope diameter of 5 cm, the stress becomes quite considerable, $P \approx 3.6$ MPa. For a reasonable value of Young's modulus, say $E = 36$ MPa, the rope will stretch by $u_{xx} \approx 10$ %.

## Poisson's ratio

Normal materials will contract in directions transverse to the direction of extension. If the transverse size, $D$, of a string changes by $y$, the transverse strain becomes of the order of $u_{yy} = y/D$ and will in general be negative for positive tension $P$. In linear materials, the transverse strain is also proportional to $P$, so that the ratio $u_{yy}/u_{xx}$ will be independent of $P$. The negative of this ratio is called *Poisson's ratio* (1829),

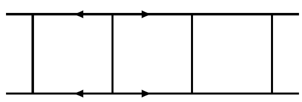
$$\nu = -\frac{u_{yy}}{u_{xx}} \,, \tag{10-4}$$

and is another constant characterizing isotropic material. It is dimensionless, and typical values lie around 0.30 in metals. We shall see below that it cannot exceed 0.5 in isotropic materials.



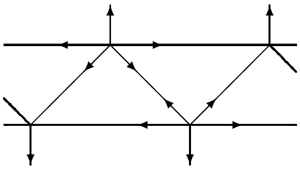

*A string normally contracts in transverse directions when pulled at the ends.*

Whereas longitudinal extension can be understood as a consequence of elastic atomic bonds being stretched, it is harder to understand why materials should contract transversally. The reason is, however, that in an isotropic material there are atomic bonds in all directions, and when bonds that are neither purely longitudinal nor purely transverse are stretched, they create a transverse tension which can only be relieved by contracting the material.



*Stretching a ladder with purely transverse rungs will not create transverse forces.*

**Example 10.1.2:** A ladder constructed from ideal springs with rungs orthogonal to the sides, will not experience a transverse contraction when stretched. If, on the



*Stretching a ladder with skew rungs creates transverse forces which must be balanced by external forces at the boundary (as here) or relieved by transverse contraction.*

other hand, some of the rungs are skew (making the ladder unusable), they will be stretched along with the ladder. But that will necessarily generate forces that tend to contract the ladder, *i.e.* a negative transverse tension, which either has to be balanced by external forces or relieved by contraction of the ladder.

## 10.2 Hooke's law in isotropic materials

Generalizing to a coordinate system with the $x$-axis along the string, Hooke's law for isotropic and homogeneous materials leads to proportionality between the tension in the $x$-direction

$$\sigma_{xx} = P \ , \tag{10-5}$$

and the strains it provokes in directions parallel and orthogonal to it,

$$u_{xx} = \frac{P}{E} \ , \qquad u_{yy} = u_{zz} = -\nu u_{xx} = -\nu \frac{P}{E} \ . \tag{10-6}$$

In an arbitrary coordinate system, the components of strain tensor will be completely mixed with each other, as illustrated by the simple rotation (2-90), and there will also arise shear stresses and strains. In order to express Hooke's law, such that it takes the same form in all coordinate systems, a linear relationship must be established between the complete stress tensor and the complete strain tensor.

In an isotropic material, there are no internal directions defined which can be used to construct such a relationship, and this means that the only tensors at our disposal are the strain tensor, $u_{ij}$, itself and the Kronecker delta, $\delta_{ij}$, multiplied with the trace $\sum_k u_{kk}$, which is the only scalar quantity that can be formed from a linear combination of strain tensor components. The most general linear tensor relation between stress and strain in an isotropic material therefore becomes (Cauchy, 1822; Lamé, 1852)

*Gabriel Lamé (1795–1870), French mathematician. Worked on curvilinear coordinates, number theory and mathematical physics.*

$$\boxed{\sigma_{ij} = 2\mu\, u_{ij} + \lambda\, \delta_{ij} \sum_k u_{kk}} \ . \tag{10-7}$$

