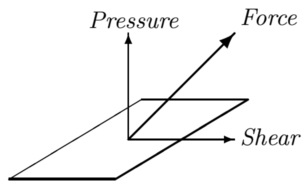
4

Fluids at rest

If the Sun did not shine, if no heat were generated inside the Earth and no energy radiated into space, all the winds in the air and the currents in the sea would die away, and the air and water on the planet would in the end come to rest in equilibrium with gravity. What distribution of pressure and density would we then find in the sea and the atmosphere?

Consider a fluid, be it air or water, under the influence of gravity. In the absence of external driving forces or time-dependent boundary conditions, and in the presence of dissipative contact forces, such a system must eventually reach a state of *hydrostatic equilibrium*, where nothing moves anymore anywhere and all fields become constant in time. This must be first approximation to the sea, the atmosphere, the interior of a planet or a star.

In hydrostatic equilibrium and more generally in mechanical equilibrium there is everywhere a balance between contact forces, such as pressure, having zero range, and body forces such as gravity with infinite range. Contact interactions between material bodies or parts of the same body take place across *contact surfaces.* A contact force acting on a tiny piece of a surface may take any direction. Its component orthogonal to the surface is called a *pressure force* whereas the tangential component is called a *shear force.* Solids and fluids in motion can sustain shear forces, whereas fluids at rest cannot. The wind will not move your parked car because of (shear) friction between the wheels and the road, but it will certainly sail your boat away if not properly moored.



The force on a small piece of a surface can always be resolved in a normal pressure force and a tangential shear force.

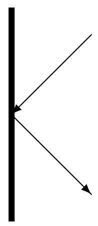
In this chapter we shall establish the basic concepts and formalism for hydrostatic equilibrium and apply it to the sea and the atmosphere. In the following three chapters we shall continue the study of hydrostatic equilibrium for fishes, icebergs and ships, the interior of planets and stars, and the shapes of gravitating fluid bodies.

4.1 Pressure

A fluid at rest can as mentioned only sustain pressure forces. If shear forces arise, the fluid will tend to flow towards a new equilibrium without shear. This expresses the most essential property of fluids and is essentially a definition of what constitutes a macroscopic fluid.

Microscopic origin of pressure

At the microscopic level, fluid pressure on a container wall arises from the incessant molecular bombardment of the wall. Consider for example a gas in which the molecules move with some average velocity v. A gas molecule impinging the wall tends to be thrown back again and thus gives up at most twice its momentum component normal to the wall. If the gas density is ρ, we may estimate the mass of the gas that hits a small area A of the wall in a small time interval dt to be of the order of $dM \approx \rho\, A\, v dt$. In this estimate we have disregarded that the molecules in fact move in random directions with random velocities. Since half the molecules move towards the wall whereas the other half moves away from it, we are certainly missing another factor 1/2. Multiplying with the velocity v and dividing by dt, we get a very rough estimate of the amount of momentum transferred to the wall area A per unit of time, namely $\mathcal{F} \approx \rho\, A v^2$. This force is evidently proportional to the area (because the volume of the gas interacting with the area is proportional to it) so that the relevant quantity must be the force per unit of area,*i.e.* $p \approx \rho v^2$, also called the *pressure*.



A molecule elastically reflected by a wall transfers twice its normal momentum component to the wall.

The unit of pressure is Pascal ($\mathsf{Pa} = \mathsf{N/m^2} = \mathsf{kg/m/s^2}$). The air pressure at sea level is a little more than 10^5 Pa = 1 bar, and the standard pressure at sea level is defined to be 1 atm = 1013.250 millibar. Modern meteorologists tend to quote air pressure in hectopascals (hPa) rather than in millibars.

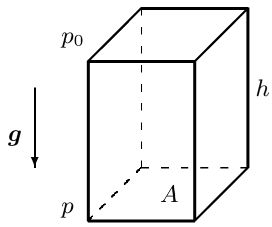
Taking the density of air at sea level to be $\rho \approx 1$ kg/m^3 and the molecular velocity to be of the order of the velocity of sound $v \approx 330$ m/s, our estimate yields $p \approx 10^5$ Pa ≈ 1 atm which is very close to the pressure at sea level. In view of the roughness of the estimate this result must be considered entirely fortuitous.

Pressure field

Blaise Pascal (1623–1662). *French mathematician and physicist. Founded probability theory. Constructed what may be viewed as the first digital calculator. He spent his later years with religious thinking in the Cistercian abbey of Port-Royal. More than one property of pressure goes under the name of Pascal's law.*

So far we have only defined the pressure acting on the walls of a container. Is it meaningful to speak about pressure in the middle of a fluid far from containing walls? We could of course put in a tiny manometer to measure the pressure, but then we would just obtain the pressure acting on the surface of the manometer, which is another "wall", and not the true internal pressure in the fluid.

In spite of these conceptual difficulties we shall imagine that there is a well-defined pressure field $p(\boldsymbol{x})$ everywhere in the fluid and that it acts along the normal to any surface, real or imaginary, in the fluid (Pascal's law). From a molecular point of view, the estimate of the momentum transferred from the material on one side of the surface to the other takes the same form as before, because in this approximation there is no difference between hitting another molecule (or car!) head-on and hitting a fixed wall.



A column of sea water. The pressure difference between bottom and top must carry the weight of the water in the box.

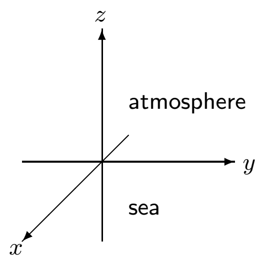
Incompressible sea

The mass of the water in all the seas is $M_{\rm sea} \approx 1.7 \times 10^{21}$ kg which is 1/3500 of the Earth itself. If no continents rose out of the seas, they would cover the planet to a uniform depth of about 3 km.

