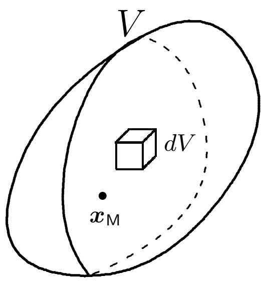
3 Gravity

The force of gravity is all around us and determines to a large extent the way we live. It is certainly the force about which we have the best intuitive understanding. We learn the hard way to rise against it as small children, to keep it at bay as adults, only to be brought down by it in the end. A few people have experienced true absence of gravity for longer periods of time in satellites orbiting the Earth or rockets coasting towards the Moon. Experience has shown that it takes great effort to live well without gravity during those travels.

It was of course Newton who gave us the theory of gravity and the mathematics to deal with it. In a world where things only seem to get done by push and pull, man suddenly had to accept that the Earth could act on the distant Moon — and the Moon back on Earth. After Newton everybody had to suppress the feeling of horror for action at a distance and accept that gravity instantaneously could jump across the emptiness of space and tug at distant bodies.

It took more than two centuries and the genius of Einstein to undo this learning. There *is* no action at a distance. As we understand it today, gravity is mediated by a field which emerges from massive bodies and as light takes time to travel through a distance. If the Sun were suddenly to blink out of existence, it would take eight long minutes before daylight was switched off and the Earth set free in space.

In this chapter we shall study the interplay between mass and the instantaneous Newtonian field of gravity, and derive the equations governing this field and its interactions with matter. Some basic knowledge of gravity is assumed in advance, and the presentation in this chapter aims mainly at developing elementary aspects of field theory in the comfortable environment of Newtonian gravity. More advanced concepts will be developed in chapter 6.



A volume V with a volume element dV and center of mass $\boldsymbol{x}_{\mathsf{M}}$.

3.1 Mass density

In the continuum approximation, the mass density field $\rho(\boldsymbol{x}, t)$ is assumed to be a continuous function of space and time. Knowing this field, we may calculate the mass of a material particle occupying a small volume dV around the point $\boldsymbol{x}$ at time t

$$\boxed{dM = \rho(\boldsymbol{x}, t)\, dV} \,. \tag{3-1}$$

We shall permit ourselves to suppress the space and time variables and just write $dM = \rho\, dV$, whenever such a notation is unambiguous. Although mass in principle is assumed to be distributed continuously throughout space it is sometimes convenient to allow for discontinuous boundaries in material bodies (see Fig. 3.1). Often these discontinuities are "real" in the sense that the transition between different materials or different states of the same material may happen at the molecular scale, as for example at the interface between two solid bodies.

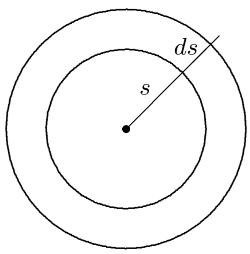
In chapter 1 we discussed the continuum approximation and concluded that the molecular structure of matter could be disregarded within a given measurement precision, provided we imposed a lower limit to the size of the smallest bodies to be considered. This means that although mathematically the volume dV is truly infinitesimal, the above relation loses its physical significance when the linear dimensions of the volume reach this lower limit. For smaller dimensions the actual mass of the molecules present in the volume dV does not obey (3-1) but fluctuates randomly around this value (see fig. 1.1).

Total mass

Mass density is a *local* quantity, defined in every point of space. The total mass in a volume V is a global quantity, given by the volume integral

$$M = \int_V dM = \int_V \rho\, dV \,. \tag{3-2}$$

Physically the integral should be viewed as an approximation to a huge sum over tiny, but not infinitesimal, material particles. If the mass density is constant, $\rho(\boldsymbol{x}) = \rho_0$, in V, the total mass becomes $M = \rho_0 V$.



The volume of a spherical shell of thickness ds and radius s is $4\pi s^2\, ds$.

For a spherical system with mass density $\rho(r)$, where $r = |\boldsymbol{x}|$ is the distance to the center, the total mass within a sphere of radius r becomes

$$M(r) = \int_0^r \rho(s)\, 4\pi s^2 ds \,. \tag{3-3}$$

For a "planet" with constant mass density $\rho(r) = \rho_0$ and radius a we get

$$M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r \le a \\ M & r > a \end{cases} \,. \tag{3-4}$$

where $M = \frac{4}{3}\pi a^3 \rho_0$ is the total mass.

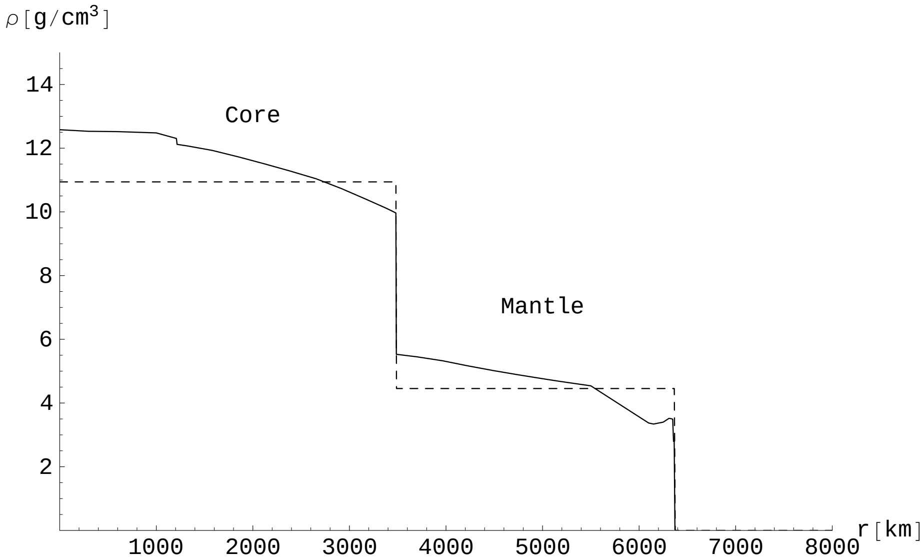


Figure 3.1: *The mass density of the Earth as a function of distance r from the center (the standard Earth model [2]) with the surface at* $r = 6371$ km. *There is a sharp break in the density at the transition between core and mantle at* $r = 3485$ km*, a smaller break at* $r = 1216$ km *between the outer liquid core and the inner solid core, which is comparable to the Moon in size.*

