15 Nearly ideal flow

The most important fluids of our daily life, air and water, are lively and easily set into irregular motion. Getting out of a bathtub creates visible turbulence in the soapy water, whereas we have to imagine the unruly air behind us when we jog. Internal friction, or *viscosity*, seems to play only a minor role in these fluids. Other fluids, like honey and grease, are highly viscous, do not easily become turbulent, and would certainly be very hard to swim in.

Being easily movable or sluggish is, however, not an absolute property of a fluid, but rather a condition of the circumstances under which it flows. Lava may be highly viscous in small amounts, but when it streams down a mountainside it appears to be quite lively. We shall later see that there is a way of characterizing fluid flow by means of a single real number, called the *Reynolds number*, which is typically large for lively and small for sluggish flow.

The earliest quantitative model of fluid behavior goes back about 250 years and did not include viscosity. Although already Newton knew the concept, viscosity first entered fluid mechanics in its modern formulation three quarters of a century later. Fluids with no viscosity have been called *ideal* or perfect, and sometimes "dry", because they do not hang on to containing surfaces. An ideal fluid is able to slip along container walls with finite velocity, whereas a real fluid has to adjust its velocity field so that it matches the speed of the container walls and the surfaces of moving objects.

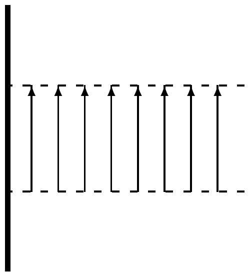
Although ideal fluids do not really exist, except for a component of superfluid helium close to zero kelvin, viscous fluids may nevertheless flow with such high Reynolds number that they behave as nearly ideal. But around solid obstacles and near the walls of fluid conduits there will always be boundary layers in which viscosity becomes dominant. In this chapter we focus on nearly ideal flow, and postpone the discussion of viscosity to chapter 17.

15.1 Euler's equation

Leonhard Euler (1707–1783). Swiss mathematician who made fundamental contributions to calculus, geometry, number theory, and to practical ways of solving mathematical problems. His books on differential calculus (1755) and integral calculus (1768–70) have been especially useful for physics.

In 1755 Euler was the first to write down Newton's second law of motion for fluids without viscosity. In such an *ideal* or *perfect* fluid the only forces at play are pressure and gravity, but in distinction to hydrostatics, these two forces are no more in balance, but give rise to an effective density of force $\boldsymbol{f}^* = \rho\boldsymbol{g} - \boldsymbol{\nabla}p$. Inserting this into the dynamic equation (14-32) and dividing by the density ρ, we obtain *Euler's equation* for ideal fluids,

$$\boxed{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{g}}\,. \tag{15-1}$$

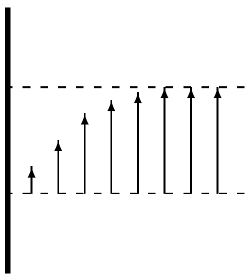


In an ideal fluid the tangential component of the velocity field jumps abruptly at a boundary.

Together with the equation of continuity (14-25) and a barotropic equation of state $p = p(\rho)$, we have a closed set of four equations for the four fields, ρ and $\boldsymbol{v}$.

As in hydrostatics, the pressure must be continuous across any material interface. Furthermore, since moving fluids are normally contained in tubes, pipes, or other kinds of conduits that are impenetrable to the fluid, it follows that the velocity component normal to a containing surface must vanish, *i.e.* $\boldsymbol{n}\cdot\boldsymbol{v} = 0$. There are, however, no conditions on the tangential velocity in ideal flow.

> It is precisely here that the greatest difference between ideal and real viscous fluids come in. Ideal fluids are as mentioned able to slip along container surfaces with finite tangential velocity, whereas the omnipresent viscosity of real fluids demands that also the tangential velocity must vanish on containing surfaces at rest. Viscosity thereby causes boundary layers to arise even in almost ideal fluids to soften sharp transitions in tangential velocity (see chapter 24).



In a real fluid the tangential component of the velocity field rises linearly at a boundary and joins smoothly with the general flow.