Here $\lambda$ and $\mu$ are material constants, called *elastic moduli* or *Lamé coefficients*, and we shall see below that they are directly related to Young's modulus and Poisson's ratio. Explicitly, we find for the diagonal elements of the stress tensor

$$\sigma_{xx} = (2\mu + \lambda)u_{xx} + \lambda(u_{yy} + u_{zz}) \ , \tag{10-8a}$$
$$\sigma_{yy} = (2\mu + \lambda)u_{yy} + \lambda(u_{zz} + u_{xx}) \ , \tag{10-8b}$$
$$\sigma_{zz} = (2\mu + \lambda)u_{zz} + \lambda(u_{xx} + u_{yy}) \ , \tag{10-8c}$$

and for the off-diagonal elements

$$\sigma_{xy} = \sigma_{yx} = 2\mu\, u_{xy} \ , \quad \sigma_{yz} = \sigma_{zy} = 2\mu\, u_{yz} \ , \quad \sigma_{zx} = \sigma_{xz} = 2\mu\, u_{zx} \ . \tag{10-9}$$

The coefficient $\lambda$ has no special name, whereas $\mu$ is called the *shear modulus* or the *modulus of rigidity*, because it controls the magnitude of shear stresses. For $\mu = 0$ there are no shear stresses and all pressures become equal $p_x = p_y = p_z = p$, just as in a fluid. Since the strain tensor is dimensionless, the Lamé coefficients are, like the stress tensor itself, measured in units of pressure.

Since Hooke's law (10-7) and Cauchy's strain tensor (9-17) are both linear relationships, successive small deformations may simply be added together. Hooke's law (10-7) can therefore also be interpreted as a relation between an *extra* stress $\sigma_{ij}$ and strain $u_{ij}$, imposed on top of an already existing stress and strain in the material. The extra stress and strain could, for example, be caused by wind forces on a bridge or a skyscraper, which is already stressed and strained by the forces of gravity.

## Young's modulus and Poisson's ratio

The relations between Young's moduls, Poisson's ratio, and the Lamé coefficients are obtained by inserting stresses and strains for simple stretching, (10-5) and (10-6), into (10-8), to obtain

$$\begin{aligned} P &= (2\mu + \lambda)\frac{P}{E} - \lambda\nu\frac{P}{E} \ , \\ 0 &= -(2\mu + \lambda)\nu\frac{P}{E} + \lambda(-\nu + 1)\frac{P}{E} \ . \end{aligned}$$

Solving for $E$ and $\nu$, we obtain

$$\boxed{E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}} \ , \tag{10-10}$$

and[1]

$$\boxed{\nu = \frac{\lambda}{2(\lambda + \mu)}} \ . \tag{10-11}$$

Conversely, we may express the Lamé coefficients in terms of Young's modulus and Poisson's ratio,

$$\lambda = \frac{E\nu}{(1 - 2\nu)(1 + \nu)} \ , \tag{10-12a}$$

$$\mu = \frac{E}{2(1 + \nu)} \ , \tag{10-12b}$$

Typical values for the Lamé coefficients in metals are thus of the same magnitude as Young's modulus, *i.e.* of the order of $10^{11}$ Pa = $10^6$ bar.

[1] The use of the symbols $E$ for Young's modulus is conventional and should of course not be confused with the use of similar symbols for the energy. Poisson's ratio is also sometimes denoted $\sigma$, but that clashes too much with the symbol for the stress tensor. Later we shall in the context of fluid mechanics also use $\nu$ for the kinematic viscosity.

## Average pressure and bulk modulus

The trace of the stress tensor (10-7) becomes

$$\sum_i \sigma_{ii} = (2\mu + 3\lambda) \sum_i u_{ii} \ , \tag{10-13}$$

because the trace of the Kronecker delta is $\sum_i \delta_{ii} = 3$. This allows us to calculate the change in average pressure (8-12) due to deformation

$$\Delta p = -\frac{1}{3} \sum_i \sigma_{ii} = -\left(\lambda + \frac{2}{3}\mu\right) \sum_i u_{ii} \ . \tag{10-14}$$

The trace of the strain tensor has previously been shown (see (9-28)) to be proportional to the relative volume change, $\sum_i u_{ii} = \mathsf{div}\, \boldsymbol{u} = \Delta V/V$, and since the bulk modulus (4-25) is defined to be minus the ratio of pressure change to relative volume change, we must have

$$\boxed{K = \frac{-\Delta p}{\Delta V/V} = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}} \ . \tag{10-15}$$

In elastic materials, the bulk modulus is always a function of the material constants. The bulk modulus is equal to Young's modulus for $\nu = 1/3$, which is a typical value for $\nu$.