Two-layer model of Earth

The most remarkable feature of the mass density of the standard model of the Earth plotted in Fig. 3.1 is the sharp transition between the core and the mantle about halfway between the center and the surface. Here we consider Earth to be completely spherical and disregard the small difference between polar and equatorial radii. In a sense the core of the Earth is an iron planet inside a stone planet!

> The mean radius of the Earth is $a = 6371$ km, the radius of the core $a_1 = 3485$ km and the thickness of the mantle $a_2 = 2886$ km. The mass of the core is $M_1 = 1.94 \times 10^{24}$ kg, of the mantle $M_2 = 4.04 \times 10^{24}$ kg, and of the whole Earth $M = M_1 + M_2 = 5.976 \times 10^{24}$ kg. This implies that the mean mass density of the core is $\rho_1 = 10.9$ g/cm^3, of the mantle $\rho_2 = 4.5$ g/cm^3, and of the whole Earth $\rho_0 = 5.517$ g/cm^3.

The simplest possible analytic model of the Earth has two layers with constant mass densities ρ_1 and ρ_2 within respectively the core ($0 \le r < a_1$) and the mantle ($a_1 < r \le a$). The density field becomes

$$\rho(r) = \begin{cases} \rho_1 & r \le a_1 \\ \rho_2 & a_1 < r \le a \\ 0 & r > a \end{cases} \tag{3-5}$$

The two-layer approximation is quite decent, even if it does not look that way in fig. 3.1, but since the strongest deviation from the standard model happens close to the center in a relatively small volume, the model is in fact quite accurate when used to calculate integrated quantities, such as the total mass, which for the two-layer model becomes

$$M(r) = \frac{4}{3}\pi \begin{cases} r^3\rho_1 & r \leq a_1 \\ a_1^3\rho_1 + (r^3 - a_1^3)\rho_2 & a_1 < r \leq a \\ a_1^3\rho_1 + (a^3 - a_1^3)\rho_2 & r > a \end{cases} \tag{3-6}$$

We shall return to this model several times in the following.

3.2 Gravitational acceleration

Galileo's observation that all bodies fall from heights in the same way shows — in Newton's language – that the force of gravity on a body is proportional to its mass. For a material particle of mass $dM = \rho dV$ in the point $\boldsymbol{x}$ at time t the force of gravity may be written

$$\boxed{d\boldsymbol{\mathcal{F}} = \boldsymbol{g}(\boldsymbol{x}, t)\, dM = \rho\boldsymbol{g}\, dV}\,, \tag{3-7}$$

where $\boldsymbol{g}(\boldsymbol{x}, t)$ is called the gravitational *acceleration* field, or just gravity. The rightmost expression shows that gravity is a *body force* which acts everywhere in a body with a *density of force* $\boldsymbol{f} = \rho\boldsymbol{g}$.

In Newtonian physics the gravitational field is the field of common acceleration for all particles. The existence of a unique $\boldsymbol{g}$ is due to the equality of the inertial mass used in Newton's second law and the gravitational mass used in the law of gravity (see (3-14)). Since all particles are subject to the same gravitational acceleration, they will — everything else being equal — follow the same orbits independent of their masses.

The universal equality of inertial and gravitational mass — or rather the proportionality — is a fundamental physical observation which is encapsulated in Einstein's Principle of Equivalence. This universality allows one to look upon the gravitational field as a property of space and time, rather than simply a vehicle for gravitational interaction. Einstein took these thoughts to their — so far — ultimate conclusion in his general theory of relativity (1915), where gravity is related to the curvature of space and time.

The gravitational acceleration at the surface of the Earth has been measured with a relative accuracy of 3×10^{-9} in an experiment using atom interferometry [18]. Galileo's law was similarly verified to within 7×10^{-9} by comparing the measured values of the gravitational acceleration for a macroscopic body and for a cesium atom, in effect a modern version of his famous 'leaning tower in Pisa' experiment [18]. The proportionality of inertial and gravitational mass has been verified to a relative accuracy of about 10^{-12} [**?**].