Although we are now in possession of the fundamental equations for ideal fluids, solving them is another matter. The bad news about partial differential equations like the one above (and the ones we shall meet later) is that they can only be solved analytically for simple flows in simple geometries, for example the asymptotically uniform steady flow around a sphere at rest to be discussed in section 15.7. It is for this reason of importance to explore their extreme limits and to make estimates of the conditions for their validity.

One such limit is hydrostatics, $\boldsymbol{v} = \boldsymbol{0}$, which has been thoroughly investigated in chapters 4-7. Another limit analyzed in section 14.5 was characterized by the complete absence of forces, $p = 0$ and $\boldsymbol{g} = \boldsymbol{0}$, and could be called the inertial limit. A slightly more complicated case with no pressure $p = 0$ but including gravity $\boldsymbol{g}$, was furnished by Newtonian cosmology in section 14.6. Having more or less exhausted the pressure-less possibilities, we now turn to the analysis of the influence of pressure on fluid motion.

Two major extreme cases present themselves. In the first case, the material acceleration is dominated by the local term $\partial\boldsymbol{v}/\partial t$, and in the second it is dominated by the inertial or advective term, $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$.

15.2 Pressure waves

When you clap your hands together, you create momentarily a small disturbance in the air which propagates to your ear and tells you that something happened. The diaphragm of the loudspeaker in your radio vibrates in tune with the music carried by the radio waves and transfers its vibrations to the air where they continue as *sound*. No bulk movement of air takes place over longer distances, but locally the air oscillates rapidly back and forth.

Wave equation

Consider a barotropic fluid in the absence of gravity. In hydrostatic equilibrium, $\boldsymbol{\nabla} p = \mathbf{0}$, both pressure and density must be constant, $\rho = \rho_0$ and $p = p_0 = p(\rho_0)$. Disturbing this hydrostatic bliss by setting fluid into motion with a tiny velocity field, $\boldsymbol{v}$, Euler's equation (15-1) simplifies to

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho}\boldsymbol{\nabla} p \ , \tag{15-2}$$

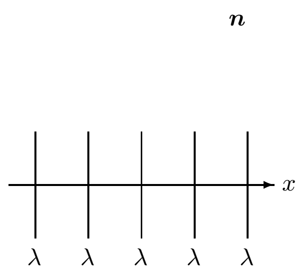
because the inertial acceleration $(\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{v}$ is of second order in the small velocity field and may be left out. By means of the definition of compressibility (4-26) and the continuity equation (14-25), we find

$$\frac{\partial p}{\partial t} = \frac{dp}{d\rho}\frac{\partial \rho}{\partial t} = -\frac{K}{\rho}\mathsf{div}\,(\rho \boldsymbol{v}) \approx -K_0 \mathsf{div}\,\boldsymbol{v} \ ,$$

where $K = K(\rho)$ is the bulk modulus. In the last step the density and the bulk modulus have been replaced by their values at rest, because the expression is already of first order in the small velocity. Differentiating once more after t, and using (15-2) with $\rho = \rho_0$, we obtain

$$\boxed{\frac{\partial^2 p}{\partial t^2} = \frac{K_0}{\rho_0}\boldsymbol{\nabla}^2 p} \ , \tag{15-3}$$

which is a standard *wave equation*.



Plane pressure wave propagating along the x-axis with wave length λ. The isobars are planes orthogonal to the direction of propagation.

Speed of sound

An elementary plane pressure wave propagating along the x-axis wave number k, angular frequency Ω, and amplitude p_1 is described by the pressure field

$$p(\boldsymbol{x}, t) = p_0 + p_1 \cos(kx - \omega t) \ . \tag{15-4}$$

The isobars are parallel with the yz-plane, and the wave crests are equally spaced one wave length, $\lambda = 2\pi/k$, apart along the x-axis. The velocity of a wave crest, called the *phase velocity* of the wave, is $c = \omega/k$. Inserting the plane wave into the wave equation (15-3), we obtain $c^2k^2 = k^2K_0/\rho_0$, or

$$c = \sqrt{\frac{K_0}{\rho_0}}\ , \tag{15-5}$$

Material	c [m/s]
Air (0° C)	*331*
Water	*1516*

Speed of sound for various materials calculated from (15-5).

which is the *speed of sound* in the material. Sound vibrations are normally so fast that temperature equilibrium is never established and the pressure variations happen adiabatically. For an isentropic gas we find by means of the isentropic bulk modulus (4-35),

$$c = \sqrt{\gamma\frac{RT_0}{M_{\rm mol}}}\ , \tag{15-6}$$

where T_0 is the temperature of the gas at rest. For air at 0° C with $\gamma = 7/5$ and $M_{\rm mol} = 29$ g/mol, this comes to the well-known $c \approx 330$ m/s.