Water is a nearly incompressible fluid with almost constant density $\rho_0 \approx 10^3$ kg/m^3. Consider a vertical box in the sea with cross-sectional area A and height h. In mechanical equilibrium the difference between the pressure p at the bottom and the pressure p_0 at the top of this box must balance the total weight of the water in the box, or

$$\rho_0 A h g_0 = (p - p_0) A \ . \tag{4-1}$$

If this equation were not fulfilled, the total force on the column of water would not vanish, and it would have to move.



The flat-earth coordinate system.

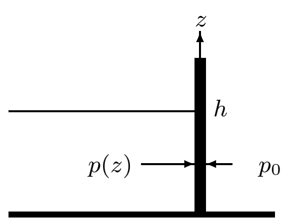
In a flat-earth coordinate system where $h = -z$, we consequently have

$$p = p_0 + \rho_0 g_0 h = p_0 - \rho_0 g_0 z \ . \tag{4-2}$$

Thus the pressure rises linearly with depth as we dive into the sea.

> The scale of the pressure increase per unit of depth in the sea becomes $\rho_0 g_0 \approx$ 100 hPa/m, corresponding to about 1 atm/10 m. At the deepest point in the sea, $z \approx -11$ km the pressure is more than 1000 atm. The assumption of constant gravity is well justified even to this depth (it changes by less than 0.5 %), whereas the density of water changes by about 5 % (see section 4.3).

The linear rise of the pressure with depth for an incompressible fluid in constant gravity may immediately be used to calculate the total pressure force on any vertical container wall. Curved and non-vertical walls and non-constant gravity requires a stronger formalism.



Water stemmed up behind a sluice gate. The pressure varies linearly with height z over the bottom.

Example 4.1.1:

Water is stemmed up behind a sluice gate of width L to height h. On the water surface and on the outer side of the gate there is atmospheric pressure p_0. On the inside of the gate the pressure is $p(z) = p_0 + \rho_0 g_0 (h - z)$, so that the total force on the gate becomes

$$\mathcal{F} = \int_0^h (p(z) - p_0)\, L dz = \frac{1}{2} \rho_0 g_0 L h^2 \ . \tag{4-3}$$

Because the pressure rises linearly with depth, this result could have been calculated without an explicit integral. The total force is simply the product of the area of the sluice gate Lh with the average pressure excess $\frac{1}{2}\rho_0 g_0 h$ acting on the gate.

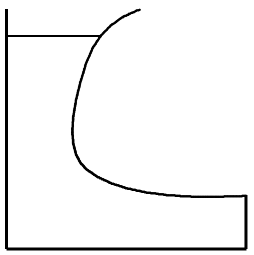
Incompressible atmosphere?

The mass of the atmosphere is about $M_{\rm atm} \approx 5 \times 10^{18}$ kg, which is smaller than a millionth of the mass of the Earth. Since air is highly compressible, it makes little sense to use the above expression (4-2) for the pressure at constant density (except for very small values of z). If we anyway do so, we find a pressure which drops linearly with height and reaches zero at

$$z = h_0 = \frac{p_0}{\rho_0 g_0} \ . \tag{4-4}$$

Using $p_0 = 1$ atm and air density $\rho_0 = 1.2$ kg/m^3 we get $h_0 = 8.6$ km, which is a bit lower than the height of Mount Everest. This is of course meaningless, since climbers have reached the summit of that mountain without oxygen masks. But as we shall see, this height is nevertheless the correct scale for major changes in the atmospheric properties (see fig. 4.3).

Paradox of hydrostatics

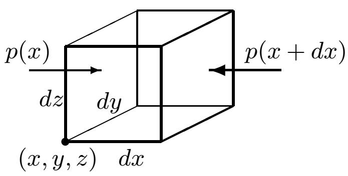


Paradox: the pressure along the vertical wall of the "boot" rises linearly because it has to carry the weight of the water above, but what about the pressure in the tip of the "toe"?

The linear rise of the pressure with depth may, as we have seen, be used to calculate the total pressure force on any vertical container wall. For a curved container wall, like that of an open vase, there seems to be no problem, except handling the necessary mathematics. But if the water column does not reach all the way to the surface, as when you fill a boot with water, what is then the pressure at the flat horizontal bottom? Will it be constant along the bottom as in the open sea, or will it vary? And if it is constant, what is it "up against", since there is only a little column of water above?

The short answer to this paradox is that the pressure is indeed constant along the horizontal bottom. If the pressure for example is lower in the "toe" than in the "heel", there will be unbalanced horizontal pressure forces directed towards the toe acting on a horizontal box of water. But that is not allowed in complete mechanical equilibrium. The only possible conclusion is that the upper wall of the toe must supply the necessary pressure force to compensate for the missing weight of the vertical water column.

Effective force on material particle



Pressure difference over a small rectangular box.

A material particle is like any other body subject to pressure from all sides, but being infinitesimal it is possible to derive a general expression for the resultant force. Let us choose a material particle in the shape of a small rectangular box with sides dx, dy, and dz, and thus a volume $dV = dxdydz$. Since the pressure is slightly different on opposite sides of the box the resultant pressure force is to leading approximation (in the x-direction)

$$d\mathcal{F}_x \approx (p(x,y,z) - p(x+dx,y,z))dydz \approx -\frac{\partial p}{\partial x} dxdydz \ .$$

Including the other coordinate directions we obtain

$$d\boldsymbol{\mathcal{F}} = -\boldsymbol{\nabla} p \, dV \ . \tag{4-5}$$

The resultant of all pressure forces acting on a tiny material particle is apparently equivalent to a volume force with a density equal to the negative gradient of the pressure. If there is also a true volume force, $\boldsymbol{f}$, for example gravity ($\boldsymbol{f} = \rho \boldsymbol{g}$), acting on the material, the total force on a material particle may be written

$$\boxed{d\boldsymbol{\mathcal{F}} = \boldsymbol{f}^* \, dV} \ , \tag{4-6}$$

where

$$\boxed{\boldsymbol{f}^* = \boldsymbol{f} - \boldsymbol{\nabla} p} \tag{4-7}$$

is the *effective force density*. We shall see in the next section that this result does not depend on the shape of the material particle.