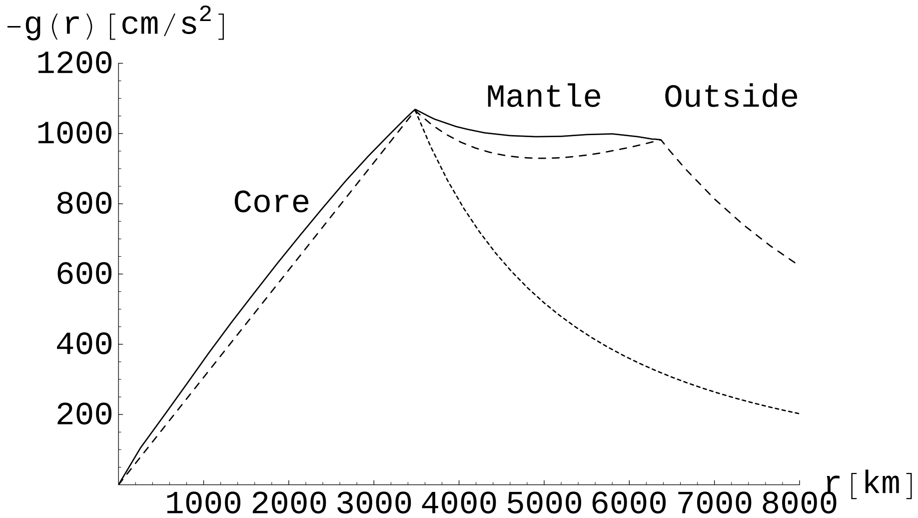


Figure 3.2: *The (negative of the) strength of gravity inside and outside the Earth as a function of distance from the center. The curves are obtained from the integrated mass using the expression (3-19) valid for spherical mass distributions. The field grows roughly linearly from the center to the core/mantle boundary at* $r = 3485$ km*, and falls slightly in the mantle due to the sharp drop in mass density at the boundary. The maximum strength* 10.7 m/s^2 *is found at the core/mantle boundary and is 9 % higher than at the surface. The dotted line is the core field, which amounts to only about* 3.2 m/s^2 *at the surface, or less than 1/3 of standard gravity.*

Force and moment

The field of gravity specifies the gravitational acceleration locally in every point of space and every instant of time. The total force on a body of volume V, the *weight* of the body, becomes

$$\boldsymbol{\mathcal{F}} = \int_V \rho \boldsymbol{g}\, dV \ , \tag{3-8}$$

and it determines how the body as a whole moves. Likewise, the total *moment of force* or *couple* of gravity relative to the origin of coordinates becomes

$$\boldsymbol{\mathcal{M}} = \int_V \boldsymbol{x} \times \rho \boldsymbol{g}\, dV \ , \tag{3-9}$$

and it determines how a body as a whole rotates around the origin.

Center of mass

In the Newtonian mechanics of point particles, the total force determines the acceleration of the center of mass. This remains of course valid in continuum mechanics, because a body may be viewed as a collection of material particles each having an unchanging mass. In continuum mechanics the sums over particles become integrals instead, and the center of mass is defined by

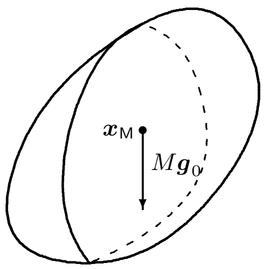
$$\boldsymbol{x}_{\mathsf{M}} = \frac{1}{M}\int_V \boldsymbol{x}\, dM = \frac{1}{M}\int_V \boldsymbol{x}\, \rho(\boldsymbol{x})\, dV \ . \qquad (3\text{-}10)$$

The shape of a body may, however, change drastically over longer time-spans — think of a bucket of oil thrown into a water fall — so although it is formally correct it may not always be physically meaningful to speak about the motion of the center-of-mass of a fluid body.

Constant field

At the surface of Earth gravity is very close to the standard acceleration, defined to be exactly $g_0 = 9.80665\ \mathsf{m/s^2}$ with no uncertainty. In a constant field $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$, the weight (3-8) becomes the familiar

$$\boldsymbol{\mathcal{F}} = M\boldsymbol{g}_0 \ , \qquad (3\text{-}11)$$



In a constant gravitational field $\boldsymbol{g}_0$, the weight of a body may be viewed as acting at the center of mass $\boldsymbol{x}_{\mathsf{M}}$.

where M is the total mass. Analogously, the moment of force may be written

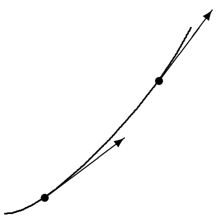
$$\boldsymbol{\mathcal{M}} = \int_V \boldsymbol{x} \times \rho \boldsymbol{g}_0\, dV = \left(\int_V \boldsymbol{\rho}\, \boldsymbol{x}\, dV\right) \times \boldsymbol{g}_0 = \boldsymbol{x}_{\mathsf{M}} \times M\boldsymbol{g}_0 \ . \qquad (3\text{-}12)$$

Just as in particle mechanics, the total moment of force is the same as that of point particle situated at the center-of-mass and having the same weight as the body itself. In a constant field the moment of force calculated relative to the center of mass (so that $\boldsymbol{x}_M = \boldsymbol{0}$) must accordingly vanish.

Visualizing the field

In order to give a direct visual impression of the structure of the gravitational field at a given instant (see fig. 3.3) one may picture the *field lines*, defined to be families of curves that everywhere at a given instant t_0 have the gravitational field as tangent. This means that the curves are solutions to the first order differential equation

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{g}(\boldsymbol{x}, t_0) \ , \qquad (3\text{-}13)$$



Field lines follow the field everywhere.

where s is a running parameter along the curve. The solutions are of the form $\boldsymbol{x} = \boldsymbol{x}(s, \boldsymbol{x}_0, t_0)$ with $\boldsymbol{x}_0$ being the starting point at $s = 0$. The field lines form an instantaneous picture of the field at time t_0, and cannot be directly related to particle orbits, except for time-independent (static) fields.

Field lines have the very important property that they can never cross. For if two field lines crossed in a point $\boldsymbol{x}$, then by (3-13) there would have to be two different values of the gravitational field in the same point, and that is impossible. As will be shown in the following section, all gravitational field lines have to come in from infinity and end on masses, and we shall also see that field lines do not form closed loops.