Inserting the plane pressure wave (15-4) into Euler's equation (15-2), we may solve for the velocity field and find

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}_0 + \boldsymbol{e}_x v_1 \cos(kx - \omega t)\ , \tag{15-7}$$

where $\boldsymbol{v}_0$ is a constant. The amplitude of the oscillating field is

$$v_1 = \frac{p_1}{\rho_0 c} = \frac{p_1 c}{K_0}\ . \tag{15-8}$$

The oscillating velocity field of a pressure wave is always *longitudinal*, *i.e.* parallel to the direction of wave propagation. The amplitude of the corresponding spatial displacement of material is $x_1 \approx v_1/\omega$.

Example 15.2.1: A certain loudspeaker transmits sound to air at a frequency $\omega/2\pi = 1000$ Hz with diaphragm movement amplitude of about $x_1 = 1$ mm. The corresponding velocity amplitude is $v_1 = \omega x_1 \approx 6$ m/s and the pressure amplitude $p_1 = \rho_0 c v_1 \approx 2000$ Pa.

Validity of the approximation

Sound waves in fluids will be analyzed in greater detail in chapter 20. Here it only remains to find the condition for the small velocity approximation. In order to check its validity, we calculate the ratio between inertial acceleration $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ and local acceleration $\partial\boldsymbol{v}/\partial t$ for the plane wave,

$$\frac{|(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}|}{|\partial\boldsymbol{v}/\partial t|} \approx \frac{kv_1^2}{kcv_1} = \frac{v_1}{c} = \frac{p_1}{K_0} \tag{15-9}$$

The condition for the approximation is thus that the amplitude of the pressure wave must be much smaller than the bulk modulus, $p_1 \ll K_0$, or equivalently that the velocity amplitude should be much smaller than the speed of sound, $v_1 \ll c$. This is well fulfilled in the above example, where $v_1/c \approx 1/50$.

15.3 Steady flow

Steady or *stationary* flow is characterized by the vanishing of the local acceleration $\partial \boldsymbol{v}/\partial t = \mathbf{0}$, such that Euler's equation takes the form

$$\boxed{(\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \boldsymbol{g}}\ . \tag{15-10}$$

But steady flow is also an approximation that is only valid under certain conditions. A river may flow steadily for days and weeks, while over several months seasonal changes in rainfall makes its water level rise and subside again. If one empties a cistern filled with water through a narrow pipe, the flow is almost steady in the pipe for a time, except that the level of water in the cistern slowly goes down and thereby reduces the flow speed in the pipe. In such cases, the flow should rather be called *nearly steady* or *quasistationary.*

Real and effective incompressibility

Most liquids are incompressible under normal circumstances with constant mass density, $\rho = \rho_0$, and vanishing velocity divergence, $\mathsf{div}\, \boldsymbol{v} = 0$. Whereas we usually think of gases as being highly compressible, they may nevertheless under steady flow conditions be effectively incompressible. We shall now see that *a compressible fluid in steady flow behaves effectively like an incompressible fluid, when the flow velocity is much smaller than the velocity of sound.*

> Waving your hands in the air, you generate flow velocities at most of the order of meters per second. Driving a car at 100 km/h involves flow velocities of the order of 30 m/s. Flying in a passenger jet at 900 km/h entails on the other hand flow velocities around 250 m/s that are comparable to the speed of sound $c \approx 300$ m/s, and considerable compression of the air must occur especially at the front of the wings and body of the aircraft.