Local equation of hydrostatic equilibrium

In complete mechanical (hydrostatic) equilibrium, the total force on an arbitrary body has to vanish. Applying this to all the material particles in the body, it follows that the effective density of force must vanish everywhere,

$$\boxed{\boldsymbol{f} - \boldsymbol{\nabla} p = \boldsymbol{0}} \ . \tag{4-8}$$

This is the *local equation of hydrostatic equilibrium*. It is a differential equation valid everywhere in a fluid at rest and encapsulates in a convenient way all the physics of hydrostatics.

The flat earth case

With constant gravity in a flat-earth coordinate system we have $\boldsymbol{f} = \rho \boldsymbol{g}_0 = \rho(0, 0, -g_0)$ and the local equilibrium equation takes the form

$$\frac{\partial p}{\partial x} = 0 \ , \tag{4-9a}$$

$$\frac{\partial p}{\partial y} = 0 \ , \tag{4-9b}$$

$$\frac{\partial p}{\partial z} = -\rho g_0 \ . \tag{4-9c}$$

The first two equations tell us that the pressure does not depend on x and y, but only on z. This confirms our previous conclusion and resolves the hydrostatic paradox. We now know that independently of the shape of the container, the pressure will always be the same at a given depth below an open surface. For the special case of constant density, $\rho(z) = \rho_0$, the last equation is immediately integrated to yield the previous result (4-2).

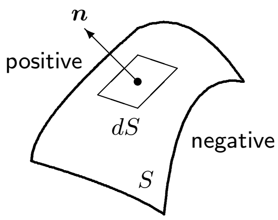
∗ Gradient fields

The local equation of hydrostatic equilibrium (4-8) equals the gradient of the pressure to the true density of force. If the density of force is given, one may calculate the pressure everywhere in the fluid in the same way as the gravitational potential may be calculated from the gravitational field (see eq. (3-37)).

It seems conversely as if the local equation (4-8) demands that the true body force density must always be a gradient field. But what happens if a force field, say $\boldsymbol{f} = \boldsymbol{a} \times \boldsymbol{x}$, is manifestly incapable of being expressed as a gradient? In that case the only possible conclusion is that hydrostatic equilibrium, with pressure being the only contact force, cannot be established, and the fluid must start to move and generate shear forces. This is however the exception, and we shall see in section 7.1 that under very general conditions the gravitational force density $\boldsymbol{f} = \rho \boldsymbol{g}$ is always a gradient field, even when ρ is not a constant.

4.2 Formal definition of pressure

Life is apparently not always as simple as in the incompressible sea, where the pressure is constant at constant depth and where we may disregard horizontal forces. Both gravity and pressure may vary from place to place and the bodies of fluid that we want to study may not be as regular as a vertical column of incompressible water. We need to establish a mathematical formalism that is capable of handling all the different situations we may meet in practice.



All normals to an oriented open surface have a consistent orientation with common positive and negative sides.

Oriented surfaces

Let S be a surface dividing a body into two parts. This surface needs not be a real surface where material properties change dramatically but may just be an imaginary surface separating two parts of the same body from each other. A tiny surface element is characterized by its area dS and the direction of its normal $\boldsymbol{n}$. It is convenient to combine these in the vector surface element

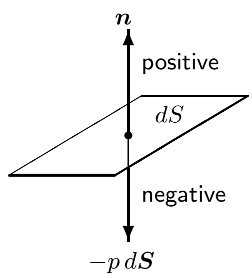
$$d\boldsymbol{S} = (dS_x, dS_y, dS_z) = \boldsymbol{n}\, dS \; . \tag{4-10}$$

There is nothing intrinsic in a surface which defines the orientation of the normal, *i.e.* whether the normal is really $\boldsymbol{n}$ and not $-\boldsymbol{n}$. A choice must, however, be made, and having done that, one may call the side of the surface element into which the normal points, positive (and the other of course negative). Usually neighboring surface elements are required to be oriented consistently, *i.e.* with the same positive sides. For a closed surface the normal is conventionally chosen to be directed out of the enclosed volume, so that the enclosed volume always lies at the negative side of its surface.

Definition of pressure

Although macroscopically of zero range, the contact forces due to microscopic molecular interactions of electromagnetic origin have a finite range at the molecular scale. Across a tiny but still macroscopic piece of surface, the number of neighboring molecules participating in the interaction as well as the force they exert may for this reason be expected to be proportional to the area of the surface. In a fluid at rest the only contact force is the pressure force acting along the normal to the surface. The force exerted by the material at the positive side of a surface element on the material at the negative side must therefore be of the form

$$\boxed{d\boldsymbol{\mathcal{F}} = -p\, d\boldsymbol{S}}\ , \tag{4-11}$$



The pressure force on a vector surface element.

where the coefficient of proportionality p is called the *pressure*. Convention dictates that a positive pressure exerts a force directed *towards* the material on the negative side, and that explains the minus sign.

Total pressure force

The total pressure force acting on a surface S is obtained by summing up all the little vector contributions from each surface element,

$$\boldsymbol{\mathcal{F}} = -\int_S p\, d\boldsymbol{S}\ . \tag{4-12}$$

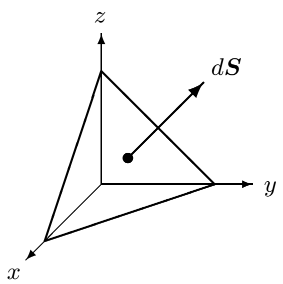
This is the force which acts on the cork in the champagne bottle, moves the pistons in the cylinders of your car engine, breaks a dam, and sends off a bullet from a canon. It is also this force that buoys fishes, ships, and balloons and thereby cancels their weight so that they are able to float (see chapter 5).

Same pressure in all directions?

Newton's third law guarantees that the material on the negative side of a surface element reacts with an equal and opposite force on the material on the positive side (provided there is no surface tension). Since $-d\boldsymbol{\mathcal{F}} = -p(-d\boldsymbol{S})$, the pressure takes the same numerical value, independently of the orientation of the surface element, *i.e.* which side of the surface element we take to be positive.