## Inverting Hooke's law

Hooke's law (10-7) may be inverted so that strain instead is expressed as a linear function of stress. Solving (10-7) for $u_{ij}$ and inserting $\sum_k u_{kk} = \sum_k \sigma_{kk}/(3\lambda+2\mu)$ from (10-13), we get

$$u_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij} \sum_k \sigma_{kk} \ . \tag{10-16}$$

Introducing Young's modulus (10-10) and Poisson's ratio (10-11), this takes the much simpler form

$$\boxed{u_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij} \sum_k \sigma_{kk}} \ . \tag{10-17}$$

Explicitly, we find for the diagonal components

$$u_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz}) \ , \tag{10-18a}$$
$$u_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}(\sigma_{zz} + \sigma_{xx}) \ , \tag{10-18b}$$
$$u_{zz} = \frac{1}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \ , \tag{10-18c}$$

and

$$u_{xy} = \frac{1+\nu}{E}\sigma_{xy} \ , \quad u_{yz} = \frac{1+\nu}{E}\sigma_{yz} \ , \quad u_{zx} = \frac{1+\nu}{E}\sigma_{zx} \ , \tag{10-19}$$

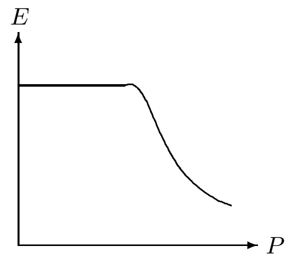
for the off-diagonal ones. Evidently, if the only stress is $\sigma_{xx} = P$, we obtain immediately from (10-16) the correct relations for simple stretching, (10-5) and (10-6).

## Positivity constraints

The bulk modulus $K = \lambda + 2\mu/3$ cannot be negative, because a material with negative $K$ would expand when put under pressure in a closed vessel containing normal material, say air or water. This would make the pressure increase, causing further expansion until the whole thing blew up. Likewise, materials with negative shear modulus, $\mu$, would behave strangely and mimosa-like pull away from a shearing force instead of yielding to it. Formally, it may be shown (see section 10.4) that the conditions $3\lambda + 2\mu \geq 0$ and $\mu \geq 0$ follow from requiring the elastic energy density to be bounded from below. Although $\lambda$ in principle may assume negative values, natural materials always have $\lambda \geq 0$.

Young's modulus cannot be negative because of these constraints, and this confirms our intuition that strings always stretch when pulled at the ends. If there were materials with the ability to contract when pulled, one could get magical-looking behavior out of them. At the moment you began climbing up a rope made from such material, it would pull you further up. Presumably, if such materials were ever created, they would spontaneously contract into nothingness at first possible occasion, or at least into a state with a normal relationship between stress and strain.

Poisson's ratio depends only on the ratio $\lambda/\mu$ and reaches its maximum $\nu = 1/2$ for $\lambda/\mu = \infty$ and its minimum $\nu = -1$ for $\lambda/\mu = -2/3$. The definition of Poisson's ratio reflects the fact that most bodies shrink in the transverse directions when stretched, but since $\lambda$ in principle could be negative, bodies might actually expand transversally without violating the laws of physics. In practice, all bodies do undergo a transverse contraction when stretched, so Poisson's ratio may always be taken to be positive. The extreme value, $\nu = 1/2$, corresponds to $\mu = 0$, implying that there are no shear stresses in the material which therefore behaves like a fluid at rest.