To prove the statement we consider a compressible fluid at rest in hydrostatic balance with pressure $p_0(\boldsymbol{x})$ and density given by a barotropic equation of state, $\rho_0(\boldsymbol{x}) = \rho(p_0(\boldsymbol{x}))$. If we perturb the fluid with a "small" steady velocity field $\boldsymbol{v}$ will create pressure variations $\Delta p = p - p_0$ everywhere in the fluid. The corresponding density variations may as before be written

$$\Delta\rho = \frac{d\rho}{dp}\Delta p = \rho_0 \frac{\Delta p}{K_0} \tag{15-11}$$

where K_0 is the bulk modulus, defined in (4-26). The fluid is effectively incompressible, if the density variations are everywhere small compared to the density itself, $|\Delta\rho| \ll \rho_0$, which by the above equation is equivalent to $|\Delta p| \ll K_0$. Assuming that gravity is not changed, the typical scale of the pressure changes can be estimated from Euler's equation (15-10) to be of magnitude $|\Delta p| \approx \rho_0 v^2$ (in the following section we shall make a more precise derivation). Inserting the estimate, the condition of effective incompressibility becomes $|v| \ll c$ where $c = \sqrt{K_0/\rho_0}$ is the speed of sound.

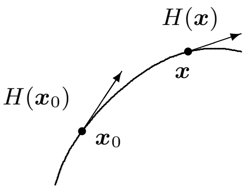
15.4 Bernoulli's theorem

The negative sign of the pressure gradient in Euler's equation (15-10) shows that a steady flow accelerating in a certain direction must be accompanied by a drop in pressure in the same direction, so that regions of high fluid velocity must generally have a lower pressure than regions of low velocity.

Daniel Bernoulli, 1700–82.

He pointed out that there must be such a dynamic relationship between velocity and pressure[1].

In 1738 Daniel Bernoulli published the world's first book on hydrodynamics. For an incompressible fluid with constant density $\rho = \rho_0$, Bernoulli's theorem states that the field,



H is constant along a streamline in steady flow, i.e. $H(\boldsymbol{x}) = H(\boldsymbol{x}_0)$.

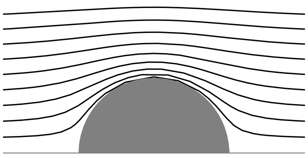
$$\boxed{H(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{v}^2 + \frac{p}{\rho_0} + \Phi} \; , \tag{15-12}$$

is *constant along streamlines*. In the absence of gravity, the constancy of H implies as we foresaw that an increase in velocity along a streamline must be a compensated by a decrease in pressure, and conversely.

The proof is straightforward. The comoving rate of change of H along a particle orbit is given by the material derivative (14-28) and since $\partial H/\partial t = 0$ in steady flow, we get

$$\begin{aligned}\frac{\delta H}{\delta t} &= \boldsymbol{v}\cdot\boldsymbol{\nabla} H = \boldsymbol{v}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \frac{1}{\rho_0}\boldsymbol{v}\cdot\boldsymbol{\nabla} p + \boldsymbol{v}\cdot\boldsymbol{\nabla}\Phi \\ &= \boldsymbol{v}\cdot\left((\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \frac{1}{\rho_0}\boldsymbol{\nabla} p - \boldsymbol{g}\right) = 0 \; ,\end{aligned}$$

where in the last step we have used Euler's equation (15-10). This shows that H is constant along particle orbits which in a steady flow coincide with the streamlines.



The flow has to quicken around an obstacle (here a half sphere) on the bottom of a stream and by Bernoulli's theorem, there will be a lower pressure above the obstacle, i.e. a lift.

Bernoulli's theorem is highly useful because most of the flows that we deal with in our daily lives are nearly ideal. Bernoulli's theorem often provides us with a first idea about the properties of a flow in a given geometry. The drop in pressure accompanying an increase in flow velocity lies, for example, at the root of lift generation for both animals and machines, whether they swim or fly.

Example 15.4.1: Lift is usually thought of as beneficial, but that may not always be the case. Some fish hide by burrowing superficially into the sandy bottom of a stream. The fish's curved upper surface forces the passing water to speed up, leading to a pressure drop above the fish that grows with the square of the flow velocity. If the stream velocity increases, the fish may be lifted out of the sand, whether it wants to or not. That is probably why flatfish are indeed — flat.

[1] It was actually not Bernoulli who formulated the theorem which now bears his name, but rather Lagrange in his famous book on analytic mechanics from 1788 [6].