This is part of a much stronger result, called Pascal's law, which we shall prove below: *the pressure in a fluid at rest is independent of the direction of the surface element on which it acts.* It implies that pressure cannot depend on the normal $\boldsymbol{n}$, but only on the location $\boldsymbol{x}$ of a surface element, and is therefore a (scalar) field $p(\boldsymbol{x})$.

The simple reason for pressure being the same in all directions in hydrostatic equilibrium is that the pressure acts on the surface of a body whereas a body force by definition acts on the volume. If we let the body shrink, the contribution from the body force will vanish faster than the contribution from the surface force because the volume vanishes faster than the surface area. In the limit of vanishing body size only the surface force is left, but it must then itself vanish because in mechanical (and thus hydrostatic) equilibrium the total force on all parts of a body must be zero.



A body in the shape of a tetrahedron. Any body shape can be built up from sufficiently many and sufficiently small tetrahedrons.

∗ Proof of Pascal's law

Assume first that the pressure is actually different in different directions. We shall then show that for physical reasons this assumption cannot be maintained. Consider a tiny body in the shape of a tetrahedron with three sides parallel to the coordinate planes. The total pressure force acting on the body is

$$d\boldsymbol{\mathcal{F}} = -p d\boldsymbol{S} + p_x d\boldsymbol{S}_x + p_y d\boldsymbol{S}_y + p_z d\boldsymbol{S}_z \ , \tag{4-13}$$

where we have denoted by p, p_x, p_y, and p_z the pressures acting on the different faces of the tetrahedron. It is sufficient to consider infinitesimal bodies of this kind, because an arbitrary body shape can be put together from these.

The different signs in (4-13) are due to the chosen orientation of the vector surface elements. Whereas the surface element of the 'front' face $d\boldsymbol{S} = (dS_x, dS_y, dS_z)$ points out of the volume, each of the tetrahedron faces lying in the three coordinate planes have been chosen to have their vectors pointing along the axes of the coordinate system. Each of these little triangles is in fact the projection of the front face onto that plane, and consequently by elementary geometry the areas of the three projected triangles are dS_x, dS_y and dS_z, so that their vector surface elements become $d\boldsymbol{S}_x = (dS_x, 0, 0)$, $d\boldsymbol{S}_y = (0, dS_y, 0)$, and $d\boldsymbol{S}_z = (0, 0, dS_z)$.

Inserting this in the above equation we find the total force

$$d\boldsymbol{\mathcal{F}} = ((p_x - p)dS_x, (p_y - p)dS_y, (p_z - p)dS_z) \ . \tag{4-14}$$

In mechanical (hydrostatic) equilibrium, which is all that we are concerned with here, the contact forces must balance body forces (gravity),

$$d\boldsymbol{\mathcal{F}} + \boldsymbol{f}\, dV = \boldsymbol{0} \ , \tag{4-15}$$

where $\boldsymbol{f}$ is the density of body forces. The idea is to show that for sufficiently small tetrahedrons the body forces can be neglected and the surface forces $d\boldsymbol{\mathcal{F}}$ must consequently vanish.

Consider now a geometrically congruent tetrahedron with all lengths scaled by a factor λ. Since the volume scales as the third power of λ whereas the surface areas only scale as the second power, the hydrostatic equation for the scaled tetrahedron becomes $\lambda^2 d\boldsymbol{\mathcal{F}} + \lambda^3 \boldsymbol{f} dV = \boldsymbol{0}$ or $d\boldsymbol{\mathcal{F}} + \lambda \boldsymbol{f} dV = \boldsymbol{0}$. In the limit of $\lambda \to 0$ it follows that the total contact force must vanish, *i.e.* $d\boldsymbol{\mathcal{F}} = 0$, and using (4-14) we find

$$\boxed{p_x = p_y = p_z = p} \ , \tag{4-16}$$

also called *Pascal's law.* As promised, the pressure must indeed be the same in all directions. The fact that a confined liquid has the same pressure everywhere is also due to Pascal and is called Pascal's law or principle.

4.3 Equation of state

In *thermodynamic equilibrium* the thermodynamic parameters — pressure p, temperature T, and mass density ρ — are independent of time. This does not mean that these quantities have to take the same values everywhere in space, but only that the system if left to itself will display no macroscopic changes in these parameters over time. Under very general conditions the density can be calculated from pressure and temperature, *i.e.* there will be a relationship, called the *equation of state*, of the form (see for example [5]).

$$\boxed{\rho = f(p, T)} \ . \tag{4-17}$$

To be precise, this should be understood as a local relation between the fields,

$$\rho(\boldsymbol{x}) = f(p(\boldsymbol{x}), T(\boldsymbol{x})) \ , \tag{4-18}$$

valid everywhere in the system. The condition of constant density $\rho(\boldsymbol{x}) = \rho_0$ which we used to calculate the pressure in the sea (page 55) may be viewed as an extreme case of an equation of state.

The precise form of the equation of state depends on the details of the molecular forces. The derivation of the equation of state from atomic properties of matter is a subject that falls outside the scope of continuum physics.

The ideal gas law

The oldest and most famous equation of state is the ideal gas law, which is usually presented in the form

$$pV = nRT \ , \tag{4-19}$$

where V is a volume of gas, p its pressure, T its absolute temperature, n the number of moles in the volume, and the molar gas constant $R = 8.31451(6)$ J/K/mol.

In classical thermodynamics, the physical system is assumed to have the same temperature, pressure and density everywhere. In continuum physics, these quantities are permitted to vary across space, and the ideal gas law should be applied to volumes of gas that are small on the macroscopic scale but large on the molecular scale. In order to cast the ideal gas law into an equation of state of the form (4-17), we calculate the mass density of the gas in a small volume V around $\boldsymbol{x}$

$$\boxed{\rho = \frac{nM_{\text{mol}}}{V} = \frac{M_{\text{mol}}}{R}\frac{p}{T}} \ . \tag{4-20}$$

It grows proportionally with the pressure and falls inversely with the absolute temperature. In the beginning of this chapter we estimated the pressure in a simple gas to be directly proportional to the density, $p \approx \rho v^2$. The ideal gas law shows that the absolute temperature must be related to the square of the molecular velocities, or more correctly to the average molecular kinetic energy.