*Sketch of how Young's modulus might vary as a function of increased tension. Beyond the proportionality limit, its effective value becomes generally smaller.*

## Limits to Hooke's law

Hooke's law in isotropic materials, expressed by the linear relationships, (10-7) or (10-17), between stress and strain, is only valid for stresses up to a certain value, called the *proportionality limit*. Beyond the proportionality limit, nonlinearities set in, and the present formalism becomes invalid. Eventually, one reaches a point, called the *elasticity limit*, where the material ceases be elastic and undergoes permanent deformation, or even fracture, without much further increase of stress. Hooke's law is, however, a very good approximation for most metals under normal conditions where stresses are tiny compared to the elastic moduli.

### ∗ Anisotropic materials

Anisotropic (also called *aeolotropic*) materials having different properties in different directions are of great technical importance. In an anisotropic material, a linear relationship connecting the 6 independent components of the symmetric stress tensor with the 6 independent components of the symmetric strain tensor could in principle require $6 \times 6 = 36$ independent coefficients, but an energy argument (Green, 1837) requires that this $6 \times 6$ array of coefficients must be symmetric, and thus reduces the number of independent parameters to 21 (see problem 10.7). The orientation of the material relative to the coordinate system requires 3 parameters (angles), so altogether there may be up to 18 independent constants characterizing the elastic properties of a general anisotropic material, a number actually realized by triclinic crystals [9, 28]. We shall, however, in this book limit ourselves to the isotropic ones, for which Hooke's law takes the simple form (10-7).

## 10.3 Static uniform deformation

To see how Hooke's law works for continuous systems, we now turn to the extremely simple case of a *static uniform deformation*, for which the strain tensor, $u_{ij}$, takes the same value everywhere in a body at all times. Hooke's law (10-7) then ensures that the stress tensor is likewise constant everywhere in the body, so that all its derivatives vanish, $\nabla_k \sigma_{ij} = 0$. Comparing with the condition for mechanical equilibrium (8-21), it follows that uniform deformation excludes body forces such as gravity. Conversely, in the presence of gravity, there must always be non-uniform deformation of an isotropic material, a quite reasonable conclusion.

Furthermore, at the boundary of a uniformly deformed body, the stress vector is as always required to be continuous, and this puts strong restrictions on the form of the external forces that may act on the surface of the body. Uniform deformation is for this reason only possible under very special conditions.

### Uniform compression

Consider a fluid with a constant pressure $P$, so that the stress tensor is $\sigma_{ij} = -P\delta_{ij}$ everywhere in the fluid. If a solid body made from isotropic material is immersed into this fluid, the natural guess is that the pressure will also be $P$ inside the body. Inserting $\sigma_{ij} = -P\delta_{ij}$ into (10-17) and using that $\sum_k \sigma_{kk} = -3P$, we obtain

$$u_{ij} = -\frac{P}{3K}\delta_{ij} \ . \tag{10-20}$$

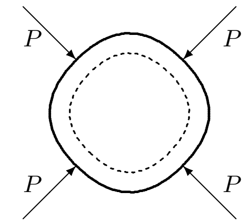
 

Since

$$u_{xx} = \nabla_x u_x = -\frac{P}{3K} \;, \tag{10-21}$$

we may immediately integrate this equation (and the similar ones for $u_{yy}$ and $u_{zz}$) and obtain a particular solution to the displacement field,

$$\boxed{\begin{aligned} u_x &= -\frac{P}{3K}\, x \;, \\ u_y &= -\frac{P}{3K}\, y \;, \\ u_z &= -\frac{P}{3K}\, z \;. \end{aligned}} \tag{10-22}$$



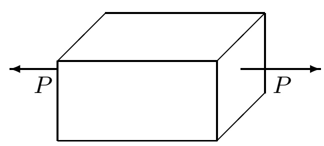
*A body made from isotropic, homogenous material subject to a uniform external pressure will be uniformly compressed.*

The most general solution is obtained by adding an arbitrary small rigid body displacement to this solution.