Effective pressure

The Bernoulli field H is a generalization of the hydrostatic quantity (also called H) introduced on page 107, and becomes identical to it for $\boldsymbol{v} = \boldsymbol{0}$. The first term in H is the kinetic energy of a unit mass fluid particle, whereas the last is its potential energy in a gravitational field. Bernoulli's theorem looks like energy conservation for a unit mass fluid particle in an effective gravitational potential $\Phi + p/\rho_0$.

Often one introduces instead the so-called *effective pressure*

$$p^* = p + \rho_0\Phi \ , \tag{15-13}$$

which is the pressure that is left over after cancelling the hydrostatic pressure due to gravity (the effective pressure is constant in hydrostatic equilibrium). The Bernoulli field then becomes,

$$H = \frac{1}{2}\boldsymbol{v}^2 + \frac{p^*}{\rho_0} \ , \tag{15-14}$$

and Bernoulli's theorem expresses that a change in velocity along a streamline must be compensated by an opposite change in effective pressure.

Terminology

The importance of Bernoulli's theorem for many practical hydrodynamical applications has led to several different terminologies. Since $\rho_0 H$ has the dimension of pressure, the term $\frac{1}{2}\rho_0 v^2$ is often called the *dynamic pressure* as opposed to the *static pressure p*. A point where the fluid has zero velocity is called a *stagnation point* for the flow. At such a point the effective pressure is $\rho_0 H$, and for this reason $\rho_0 H$ is also called the *stagnation pressure*. In constant gravity, g_0, the Bernoulli field becomes

$$H = \frac{1}{2}v^2 + \frac{p}{\rho_0} + g_0 h \tag{15-15}$$

where h is the vertical height of a point on the streamline above some fixed reference point. Engineers often call this height the *static head*, and similarly $p/\rho_0 g_0$ is called the *pressure head* and $v^2/2g_0$ is called the *velocity head*. The total head, H/g_0, is thus the same everywhere along a streamline.

∗ Compressible fluid

Bernoulli's theorem is not limited to incompressible fluids, but is also valid for nearly ideal flow in a compressible fluid with barotropic equation of state (also called an *elastic fluid*), provided we take

$$H = \frac{1}{2}\boldsymbol{v}^2 + w(p) + \Phi \ , \tag{15-16}$$

where $w(p)$ is the pressure function defined in (7-3) (see problem 15.3). Since $\boldsymbol{\nabla} w = \boldsymbol{\nabla} p/\rho$, the proof is elementary.

15.5 Using Bernoulli's theorem

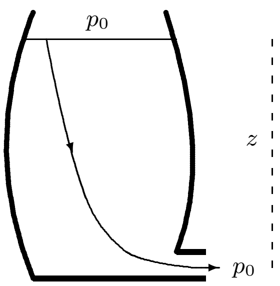
Bernoulli's theorem is as mentioned before highly useful because the fluids that we ordinarily deal with are nearly ideal. Viscosity is, however, never completely absent, but mostly it is negligible well away from the boundaries of the ducts and containers that we use to handle fluids.

Bernoulli's theorem only furnishes a partial solution of Euler's equation (15-10). Although it relates the pressure and the magnitude of the velocity along a streamline, nothing is said about the direction of the velocity or how different streamlines relate to each other. To determine that one needs to solve the Euler equation with the boundary conditions imposed by the problem at hand. Nevertheless, conservation of mass may relate the average flow velocities in different parts of a system and Bernoulli's theorem can then be used to estimate the corresponding variations in the pressure. This method is for example used in the Venturi tube (problem 15.4).

In many cases of technological interest, fluids flow through a system between inlets and outlets that are controlled externally. Since there must be streamlines connecting inlets with outlets, Bernoulli's theorem is typically used to relate the velocities and pressures at inlets and outlets, even if nothing is known about details of the the internal flow in the system. We shall now analyze a couple of simple examples of this kind.

Torricelli's Law

Evangelista Torricelli (1608 – 1647). *Italian mathematician.*



Wine running out of a barrel. The wine emerges with the same speed as it would have obtained by falling freely through the height z of the fluid in the barrel.