The ideal gas law is also valid for mixtures of pure gases, such as air, for which one must use the average molar weight $M_{\rm air} = 28.9635 \ \mathsf{g/cm^3}$. The equation of state for real air is only approximately given by the ideal gas law and needs non-linear corrections (in p/T) and other corrections due to humidity (see [1]).

Isothermal equilibrium

All materials conduct heat to some extent. A small volume of a fluid is always in thermal contact with the fluid surrounding it and will eventually in equilibrium obtain the same temperature as its environment. If the walls containing a fluid are held at a fixed temperature T_0, all of the fluid will eventually in hydrostatic equilibrium arrive at this temperature everywhere, $T(\boldsymbol{x}) = T_0$.

In isothermal hydrostatic equilibrium we may suppress the dependence on the (constant) temperature and assume that the equation of state takes the form

$$\rho = \rho(p) \ . \tag{4-21}$$

An equation of state of this form which only depends on the pressure is said to be *barotropic*. In section 4.5 we shall discuss another type of thermal equilibrium for ideal gases, called isentropic or adiabatic equilibrium, in which the temperature is *not* the same everywhere, but the dependence on temperature in the equation of state may in that case also be eliminated.

Isothermal atmosphere

Everybody knows that the atmosphere is not at constant temperature, but if we nevertheless assume it to be so we obtain from the equation of hydrostatic equilibrium (4-9) and the ideal gas law (4-20)

$$\frac{dp}{dz} = -\rho g_0 = -\frac{p}{h_0} \tag{4-22}$$

with

$$h_0 = \frac{RT_0}{g_0 M_{\rm mol}} = \frac{p_0}{\rho_0 g_0} \ . \tag{4-23}$$

In the last equation we have used the ideal gas law once more to arrive at exactly the same expression as eq. (4-4) for the scale height. The differential equation has the solution

$$p = p_0 e^{-z/h_0} \ . \tag{4-24}$$

In this model of the atmosphere the pressure decreases exponentially with height with a characteristic length scale again set by h_0. This time the pressure at $h = h_0$ (the top of Mount Everest) is finite and predicted to be 373 hPa, which is a little more than 1/3 of the pressure at sea level.

Convection

If different parts of the walls containing a fluid are kept at different temperatures there may arise a stationary heat flow in the fluid — transporting heat from higher to lower temperatures — in spite of the fluid itself being in hydrostatic (mechanical) equilibrium. Since materials expand when heated the hydrostatic equilibrium may not be exactly the same as if there were no heat flow, because the mass density will have decreased in warmer and increased in colder regions.

In the presence of gravity, the warmer parts of the fluid will experience a buoyancy relative to the colder regions. Eventually if the temperature gradients become sufficiently large this will give rise to *convective instability*, and the warmer fluid will start to move up (against the direction of gravity) while the colder fluid sinks down.

These reflections warn us that hydrostatic stability may be compromised by temperature gradients in combination with gravitational forces. This is all too well known in daily life, from the boiling water in a pot to drafts arising near heat radiators in our homes. In moving fluids other instabilities may arise from the internal friction in the fluid. We shall return to the study of instabilities, both convective and dynamic, in later chapters.

4.4 Compressibility

Liquids tend to be rather incompressible, so that a homogenous liquid body has constant density everywhere and at all time, $\rho(\boldsymbol{x}, t) = \rho_0$. Gases, on the other hand, are compressible, and by and large obey the ideal gas law, $\rho \sim p$, under isothermal conditions. We shall now clarify what is meant by compressibility (and incompressibility) and calculate the pressure in the sea taking into account that water may be compressed.

Bulk modulus

When the volume of a fluid is decreased, the pressure must necessarily increase. If this were not the case, pumping your bicycle would take no effort and the piston would be sucked into the air chamber. The *compressibility* of a fluid is therefore defined to be the negative of the fractional volume change, $-dV/V$, per unit of pressure increase dp

$$\beta \equiv \frac{1}{K} = -\frac{1}{V}\frac{dV}{dp} \ . \tag{4-25}$$

It is convenient to define the *bulk modulus* $K = 1/\beta$ which is really a measure of *incompressibility*. The greater it is, the less a volume contracts for a a given pressure increase. The bulk modulus is also used to characterize the behavior of elastic solids under a uniform compression, although they are more complicated than fluids at rest (see chapter 10).

Material	K[GPa]
Plain steel	*205*
Nickel	*200*
Copper	*120*
Titanium	*110*
Brass	*100*
Zirconium	*95*
Cast iron	*90*
Aluminium	*70*
Magnesium	*45*
Lead	*13*

Typical values for the bulk modulus of various metals and alloys [2].

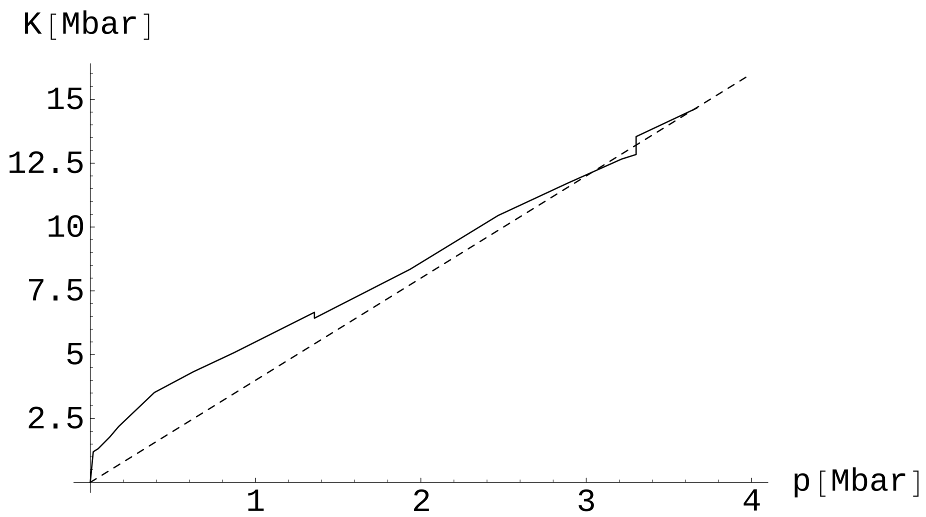


Figure 4.1: *Bulk modulus as a function of pressure in the Earth. The dashed line corresponds to $K = 4p$ (data from [1]). The surface of the Earth is to the left and the centre to the right in this figure.*