The result was obtained by making an educated *guess* on the form of the stress tensor inside the body. It could in principle be wrong, but is in fact correct due to a uniqueness theorem to be derived in section 10.4. The theorem guarantees in analogy with the uniqueness theorems of electrostatics, that provided the equations of mechanical equilibrium and the boundary conditions are fulfilled (which they are here), there is essentially only one solution to any *elastostatic* problem. The only liberty left is an arbitrary rigid body displacement which may always be added to the solution.

## Uniform stretching

Consider now as in the beginning of the chapter a string-like body which is stretched along its main axis, say the $x$-direction, by means of a tension $\sigma_{xx} = P$ acting uniformly over its cross section. If there are no other external forces acting on the body, the natural guess is that the only non-vanishing component of the stress tensor is $\sigma_{xx} = P$ everywhere in the body. Inserting that into (10-16), we obtain as before the strains (10-6).



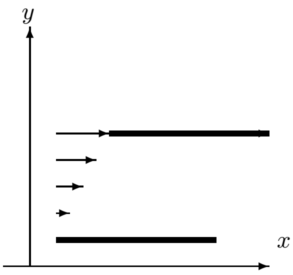
*Uniformly stretched body with a constant tension $P$.*

The corresponding displacement field is again obtained by integrating $\nabla_x u_x = u_{xx}$ etc, and we find the particular solution

$$\boxed{\begin{aligned} u_x &= \phantom{-\nu}\frac{P}{E}\, x \;, \\ u_y &= -\nu\frac{P}{E}\, y \;, \\ u_z &= -\nu\frac{P}{E}\, z \;. \end{aligned}} \tag{10-23}$$

The solution describes as expected a simple dilatation along the $x$-axis and a contraction along the other axes.



*Clamped slab of homogeneous material under shear stress. The displacement grows linearly with y.*

### Uniform shear

Finally we return to the example from section 8.2 of a clamped slab of homogeneous, isotropic material in the $xz$-plane, subjected to a shear force in the $x$-direction. As we argued, the shear stress $\sigma_{xy} = P$ must be constant everywhere in the material. Assuming that there are no other stresses, the only strain component becomes $u_{xy} = P/2\mu$, and using that $2u_{xy} = \nabla_x u_y + \nabla_y u_x$, we find a particular solution

$$u_x = \frac{P}{\mu}\, y \ , \qquad u_y = u_z = 0 \ . \tag{10-24}$$

As expected, the displacement in the $x$-direction vanishes for $y = 0$ and grows linearly with $y$. In problem 10.5 the displacement field is calculated without making the assumption of small strains.

## ∗ 10.4 Energy of deformation

The work performed by the external force in extending a spring further by the amount $dx$ is $dW = Fdx = kxdx$. Integrating this expression, we obtain the total work $W = \frac{1}{2}kx^2$, which is (of course) the well-known expression for the elastic energy, $E = W$, stored in a stretched spring. Calculated per unit of volume $V = AL$ for a material string, we find the density of elastic energy in the material

$$\frac{\mathcal{E}}{V} = \frac{kx^2}{2V} = \frac{1}{2}Eu_{xx}^2 = \frac{P^2}{2E} \ . \tag{10-25}$$

The transverse contraction controlled by Poisson's ratio, $\nu$, can play no role in building up the energy, because there are no forces acting on the sides of the string.

### Elastic energy

In the general case, strains and stresses vary over the body, and the calculation becomes more complicated. In section 9.4 we determined the work that must be performed in order to change the strain infinitesimally.

If the strain $u_{ij}$ describes the extra deformation of a body already pre-stressed by $\sigma^0_{ij}$, assumed to be independent of the strain, we may immediately integrate (9-31) to obtain the work performed against the already existing stress

$$W_0 = \int_V \sum_{ij} \sigma^0_{ij} u_{ij}\, dV \ . \tag{10-26}$$

Alternatively, one may view $\mathcal{E}_0 = W_0$ as the potential energy of the deformation in the given stress field. It is, for example, the energy you must spend against the already existing strong tension in a bow when you wish to shoot an arrow.