A barrel of wine with a little spout close to the bottom is a prototypical example of a fluid container. If the plug in the spout is suddenly removed, gravity makes wine emerge with considerable speed. Provided the spout is narrow compared to the barrel, a nearly steady flow will soon establish itself. This is a case where Bernoulli's theorem readily gives us the answer.

Consider a streamline starting near the top of the barrel and running with the flow down through the middle of the spout. Near the top at a height z over the position of the spout, the fluid is almost at rest, *i.e.* $v \approx \mathbf{0}$. The pressure is atmospheric, $p = p_0$, and the gravitational potential may be taken to be $g_0 z$ so that

$$H_{\text{top}} = \frac{p_0}{\rho_0} + g_0 z \ . \tag{15-17}$$

Just outside the spout the fluid has some velocity v, and the pressure is also atmospheric, $p = p_0$, with no contribution from gravity, because the potential has been chosen to vanish here. Hence

$$H_{\text{bottom}} = \frac{1}{2}v^2 + \frac{p_0}{\rho_0} \ . \tag{15-18}$$

Equating the values of H at the top and the bottom we find

$$\frac{1}{2}v^2 + \frac{p_0}{\rho_0} = \frac{p_0}{\rho_0} + g_0 z \ ,$$

which has the solution

$$\boxed{v = \sqrt{2g_0 z}}\ . \tag{15-19}$$

Surprisingly, this is exactly the same velocity as a drop of wine would have obtained by falling freely from the top of the barrel to the spout.

In a sense the barrel acts as a device for diverting the vertical momentum of the falling liquid into the horizontal direction (see chapter 16). The same result holds if the spout is replaced by a pipe that may not even be horizontal, but may turn and twist. As long as friction can be ignored in the pipe, the exit velocity will be equal to the free-fall velocity from the fluid surface at the top of the barrel to the exit from the pipe. This result is called Torricelli's Law, and was in its time a major step forward in the understanding of fluid mechanics.

We have not specified precisely which streamline to take. Other streamlines will start out in different places at the top of the barrel, but all will begin with essentially zero velocity, be subject to the same gravitational potential, and end up in roughly the same place. The calculation of the velocity must therefore lead to the same result, except for streamlines running very near to the walls of the barrel and spout, where the unavoidable viscosity slows down the flow of wine.

Time to empty a container

If the barrel is cylindrical with cross section A_0, Leonardo's law (14-12) tells us that the average vertical flow velocity in the barrel is given by $v_0 = Av/A_0$ where A is the cross section of the spout and $v = \sqrt{2g_0 z}$ the average horizontal flow velocity through it. Since $v_0 = -dz/dt$, we obtain the following differential equation for quasistationary emptying of the barrel,

$$\frac{dz}{dt} = -\frac{A}{A_0}\sqrt{2g_0 z}\ . \tag{15-20}$$

The solution is

$$z = -\frac{1}{2}g_0\left(\frac{A}{A_0}\right)^2 t^2 \tag{15-21}$$

The surface of the wine in the barrel falls like a free particle with a reduced acceleration. The time it takes to empty the barrel from a given height is correspondingly the free-fall time from this heigth multiplied with the ratio of the barrel and spout cross sections.

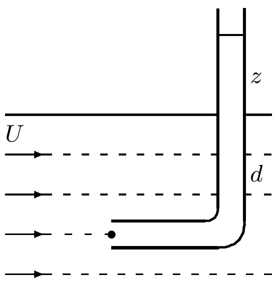
Example 15.5.1: A cylindrical barrel has diameter 1 m and height 2 m. Wine will emerge with a speed of about 6 m/s, and it takes $t \approx 250$ s or 4 minutes to empty the barrel through a spout with diameter 5 cm.

Pitot's tube

Fast aircraft often have a sharply pointed nose which on closer inspection is seen to end in a little open tube. On other aircraft the tube mostly sticks orthogonally out from the side and is then bent forward into the air stream. This device is called a *Pitot tube*, and in today's technology it is used in many variant forms to measure flow speeds in gases and liquids.

Henri Pitot, 1695–1771, French hydraulic engineer. Invented the Pitot tube around 1732.