The definition of compressibility shows that the bulk modulus is measured in the same units as pressure. For air at normal pressure we have $K \approx 1$ bar, and for sea water we have $K \approx 23$ kilobar, whereas typical values for metals are measured in megabars (10^{11} Pa or 100 GPa). This means that it requires about a million atmospheres of pressure to squash iron. Pumping real iron takes more than human muscles!

Equation of state

Since the density varies inversely with the volume of an amount of material, $\rho \sim 1/V$, we have $dV/V = -d\rho/\rho$ and may write (4-25) in the form

$$\beta = \frac{1}{K} = \frac{1}{\rho}\frac{d\rho}{dp} . \tag{4-26}$$

This may be viewed as a direct definition of the compressibility $\beta = \beta(p)$, or bulk modulus $K = K(p)$, in terms of the barotropic equation of state $\rho = \rho(p)$. For an ideal gas we find, for example, by inserting the equation of state (4-20)

$$K_T = p , \qquad \text{(ideal gas)}, \tag{4-27}$$

where we have used an index T to emphasize that this is the isothermal compressibility. Conversely, we may integrate (4-26) to obtain the equation of state

$$\rho(p) = C \exp\left(\int \frac{dp}{K(p)}\right) , \tag{4-28}$$

where C is an integration constant. The correctness of this expression may be verified by inserting it into (4-26).

In fig. 4.1 the bulk modulus of the Earth is plotted as a function of the pressure. The pressure rises monotonously as one approaches the centre of the Earth, but the interesting result is that the bulk modulus varies approximately linearly with pressure, $K \approx \gamma p$, with $\gamma \approx 4$. The dramatic change in density at the core/mantle boundary (see fig. 3.1) is barely visible in the bulk modulus. In both mantle and core we have $\rho \sim p^{1/\gamma}$, but with different coefficients of proportionality in front.

Sea with constant compressibility

The bulk modulus of water varies somewhat with pressure, doubling in value between 1 and 3000 atmospheres [1], but we shall nevertheless assume that it is constant, $K \approx 2.3$ GPa, in order to get an idea of how much sea water is compressed at the bottom of the sea. In that case we obtain from (4-28)

$$\rho = Ce^{p/K} \ . \tag{4-29}$$

Inserting this into (4-9) we obtain a differential equation for the pressure

$$\frac{dp}{dz} = -Cg_0 e^{p/K} \ ,$$

which upon bringing the exponential to the other side is seen to have the solution

$$-Ke^{-p/K} = -Cg_0 z + A \ ,$$

where A is another integration constant.

The integration constants are determined by the known values of the pressure p_0 and density ρ_0 at the surface, $z = 0$. Inserting these values we get

$$\begin{aligned} \rho_0 &= Ce^{p_0/K} \ , \\ A &= -Ke^{-p_0/K} \ , \end{aligned}$$

and eliminating A and C we find

$$p = p_0 - K \log\left(1 + \frac{\rho_0 g_0 z}{K}\right) \ . \tag{4-30}$$

The characteristic scale for variations in pressure is

$$h_1 = \frac{K}{\rho_0 g_0} = \frac{K}{p_0} h_0 \ , \tag{4-31}$$

so that the pressure and density finally takes the form

$$p = p_0 - K \log\left(1 + \frac{z}{h_1}\right) \ , \tag{4-32}$$

$$\rho = \frac{\rho_0}{1 + \dfrac{z}{h_1}} \ . \tag{4-33}$$

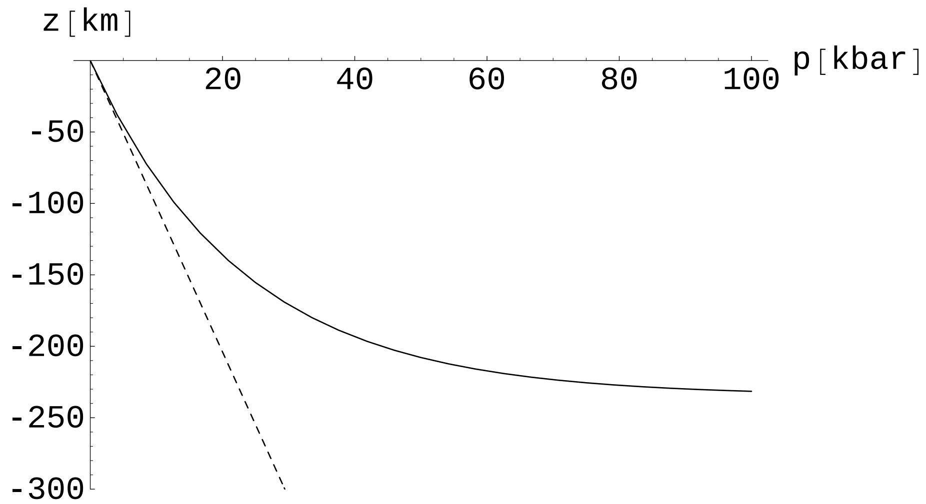


Figure 4.2: *Two models of the flat-earth pressure rise in the sea: constant density (dashed) and constant compressibility (fully drawn). In the latter case, there is an unphysical singularity at $z = -235$ km.*

Notice that for $|z| \ll h_1$ the pressure and density become identical to the pressure (4-2) in a sea of constant density, as it ought to.