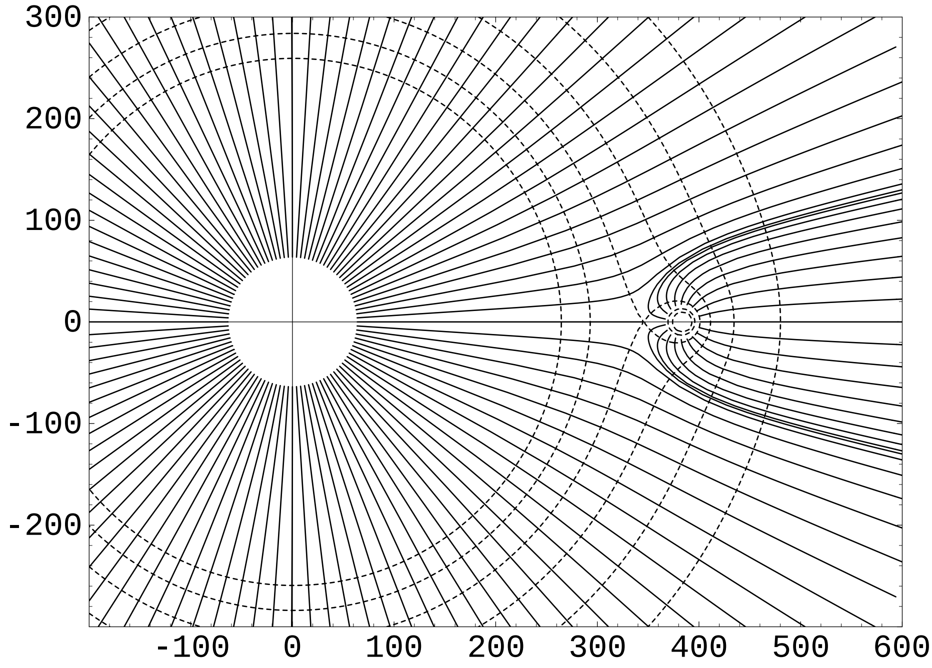


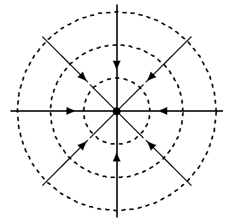
Figure 3.3: *The gravitational field and some equipotential surfaces between Earth and Moon. You should imagine rotating this picture around the Earth-Moon axis. The drawing is to scale, except for two regions of 10 times the sizes of the Earth and the Moon which have been cut out for technical reasons. The field lines are everywhere plotted with a density proportional to the field strength. The numbers on the frame are coordinates centered on Earth in units of 1000 km.*

3.3 Sources of gravity

The gravitational field tells us how the force of gravity acts on material bodies. But what generates the gravitational field? What are its sources? The answer is — as most people are aware — that the field is generated by mass. Newton's law of gravity says in fact that the field from a point particle of mass M at the origin of the coordinate system is

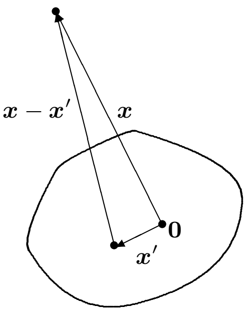
$$\boldsymbol{g}(\boldsymbol{x}) = -GM\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} , \tag{3-14}$$

where G is the universal gravitational constant. The negative sign asserts that gravitation is always attractive, or equivalently that field lines always run towards masses.



Field lines around a point particle all come in from infinity and converge on the particle. The corresponding equipotential surfaces (dashed) are spheres.

The gravitational constant is hard to determine to high precision. The quoted value [1] of $G = 6.6726(9)\times10^{-11}$ $\mathsf{Nm^2/kg^2}$ has an uncertainty which is larger than one part in 10^4. The force of gravity is terribly weak compared to other forces. In the hydrogen atom the ratio of the force of gravity $F_g = GmM/r^2$ (between electron and proton) to the electrostatic force $F_e = e^2/4\pi\epsilon_0 r^2$ is $F_g/F_e = 4.4 \times 10^{-40}$. The only reason gravity can be observed at all is the almost exact electric neutrality of most macroscopic bodies. Although electrostatic and Newtonian gravitational forces behave analogously as a function of distance, they are very different in a deeper sense. There are, for example, no "neutral" bodies unaffected by gravity, nor bodies that are repelled by the gravity of other bodies.



The vectors involved in calculating the field in the position $\boldsymbol{x}$

Another basic property of gravity is that it is *additive*, so that matter — also from the point of view of its gravity — may be viewed as a collection of material particles, each contributing its own little field to the total. Using (3-14), but shifting the origin to $\boldsymbol{x}'$, the collective gravitational field due to all material particles in a volume V becomes

$$\boxed{\boldsymbol{g}(\boldsymbol{x}) = -G \int_V \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \rho(\boldsymbol{x}')\, dV'} \ . \tag{3-15}$$

Although for $\rho(\boldsymbol{x}) \neq 0$ the integral has a singularity at $\boldsymbol{x}' = \boldsymbol{x}$, this singularity is integrable and causes no problems (see problem 3.3).

Forces between extended bodies

The above result brings us full circle. We are now able to calculate the gravitational field from a mass distribution, and the force that this field exerts on another mass distribution, or even on itself. In fact, by substituting the field into the equation for the force (3-8), the total force by which a mass distribution ρ_2 in the volume V_2 acts on a mass distribution ρ_1 in V_1 is found to be

$$\boldsymbol{\mathcal{F}}_{12} = -G \int_{V_1} \int_{V_2} \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^3} \rho_1(\boldsymbol{x}_1)\rho_2(\boldsymbol{x}_2)\, dV_1\, dV_2 \ . \tag{3-16}$$

Newton's third law is fulfilled, $\boldsymbol{\mathcal{F}}_{12} = -\boldsymbol{\mathcal{F}}_{21}$, because the integrand is antisymmetric under exchange of $1 \leftrightarrow 2$. Consequently, the force from an isolated mass distribution acting on itself — the self-force — vanishes, as it ought to, for otherwise a body could, so to speak, lift itself by its bootstraps.