For the part of the stress tensor which depends linearly on the strain tensor one must build up the deformation in infinitesimal steps, as we did for the gravitational self-energy in section 6.5. The elastic self-energy, *i.e.* the energy of a deformation in its own stress field, becomes in this way quadratic in the small strain tensor, and we expect as in the gravitational case also a factor 1/2 here. For isotropic materials, the work performed in building up a deformation, or equivalently the total elastic energy, thus becomes

$$\boxed{E = \frac{1}{2}\int_V \sum_{ij} \sigma_{ij} u_{ij}\, dV} \,, \tag{10-27}$$

where $\sigma_{ij}$ is given by Hooke's law (10-7). For anisotropic materials the existence of such an energy function will impose a further condition on the coefficients in the generalized Hooke's law (see problem 10.7).

## Energy density

The quantity

$$\epsilon = \frac{1}{2}\sum_{ij} \sigma_{ij} u_{ij} = \mu \sum_{ij} u_{ij}^2 + \frac{1}{2}\lambda \left(\sum_i u_{ii}\right)^2 \tag{10-28}$$

must be interpreted as the elastic energy density in a deformed isotropic material. It is instructive to write out the sums explicitly to get

$$\epsilon = \mu(u_{xx}^2 + u_{yy}^2 + u_{zz}^2 + 2u_{xy}^2 + 2u_{yz}^2 + 2u_{zx}^2) + \frac{1}{2}\lambda(u_{xx} + u_{yy} + u_{zz})^2 \,.$$

Inserting the strains for uniform stretching (10-6), all the dependence on $\nu$ cancels out, and we obtain again the energy density (10-25).

## Positivity of the energy density

The energy density must be bounded from below, for if it were not, elastic materials would be unstable, and an unlimited amount of work could be obtained by increasing the state of deformation. Imagine for a moment how magically a body made from such material would behave when squeezed.

From the boundedness, it follows immediately that the shear modulus must be non-negative, $\mu \geq 0$, because otherwise we might let one of the off-diagonal components of the strain tensor, say $u_{xy}$, grow without limit while the energy density became more and more negative. The condition on $\lambda$ is more subtle because the diagonal components of the strain tensor are involved in both terms. For a uniform deformation with $u_{ij} = \alpha\delta_{ij}$ we get the energy density $\rho_{\text{elastic}} = \frac{3}{2}(3\lambda + 2\mu)\alpha^2$, and consequently we must demand that $3\lambda + 2\mu \geq 0$, implying that the bulk modulus (10-15) is positive, $K > 0$ (see problem 10.6 for the proof that there are no stronger conditions).

### Total energy in the gravitational field

In an external gravitational potential, $\Phi(\boldsymbol{x})$, the change in gravitational energy due to the displacement of a material particle of fixed mass $dM$ is

$$dM\,\Phi(\boldsymbol{x}+\boldsymbol{u}(\boldsymbol{x})) - dM\,\Phi(\boldsymbol{x}) \approx -\boldsymbol{g}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\,dM \ ,$$

where the last expression is obtained under the assumption that the displacement is small compared to the length scale for variations in the gravitational field. The total (potential) energy of a deformed body in a gravitational field is therefore the sum of the internal energy of deformation and the extra gravitational energy due to the deformation

$$\boxed{E = \frac{1}{2}\int_V \sum_{ij} u_{ij}\sigma_{ij}\,dV + \int_V \rho(-\boldsymbol{g}\cdot\boldsymbol{u})dV} \ , \tag{10-29}$$

where $\rho(\boldsymbol{x})$ is the mass density of the undeformed body.