The weird thing is that at a depth of $z = -h_1$ both density and pressure become infinite. The assumption of constant compressibility on a flat Earth implies that the density and pressure will increase and reach an infinite value at a finite depth. This is of course nonsense. Many other properties of water break down long before, so one may use this formula only as long as the depth is not comparable to h_1. For water with $K \approx 23$ kilobar we get $h_1 = 235$ km, which is more than 20 times deeper than any of the seas. The compression of water therefore only amounts to less than 1/20 or 5% in the deepest abyss.

For a flat Earth made of cast iron with density 7.2 g/cm^3 the singularity would occur at a depth of 1275 km which is somewhere in the mantle. At this depth the flat-earth assumption fails, however, but for all it is worth (since the material of the Earth does not have constant compressibility as witnessed by fig. 4.1), the singularity disappears for a spherical Earth (problem 6.8).

4.5 The isentropic atmosphere

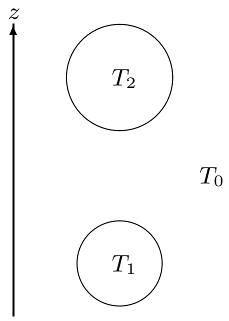
Pilot: "The temperature outside is now minus fifty degrees centigrade."

The assumption that the temperature is the same everywhere in the atmosphere is blatantly wrong, as anyone who has ever flown in a modern passenger jet knows. Temperature falls with height instead of staying constant.

So the atmosphere is *not* in isothermal equilibrium, and this is perhaps not so surprising, since the "container walls" of the atmosphere, the ground and outer space, are not at the same temperature. There must be a heat flow through the atmosphere from the ground towards space, maintained by an influx of solar radiation and an outflux of geothermal energy. But air is a bad conductor of heat, so although heat conduction does play a role, it is not directly the cause of the temperature drop in the atmosphere.

Of much greater importance are the indirect effects of solar heating which create air currents, winds, local turbulence, and convection, continually mixing different layers of the atmosphere. The lower part of the atmosphere, the *troposphere*, is quite unruly and vertical mixing happens at time scales that are much shorter than the time scales necessary for reaching isothermal equilibrium. There is therefore no true hydrostatic equilibrium state for the real atmosphere. Even if we disregard large-scale winds and weather systems, horizontal and vertical mixing always takes place at small scales, and a realistic thermodynamic model of the atmosphere must take this into account.

A thought experiment



Swapping air masses from different heights. If the air has temperature T_0 before the swap, the swapped air would be warmer $T_1 > T_0$ or colder $T_2 < T_0$.

Let us imagine that we take a small body of air and exchange it with another body of air with the same mass, but taken from a different height with a different volume and pressure. In order to fill out the correct volume, one air mass would have to be compressed and the other expanded. If this is done quickly, there will be no time for heat exchange with the surrounding air, and one air mass will consequently be heated up by compression and the other cooled down by expansion. If the atmosphere initially were in isothermal equilibrium, the temperature of the swapped air would not be the same as the temperature of the surrounding air, and the atmosphere would be brought out of equilibrium.

If, however, the surrounding air initially had a temperature distribution, such that the swapped air after the expansion and compression would arrive at the correct temperature of the new surroundings, a kind of thermodynamic and mechanical "equilibrium" could be established, in which the omnipresent local mixing had essentially no effect. Intuitively, it is reasonable to expect that the end result of fast local mixing and slow heat conduction might be precisely this state.

Isentropic processes in ideal gases

In thermodynamics one learns that if a system is isolated from its surroundings, so that heat exchange cannot take place, quasistatic processes conserve *entropy*, and are accordingly called *isentropic* or *adiabatic*.

For an ideal gas in a volume V, constant entropy implies that the expression pV^γ will be constant, where γ is the *adiabatic index*. Thermodynamics teaches us [5] that $\gamma = 1 + 2/n$ where n is the number of molecular degrees of freedom that contribute to the entropy. For monatomic gases $n = 3$ and $\gamma = 5/3$, for diatomic gases such as air $n = 5$ and $\gamma = 7/5$, whereas for multiatomic gases $n = 6$ and $\gamma = 4/3$.

Using that $\rho \sim 1/V$ we conclude that p/ρ^γ must be constant, or

$$\boxed{\rho = C p^{1/\gamma}} \,, \tag{4-34}$$

where C is a constant. This so-called *polytrope* relationship between pressure and density is of the same form as the isothermal equation of state (4-21), but for extended systems, as we have argued above, it cannot correspond to a true thermodynamic and mechanical equilibrium state.

Formally, the relationship (4-34) includes the equation of state for constant density ($\gamma \to \infty$) and the isothermal equation of state ($\gamma = 1$). The bulk modulus for isentropic compression is calculated from (4-26), and we get

$$K_S = \gamma p \,, \tag{4-35}$$

where the index S indicates that the entropy must be held constant during the compression. The isentropic bulk modulus for an ideal gas is always greater (by a factor γ) than the isothermal one. Qualitatively, the greater incompressibility can be understood from the associated heating of the isolated gas during compression. When the isolated gas is compressed it also heats up (the bicycle pump again!) which further increases the pressure.

The temperature of the isentropic ideal gas as a function of pressure is calculated from (4-20) using (4-34) and we obtain

$$\boxed{T = \frac{M_{\rm mol}}{RC} p^{1-1/\gamma}} \,. \tag{4-36}$$

This shows that there is only one free thermodynamic parameter which may be taken to be either pressure, density, or temperature.

The atmospheric temperature lapse

In order to calculate the flat-earth pressure distribution in the isentropic atmosphere we should insert the density (4-34) into (4-9) and solve the resulting differential equation for the pressure. A little trick, however, makes this job much easier. Differentiating the temperature (4-36) after z and making use of (4-9) we obtain

$$\frac{dT}{dz} = \frac{M_{\rm mol}}{RC}\left(1 - \frac{1}{\gamma}\right) p^{-1/\gamma}\frac{dp}{dz} = -\left(1 - \frac{1}{\gamma}\right)\frac{g_0 M_{\rm mol}}{R} \,, \tag{4-37}$$

which is independent of z. This means that the temperature drops (lapses) at a constant rate with height in the atmosphere.