Asymptotic behavior

For a mass distribution of finite extent, a body, we find in the limit of $|\boldsymbol{x}| \to \infty$ that $\boldsymbol{x}$ dominates over $\boldsymbol{x}'$ in the integrand of (3-15), so that $(\boldsymbol{x} - \boldsymbol{x}')/\left|\boldsymbol{x} - \boldsymbol{x}'\right|^3 \to \boldsymbol{x}/\left|\boldsymbol{x}\right|^3$. Taking this expression outside the integral we obtain

$$\boldsymbol{g}(\boldsymbol{x}) \to -GM \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} \ , \tag{3-17}$$

where M is the total mass. At sufficiently large distances the field of an arbitrary but spatially bounded mass distribution always approaches that of a point containing the total mass of body.

Field of a spherical body

The mass distribution $\rho(r)$ of a spherically symmetric body is, as discussed before, only a function of the distance $r = |\boldsymbol{x}|$ from its center, which here is taken to be at the origin of the coordinate system. Since gravity according to (3-15) is caused by the mass distribution, it should also be spherically symmetric, which for a vector field means that it is everywhere directed along the radius vector, *i.e.* of the form

$$\boldsymbol{g}(\boldsymbol{x}) = g(r)\,\boldsymbol{e}_r \ . \tag{3-18}$$

Here $\boldsymbol{e}_r = \boldsymbol{x}/r$ is the radial unit vector and $g(r)$ is a function of r alone.

In chapter 6 we shall show that spherical gravity takes an extremely simple form. We shall prove that the field strength $g(r)$ is everywhere equal to the field of a point particle situated at the center of the distribution with a mass equaling the total mass inside r. In other words,

$$\boxed{g(r) = -G\frac{M(r)}{r^2}} \ , \tag{3-19}$$

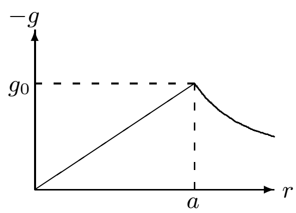
where $M(r)$ is the integrated mass defined in (3-3). It is also possible to prove this by direct integration of (3-15) (see problem 3.8).

In particular it follows that for every point in the vacuum *outside* a spherical mass distribution, the field is exactly the same as that of a point particle at the center having the total mass of the body. We have seen above that the field at great distances from an arbitrary body is always approximately that of a point particle, but now we learn that the field is of this form everywhere around a perfectly spherical body. There are no near-field corrections to the gravitational field of a spherical body, such as a planet or star. Without this wonderful property, Newton could never have related the strength of gravity at the surface of the Earth, iconized by the fall of an apple, to the strength of gravity in the Moon's orbit.

Planet with constant density

For a spherical planet with constant mass density we obtain from (3-4) and (3-19)

$$g(r) = -\frac{4}{3}\pi G \rho_0 \begin{cases} r & r \leq a \\ \dfrac{a^3}{r^2} & r > a \end{cases} \tag{3-20}$$



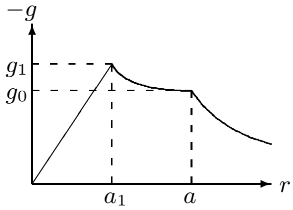
The strength of gravity for a planet with constant density.

The strength of gravity grows linearly with r inside the planet because the total mass grows with the third power whereas the field strength decreases with the second power of r.

Two-layer planet

For a two-layer spherical planet with a core and a mantle we obtain similarly from (3-6)

$$g(r) = -\frac{4}{3}\pi G \begin{cases} r\rho_1 & r \le a_1 \\ \dfrac{a_1^3}{r^2}\rho_1 + \left(r - \dfrac{a_1^3}{r^2}\right)\rho_2 & a_1 < r \le a \\ \dfrac{a_1^3\rho_1 + (a^3 - a_1^3)\rho_2}{r^2} & r > a \end{cases} \tag{3-21}$$



The strength of gravity for a two-layer planet with a dense core.

Here the field is a combination of increasing and decreasing terms as can be seen in fig. 3.2. If the core is dense enough, the field strength at the core/mantle boundary will be greater than at the surface, $|g(a_1)| > |g(a)|$, or $a_1\rho_1 > (a_1^3\rho_1 + (a^3 - a_1^3)\rho_2)/a^2$. This inequality may be rewritten in the form

$$\frac{\rho_1 - \rho_2}{\rho_2} > \frac{a^2}{a_1(a + a_1)} \tag{3-22}$$

For the Earth the left hand side becomes 1.42 and the right hand side 1.18, so the inequality is fulfilled. It comes as a surprise to many that the strength of gravity at the core/mantle boundary is about 9 % higher than standard gravity.

Physical reality of the gravitational field

Whereas the mass density only has meaning in a volume containing mass and may be defined to be zero in the vacuum, the gravitational field is assumed to exist and take non-vanishing values even in the vacuum. It specifies the force that would be exerted on a unit mass particle at a given point, but the field is assumed to be there, even if no particles are present.

In Newtonian physics, the gravitational field defined in (3-15) has no independent physical meaning and may be completely eliminated and replaced by forces between material bodies, as we just did in (3-16). The true physical objects appear to be the material bodies, and the field just a mathematical convenience for calculating the gravitational force exerted by these bodies. When the bodies move around or change their mass distributions, the field will change instantaneously with them.

In relativity, on the other hand, fields take on a completely different meaning. The reason is that instantaneous action-at-a-distance cannot take place. If the particles move, the current view is that it will take some time before the field adjusts to the new positions, because no signal can travel faster than light. Due to relativity, fields travel independently, obey their own equations of motion, and carry physical properties such as energy and momentum. Electromagnetic waves bringing radio and tv signals to us, are examples of fields thus liberated from their origin. Gravitational waves have not yet been observed directly. Indirectly they have been observed in binary neutron star systems which can only be fully understood if gravitational radiation is taken into account.