Consider now the change in total energy due to a small variation $\delta u_i(\boldsymbol{x})$ in the displacement. Using that $\sigma_{ij}$ is linear in $u_{ij}$ and symmetric, we find for the variation in the first term

$$\begin{aligned}\int_V \sum_{ij} \delta u_{ij}\sigma_{ij}\,dV &= \int_V \sum_{ij} (\nabla_j \delta u_i)\sigma_{ij}\,dV \\ &= \oint_S \sum_{ij} \delta u_i \sigma_{ij}\,dS_j - \int_V \sum_{ij} \delta u_i \nabla_j \sigma_{ij}\,dV \ ,\end{aligned}$$

where we have used Gauss' theorem in the last step. The variation in total energy thus becomes

$$\boxed{\delta E = \oint_S \sum_{ij} \delta u_i \sigma_{ij}\,dS_j - \int_V \sum_{ij} \delta u_i(\nabla_j\sigma_{ij} + \rho g_i)\,dV} \ . \tag{10-30}$$

The first term is the change in energy due to the forces acting on the surface of the body and the second the change in energy due to volume forces. The surface integral vanishes if the boundary is either fixed or free, *i.e.* $\delta u_i = 0$ or $\sum_j \sigma_{ij} n_j = 0$, whereas the volume integral vanishes in mechanical equilibrium.

This result shows that the total energy is stationary, $\delta E = 0$, under variations in displacement around mechanical equilibrium. Since the elastic energy density is a positive definite quadratic form in the strain tensor, mechanical equilibrium must correspond to an absolute minimum in the total energy.

A non-singular quadratic form can only have one minimum, and this result guarantees that there is only one solution to Navier's equation of equilibrium. We shall now explicitly prove that the minimum is indeed unique.

### Uniqueness of elastostatic solutions

We shall now prove uniqueness of the solutions to the mechanical equilibrium equations (8-21) with elastic stresses given by Hooke's law (10-7) and strains given Cauchy's expression (9-17). Let us assume that there are actually two displacement fields $u_i^{(1)}$ and $u_i^{(2)}$ which both satisfy these equations with the same external volume forces $f_i$ and the same boundary conditions. Then we have

$$\begin{aligned}
&\sum_j \nabla_j \sigma_{ij}^{(1)} + f_i = 0 && \sum_j \nabla_j \sigma_{ij}^{(2)} + f_i = 0 \\
&\sigma_{ij}^{(1)} = 2\mu u_{ij}^{(1)} + \lambda \delta_{ij} \sum_k u_{kk}^{(1)} && \sigma_{ij}^{(2)} = 2\mu u_{ij}^{(2)} + \lambda \delta_{ij} \sum_k u_{kk}^{(2)} \\
&u_{ij}^{(1)} = \frac{1}{2}(\nabla_i u_j^{(1)} + \nabla_j u_i^{(1)}) && u_{ij}^{(2)} = \frac{1}{2}(\nabla_i u_j^{(2)} + \nabla_j u_i^{(2)})
\end{aligned}$$

For the difference field $u_i = u_i^{(1)} - u_i^{(2)}$, the corresponding stress tensor $\sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$ must satisfy $\sum_j \nabla_j \sigma_{ij} = 0$, and we obtain by means of Gauss' theorem (5-3)

$$0 = \int_V u_{ij} \sum_j \nabla_j \sigma_{ij} \, dV = \oint_S \sum_{ij} u_i \sigma_{ij} \, dS_j - \int_V \sum_{ij} \nabla_j u_i \sigma_{ij} \, dV \ .$$

Here the surface integral vanishes because of the boundary conditions, which either specify the same displacements for the two solutions at the surface, *i.e.* $u_i = 0$, or the same stress vectors, *i.e.* $\sum_j \sigma_{ij} n_j = 0$. Using the symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$, we have

$$\sum_{ij} \nabla_j u_i \sigma_{ij} = \sum_{ij} \nabla_i u_j \sigma_{ji} = \frac{1}{2} \sum_{ij} (\nabla_i u_j + \nabla_j u_i) \sigma_{ij} = \sum_{ij} u_{ij} \sigma_{ij} \ ,$$

and we get,

$$\int_V \sum_{ij} u_{ij} \sigma_{ij} \, dV = 0 \ . \qquad (10\text{-}31)$$

The integrand is of the same form as the energy density (10-28), which has been shown to be strictly positive definite for $\mu > 0$ and $3\lambda + 2\mu > 0$, and consequently, the integral can only vanish if the strain tensor for the difference field vanishes everywhere in the body, *i.e.* $u_{ij} = 0$.