With $M_{\rm mol} \approx 29$ g/mol and $\gamma \approx 1.4$ the numerical value of the temperature lapse rate becomes $dT/dz \approx 10$ K/km. The temperature drops from, say, $+27^\circ$ centigrade at sea level to -60° at the top of Mount Everest, which is a bit colder than what the pilot tells the passengers in modern jet aircraft.

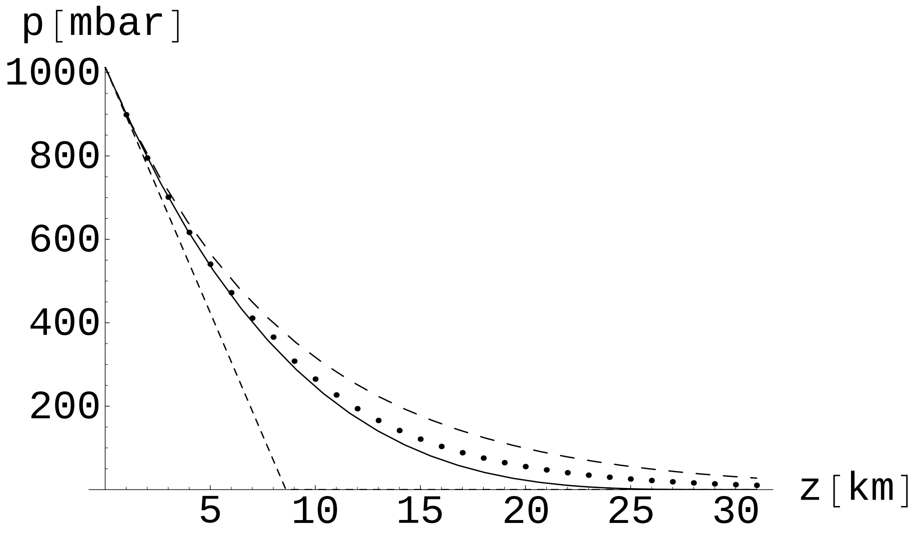


Figure 4.3: *Three different models for the atmospheric pressure: constant density (dashed), isentropic (fully drawn) and isothermal (large dashes), plotted together with the standard atmosphere data (dots)[2]. The parameters are* $h_0 = 8.6$ *km and* $\gamma = 1.4$.

Introducing the temperature T_0 at sea level the temperature at height z becomes

$$T(z) = T_0 \left(1 - \frac{z}{h_2}\right) , \tag{4-38}$$

where the scale is set by

$$h_2 = \frac{\gamma}{\gamma - 1} \frac{RT_0}{g_0 M_{\rm mol}} = \frac{\gamma}{\gamma - 1} h_0 . \tag{4-39}$$

At $z = h_2 \approx 30$ km the temperature has dropped to absolute zero, which is of course unphysical, since the atmosphere does not stop at this height. It is nevertheless a reasonable scale for the height of the atmosphere.

The pressure follows from (4-36)

$$p = p_0 \left(1 - \frac{z}{h_2}\right)^{\frac{\gamma}{\gamma-1}} , \tag{4-40}$$

and the density from (4-34)

$$\rho = \rho_0 \left(1 - \frac{z}{h_2}\right)^{\frac{1}{\gamma-1}} . \tag{4-41}$$

Both of these quantities vanish like the temperature for $z = h_2$.

The real atmosphere

In fig. 4.3 the various atmospheric models have been plotted together with the data for the standard atmosphere [2]. Even if the isentropic model gives the best fit, it fails at higher altitudes because some of the physical assumptions behind the derivation fail. The real atmosphere is in fact more complicated than this model suggests.

Water vapor is always present and will condense to clouds in rising currents of air. The latent heat of condensation heats up the air, so that the temperature lapse rate becomes smaller than 10 K/km, perhaps more like $6-7$ K/km, leading to a somewhat higher temperature at the top of Mount Everest. The clouds may eventually precipitate out as rain, and when the dried air afterwards descends again, for example on the lee side of a mountain, the air will heat up at a higher rate than it cooled during its ascent on the windward side and become quite hot, a phenomenon known as *föhn* in the Alps.

The fact that the temperature lapse is smaller in the real atmosphere than in the isentropic model has a bearing on the stability of the atmosphere. If a certain amount of air is transported to higher altitude without heat exchange and condensation of water vapor, it will behave like in the isentropic model and become cooler than the surrounding air. Consequently it will also be heavier than the surrounding air and tend to sink back to where it came from. Conversely, if the real temperature lapse becomes larger than in the isentropic model, the atmosphere becomes unstable and strong vertical currents may arise. This can, for example, happen in thunderstorms.

Problems

4.1 Consider a canal with a dock gate which is 12 m wide and has water depth 9 m on one side and 6 m on the other side.

a) Calculate the pressures in the water on both sides of the gate at a height z over the bottom of the canal.

b) Calculate the total force on the gate.

c) Calculate the total moment of force around the bottom of the gate.

d) Calculate the height over the bottom at which the total force acts.

4.2 An underwater lamp in a swimming pool is covered by a hemispherical glass with a diameter of 30 cm and is placed at a depth of 3 m. Calculate the total horizontal force from the water on the lamp, when there is air at normal pressure inside.

4.3 The isentropic equation of state for water is to a good approximation (for pressures up to 100,000 bar) given by

$$\frac{p+B}{p_0+B} = \left(\frac{\rho}{\rho_0}\right)^n \tag{4-42}$$

with $B = 3000\ \mathsf{bar}$, $n = 7$, $p_0 = 1\ \mathsf{bar}$ and $\rho_0 = 1\ \mathsf{g/cm^3}$. a) Calculate the isentropic compressibility K for water. b) Calculate the density and pressure in the isentropic sea. c) Calculate the compression of the water at the deepest point in the sea.