Even if we shall not deal with relativistic theories of the continuum, and therefore may consider the gravitational field to be merely a mathematical convenience, it may nevertheless be wise, at least in the back of our minds, to think of the field as having an independent physical existence. Then we shall have no philosophical problem endowing it with physical properties, even in matter-free regions of space.

3.4 Gravitational potential

Although the field of gravity is a vector field (see problem 3.2) with three components, there is really only one functional degree of freedom underlying the field, namely the mass distribution giving rise to it. The relationship between these two fields expressed by (3-15) is, however, *non-local*, meaning that $\boldsymbol{g}(\boldsymbol{x})$ in a point $\boldsymbol{x}$ depends on a physical quantity $\rho(\boldsymbol{x}')$ in other points that may be arbitrarily far away.

We shall now prove that the acceleration field can be derived *locally* from a scalar field $\Phi(\boldsymbol{x})$, called the gravitational *potential*. The relation between the acceleration field and the potential is

$$\boxed{\boldsymbol{g} = -\boldsymbol{\nabla}\Phi} \tag{3-23}$$

with a conventional minus sign in front. The gradient operator (nabla) has been defined in (2-17). Because of the gradient, the potential contains an undetermined additive constant.

In order to demonstrate (3-23) we first calculate the gradient of the central distance (see also problem 2.8)

$$\boldsymbol{\nabla}|\boldsymbol{x}| = \boldsymbol{\nabla}\sqrt{\boldsymbol{x}^2} = \frac{1}{2|\boldsymbol{x}|}\boldsymbol{\nabla}\boldsymbol{x}^2 = \frac{1}{2|\boldsymbol{x}|}\boldsymbol{\nabla}(x_1^2 + x_2^2 + x_3^2) = \frac{\boldsymbol{x}}{|\boldsymbol{x}|} , \tag{3-24}$$

and from this

$$\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}|} = -\frac{1}{|\boldsymbol{x}|^2}\boldsymbol{\nabla}|\boldsymbol{x}| = -\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} . \tag{3-25}$$

Comparing with (3-14) we conclude that the potential of a point particle of mass m is

$$\Phi(\boldsymbol{x}) = -\frac{GM}{|\boldsymbol{x}|} , \tag{3-26}$$

apart from an arbitrary additive constant which we choose to be zero. Replacing $\boldsymbol{x}$ by $\boldsymbol{x} - \boldsymbol{x}'$ in this expression and comparing with (3-15) we find the potential from a mass distribution in V

$$\boxed{\Phi(\boldsymbol{x}) = -G\int_V \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}dV'} , \tag{3-27}$$

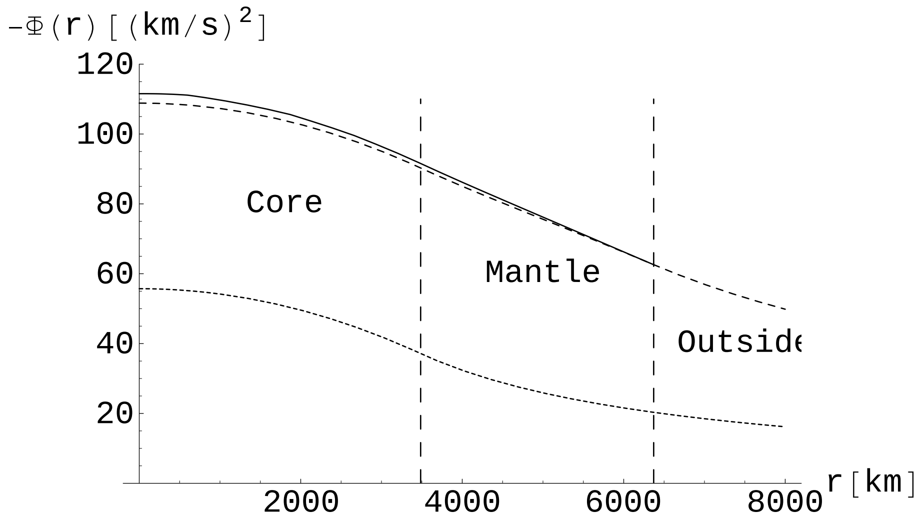


Figure 3.4: *The (negative of the) gravitational potential of the Earth as a function of distance from the center. The dotted curve is the potential of the core alone, and the dashed curve is the two-layer model discussed in the text. Notice that all signs of sharp transitions (at the vertical lines) have disappeared in this plot.*

Since the mass density is always positive, the gravitational potential is always negative, a direct consequence of the attractive nature of gravity (and the normalization to zero potential at infinity).

Even if the mass distribution may jump discontinuously, the gravitational field will always be continuous as is evident from fig. 3.2, because of the integration over the mass distribution in (3-15). The potential is still smoother, because its definition (3-23) requires it to have a continuous derivative. This is also borne out by fig. 3.4 which shows the potential of the Earth as a function of central distance. Almost all traces of the original discontinuities have vanished from sight.

Visualizing the potential

The gravitational potential may be visualized by means of surfaces of constant potential, also called *equipotential surfaces.* The field lines are always orthogonal to the equipotential surfaces, and if they are plotted with constant potential difference, the strength of the field will be inversely proportional to the distances between them. A few equipotential surfaces have been shown in the Earth-Moon plot in Fig. 3.3.

Asymptotic behavior

For a mass distribution of finite extent, the denominator will for $|\boldsymbol{x}| \to \infty$ become independent of $\boldsymbol{x}'$, so that

$$\Phi(\boldsymbol{x}) \to -G\frac{M}{|\boldsymbol{x}|} \tag{3-28}$$

where M is the total mass, in complete accordance with (3-17). At large distances the potential of a body thus approaches that of a point mass carrying the total mass of the body, and vanishes at infinity. For mass distributions of infinite extent, all bets are off, and the behavior at infinity obeys no simple rule.