Given the boundary conditions, there is essentially only one solution to the equations of mechanical equilibrium in linear elastic materials. Although the two displacement fields may in principle be different, they must correspond to identical deformations everywhere in the body, and can thus at most differ by a rigid body displacements. If we can guess a solution satisfying the equations of mechanical equilibrium and the boundary conditions, it must be the right one.

## Problems

**10.1** Two particles interact with a smooth distance dependent force $f(r)$. Show that the force obeys Hooke's law in the neighborhood of any fixed separation $r = a$.

**10.2** A one-dimensional open (in contrast to circular) elastic chain hangs vertically in a gravitational field and will be stretched by the weight of the particles (the springs are assumed to be weightless). a) Calculate the equilibrium lengths of all the springs, and the equilibrium length of the chain. b) Find the equations of motion for displacements of the particles away from the equilibrium in this situation.

**10.3** Using this to eliminate $\lambda$, we may write (10-7) in the form

$$\sigma_{ij} = 2\mu\left(u_{ij} - \frac{1}{3}\delta_{ij}\sum_k u_{kk}\right) + K\delta_{ij}\sum_k u_{kk} \; . \tag{10-32}$$

The first term gives no contribution to the average pressure, but only to the deviatoric (*i.e.* traceless) stress tensor (8-15).

**10.4** Show that if the sides of an elastic cylinder are kept fixed, while the cylinder is uniformly stretched in the $z$-direction, the only non-vanishing strain is

$$u_{zz} = \frac{1}{\lambda + 2\mu}P = \frac{(1+\nu)(1-2\nu)}{1-\nu}\frac{P}{E} \; . \tag{10-33}$$

whereas the stress in the $xy$-plane becomes

$$\sigma_{xx} = \sigma_{yy} = \lambda u_{zz} = \frac{\lambda}{\lambda + 2\mu}P = \frac{\nu}{1-\nu}P \; . \tag{10-34}$$

$*$ **10.5** Consider a shear deformation of a slab of elastic material in the $xz$-plane by a force in the $x$-direction. Assume that the sides of the slab are kept free to move, so that the only non-vanishing components of the strain tensor are $u_{xy} = u_{yx} = \alpha$. Show that the displacement becomes

$$u_x = \alpha y \; , \tag{10-35}$$

$$u_y = -\left(1 - \sqrt{1-\alpha^2}\right) y \; . \tag{10-36}$$

for a deformation which is not assumed to be small. Describe what happens for $\alpha \to 1$.

$*$ **10.6** Show that one may write the energy density (10-28) in the following form

$$E = \frac{1}{2}[\lambda - 2\mu(3\alpha^2 - 2\alpha)]u_{ii}u_{jj} + \mu\,(u_{ij} - \alpha u_{kk}\delta_{ij})(u_{ij} - \alpha u_{ll}\delta_{ij}) \tag{10-37}$$

where $\alpha$ is arbitrary. Use this to argue that $3\lambda + 2\mu > 0$ and that this is the strictest condition on $\lambda$.

∗ **10.7** The most general linear relation between stress and strain is of the form

$$\sigma_{ij} = \sum_{kl} \lambda_{ijkl} u_{kl} \tag{10-38}$$

where $\lambda_{ijkl}$ is called the *elasticity tensor*.

a) Show that the elasticity tensor is symmetric in the two first and two last indices.

b) Show that for the elastic energy (9-31) to be integrable, the tensor must obey the further symmetry relation

$$\lambda_{ijkl} = \lambda_{klij} \tag{10-39}$$

c) Show that these symmetry conditions leave only 21 free parameters in the tensor.

d) Show that 3 of these are angles that fix the orientation of the material.