Constant field

For a constant gravitational field $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$ we may take

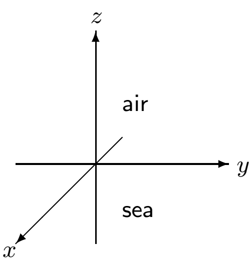
$$\Phi(\boldsymbol{x}) = -\boldsymbol{x} \cdot \boldsymbol{g}_0 \tag{3-29}$$

This seems to be at variance with the expression (3-27) and does certainly not vanish at infinity. Constant gravitational fields should, however, always be understood as approximative and existing only within limited regions of space and time, and then the difficulty disappears.

For length scales much smaller than the radius of the Earth, the surface of the sea may be considered to be flat and the gravitational field constant. It is customary to introduce a coordinate system with the z-axis vertical with the sea level at $z = 0$, so that

$$\Phi = g_0 z \tag{3-30}$$

where g_0 is the magnitude of $\boldsymbol{g}_0 = (0, 0, -g_0)$. The acceleration in the z-direction becomes $g_z = -\partial\Phi/\partial z = -g_0$ and is directed downwards everywhere, as it should be.



The flat Earth coordinate system.

The spherical case

The potential of a spherical mass distribution can only depend on $r = |\boldsymbol{x}|$. Using that $\nabla\Phi(r) = \nabla r \, d\Phi(r)/dr$ and $\boldsymbol{\nabla} r = \boldsymbol{x}/|\boldsymbol{x}| = \boldsymbol{e}_r$, we find by comparison with (3-18)

$$\boxed{g(r) = -\frac{d\Phi(r)}{dr}} \, . \tag{3-31}$$

Conversely, integrating this from r to ∞ and using that the potential vanishes at infinity, we obtain

$$\Phi(r) = \int_r^\infty g(s)\, ds = -G \int_r^\infty \frac{M(s)}{s^2}\, ds \, , \tag{3-32}$$

where we have also made use of (3-19). Performing a partial integration we obtain

$$\Phi(r) = G \int_r^\infty M(s)\, d\left(\frac{1}{s}\right) = -G\frac{M(r)}{r} - 4\pi G \int_r^\infty s\rho(s)\, ds \, . \tag{3-33}$$

The final expression is not quite as pretty as (3-19) because of the second term, which is necessary to secure the continuity of the derivative. But outside the mass distribution the second term vanishes, and the potential becomes as expected that of a point particle carrying the total mass of the body.

Planet with constant density

For a planet with constant mass density we obtain from (3-20)

$$\Phi(r) = -\frac{2}{3}\pi G\rho_0 \begin{cases} 3a^2 - r^2 & r \le a \\ 2\dfrac{a^3}{r} & r \ge a \end{cases} \tag{3-34}$$

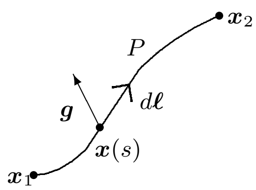
This is most easily verified by differentiation and afterwards fixing the constant term in the interior of the planet by continuity at the surface, $r = a$.

Two-layer planet

For the two-layer model (3-5) with the mass given by (3-6) the potential becomes (see fig. 3.4)

$$\Phi(r) = -\frac{2}{3}\pi G \begin{cases} (3a_1^2 - r^2)\rho_1 + 3(a^2 - a_1^2)\rho_2 & r \le a_1 \\ 2\dfrac{a_1^3}{r}\rho_1 + \left(3a^2 - r^2 - 2\dfrac{a_1^3}{r}\right)\rho_2 & a_1 \le r \le a \\ 2\dfrac{a_1^3}{r}\rho_1 + 2\dfrac{a^3 - a_1^3}{r}\rho_2 & r \ge a \end{cases} \tag{3-35}$$

3.5 Potential energy



Path $\boldsymbol{x}(s)$ with $s_1 \le s \le s_2$ running from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$.

Suppose we in a static field $\boldsymbol{g}(\boldsymbol{x})$ move a point particle of mass m from position $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ along a path P. The amount of work *we* must do *against* the force of gravity may according to the rules of mechanics be calculated as force times distance. Since we must provide a force $-m\boldsymbol{g}$ to counteract gravity, the work performed by us along a curved path is the sum of the small contributions $-m\boldsymbol{g} \cdot d\boldsymbol{\ell}$ from each little path element $d\boldsymbol{\ell}$. In this way we obtain the line integral

$$W = -\int_P m\boldsymbol{g} \cdot d\boldsymbol{\ell} = -m\int_{s_1}^{s_2} \boldsymbol{g}(\boldsymbol{x}(s)) \cdot \frac{d\boldsymbol{x}(s)}{ds}\, ds \ . \tag{3-36}$$

In the last expression the line integral along the path $\boldsymbol{x}(s)$ from $\boldsymbol{x}_1 = \boldsymbol{x}(s_1)$ to $\boldsymbol{x}_2 = \boldsymbol{x}(s_2)$ has been made explicit by means of a running parameter s varying in the interval $s_1 \le s \le s_2$ along the path. If the field is time dependent, the movement should be thought of as being *virtual*, meaning that the field is frozen in place while the particle is moved.

Because the field of gravity is derivable from a potential, the line integral may be carried out. Inserting (3-23) we find

$$W = m \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \nabla\Phi(\boldsymbol{x}) \cdot d\boldsymbol{\ell} = m\Phi(\boldsymbol{x}_2) - m\Phi(\boldsymbol{x}_1) \; . \tag{3-37}$$

Since this result is independent of the actual path of the particle, it follows that gravitational forces are *conservative*, and that the gravitational potential is indeed the potential energy of a unit mass particle in a gravitational field.

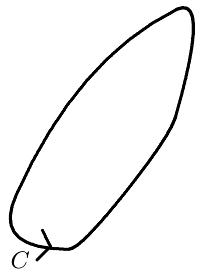
No closed loops of gravity

For a closed path, C, which begins in the same point as it ends, the integral must vanish

$$\oint_C \boldsymbol{g} \cdot d\boldsymbol{\ell} = 0 \tag{3-38}$$

because the potential is the same in the end-points of the path.

This result is true for all gradient fields, such as gravity $\boldsymbol{g}$ and the electrostatic field $\boldsymbol{E}$. It implies that there can be no closed loops of field lines. For if there were, the product $\boldsymbol{g} \cdot d\boldsymbol{\ell}$ would have the same sign all around the loop, because the tangent of a field line is everywhere proportional to the field, *i.e.* $d\boldsymbol{\ell} \sim \boldsymbol{g}$, and then the integral could not vanish.



A gravitational field line cannot form a closed path C.

Escape velocity

The total energy of a point particle in $\boldsymbol{x}$ with velocity $\boldsymbol{v}$ is the sum of its kinetic energy and its potential energy, $\frac{1}{2}m\boldsymbol{v}^2 + m\Phi(\boldsymbol{x})$. From elementary mechanics we know that it is conserved, *i.e.* constant, during the motion. In other words, a particle starting in the point $\boldsymbol{x}_0$ with velocity $\boldsymbol{v}_0$ must all the time obey the equation

$$\frac{1}{2}\boldsymbol{v}^2 + \Phi(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{v}_0^2 + \Phi(\boldsymbol{x}_0) \; . \tag{3-39}$$

Taking $\boldsymbol{x}_0$ at infinity, where the potential vanishes, it follows immediately that in order to escape the grip of gravity (*i.e.* with $|\boldsymbol{v}_0| \geq 0$) from a point with potential Φ, an object must be given at least a velocity

$$v_{\text{esc}} = \sqrt{-2\Phi} \; . \tag{3-40}$$

Knowing the potential in a point is simply equivalent to knowing the escape velocity from that point.

Place	*km/s*
Earth surface	*11.2*
Mars surface	*5.0*
Moon surface	*2.4*
Sun surface	*617.6*
Earth orbit	*42.1*
Moon orbit	*1.4*
Neutron star	1×10^5
Black hole	3×10^5

Escape velocities from some places in the solar system, and a couple of exotic ones.

A particularly interesting case happens when the potential becomes so deep that the escape velocity equals or surpasses the velocity of light c. In that case the body has turned into a black hole. Using the potential of a point mass (3-26) we find that this happens when the radius a of a spherical mass distribution satisfies

$$a < \frac{2GM}{c^2} \; . \tag{3-41}$$

Being a non-relativistic calculation this is of course highly suspect, but accidentally it is exactly the same as the correct condition obtained in general relativity [3], where the right hand side is called the Schwarzchild radius. For the Earth the Schwarzchild radius is about a centimeter, and for the Sun three kilometers.

Problems

3.1 Total mass is a scalar, whereas mass density is a scalar field, transforming slightly differently from a true scalar. Show that under a rotation $\boldsymbol{x}' = \mathbf{a}\cdot\boldsymbol{x}$ the mass density transforms according to the rule

$$\rho'(\boldsymbol{x}') = \rho(\boldsymbol{x}) \tag{3-42}$$

3.2 Total force is a vector, whereas gravity is a vector field, transforming slightly differently from a true vector. Show that under a rotation $\boldsymbol{x}' = \mathbf{a}\cdot\boldsymbol{x}$ the gravitational field transforms according to the rule

$$\boldsymbol{g}'(\boldsymbol{x}') = \mathbf{a}\cdot\boldsymbol{g}(\boldsymbol{x}) \tag{3-43}$$

3.3 Show by direct integration in a small spherical volume around the singularity in (3-15) that it gives a finite contribution to the integral.

3.4 A comet consisting mainly of ice falls to Earth. a) Estimate the minimum energy released in the fall per unit of mass. b) Compare with the an estimate of the energy needed to evaporate the comet.

3.5 A stone is set in free fall from rest through a mine shaft going right through the center of a planet with constant density. a) Calculate the speed with which the stone passes the center. b) Calculate the time it takes to fall to the center. c) Use the example of a constant density Earth.

3.6 A spherical planet with mass distribution of the form $\rho(r) = Ar^\alpha$ for $r \leq a$. a) Calculate the gravitational field strength and the potential inside the planet for this distribution. b) For what values of α is the problem solvable with finite planet mass. c) For what value of α does gravity grow stronger towards the center.

3.7 An "exponential star" has a mass density $\rho = \rho_0 e^{-r/a}$, where ρ_0 is the central mass density and a is the "radius". Calculate the gravitational field and potential.

∗ **3.8** Show that gravitational field of a spherical body (3-19) may be derived by integration of (3-15).

∗ **3.9** a) Calculate the gravitational potential and field from a mass distribution shaped like a very thin line (a model of a cosmic string) of length $2a$ with uniform mass λ per unit of length. b) Calculate the behaviour of the potential at infinity orthogonally to the line. c) Discuss what happens in the limit of $a \to \infty$.

∗ **3.10** a) Write an expression for the gravitational potential from a mass distribution shaped like a very thin circular plate of radius a with uniform mass σ per unit of area (a model of the luminous matter of a galaxy). b) Calculate the value of the potential along the central normal of the plate. c) Calculate its form far from the disk. d) What happens for $a \to \infty$?