In its simplest and original form, the Pitot tube is just an open glass tube bent through a right angle. The tube is lowered into a river, streaming steadily with speed U, with one end turned horizontally towards the current and the other vertically in the air above. The flow will stem water up into the vertical part of the tube, until the hydrostatic pressure of the water column balances the dynamic pressure from the flow. After the flow has steadied, the water in the tube rises to a height z above the river surface.



The principle of the Pitot tube. The pressure increase along the stagnating streamline must equal the weight of the raised water column.

Since the water speed must be zero at the entrance to the horizontal part of the tube, a horizontal streamline arriving here from afar must come to an end in a stagnation point. The gravitational potential is constant everywhere along the horizontal streamline, and can be disregarded. The hydrostatic pressure at the streamline far away from the stagnation point is $p_0 + \rho_0 g_0 d$ where p_0 is the atmospheric pressure and d is the depth of the streamline. At the stagnation point the hydrostatic pressure is $p_0 + \rho_0 g_0 (d + z)$ because of the raised water column. Bernoulli's theorem then implies that

$$\frac{1}{2}U^2 + \frac{p_0 + \rho_0 g_0 d}{\rho_0} = \frac{p_0 + \rho_0 g_0 (d+z)}{\rho_0} , \tag{15-22}$$

or

$$U = \sqrt{2 g_0 z} . \tag{15-23}$$

Again we find the simple and surprising result that the speed of the water is exactly the same as it would be after a free fall from the height z. At a typical flow speed of 1 m/s the water in the tube is raised 5 cm above the river level.

Example 15.5.2: Farmers know better than to leave the barn door open towards the wind in a storm, even if they are not familiar with Bernoulli's theorem. A gust of wind will not only decrease the pressure above the barn roof because the wind has to move faster above the roof than at the ground, but the Pitot effect will also increase the pressure inside, so that the roof may blow out.

Water scoop

The idea behind the Pitot tube has a completely different application. Forest fires are sometimes combated by aircraft dropping large amounts of sea or lake water. To avoid excessive landing and take-off, the aircraft collects water by lowering a scoop into the water while flying slowly at very low altitude. If the scoop turns directly forward and the aircraft velocity is U, it can like the Pitot tube raise the water to a maximal height, $z = U^2/2g_0$. Even for a speed as low as $U = 120$ km/h ≈ 33 m/s, this comes to $z = 55$ m.

In practice, the height of the water tank over the lake surface is much smaller, so that the water ideally could arrive in the tank with nearly maximal speed U. A scoop with an opening area of just $A \approx 300$ cm^2 can deliver water at a rate of $Q = UA \approx 1$ m^3/s. Turbulence and friction lowers this somewhat, but typically such an aircraft can collect 6 m^3 in just 12 s.

15.6 Vorticity

The value of the Bernoulli field $H(\boldsymbol{x})$ in a point $\boldsymbol{x}$ is only a function of the streamline going through this point. Different streamlines will in general have different values of H. This can be illustrated by considering the case of Newton's rotating bucket (see page 110).

In the corotating (bucket) coordinate system, the fluid is at rest and the Bernoulli function becomes

$$H_0 = \frac{p}{\rho_0} + g_0 z - \frac{1}{2}\Omega^2 r^2 \qquad (15\text{-}24)$$

where we have included the centrifugal potential (7-10) and $r = \sqrt{x^2 + y^2}$ is the distance from the axis of rotation. Hydrostatic balance ensures that $\boldsymbol{\nabla} H_0 = 0$ so that H_0 is independent of both r and z.

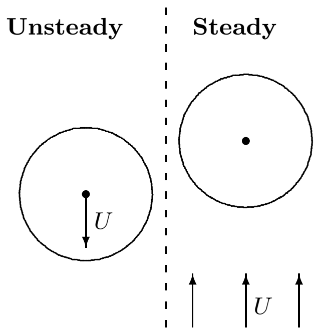
In the non-rotating (laboratory) system, there is no centrifugal force. The fluid moves steadily with velocity $v = r\Omega$, and the streamlines are concentric circles. Since the pressure and the gravitational field must be the same as in the corotating system, the Bernoulli field becomes

$$H(r) = \frac{1}{2}v^2 + \frac{p}{\rho_0} + g_0 z = \Omega^2 r^2 + H_0 \ . \qquad (15\text{-}25)$$

It depends on the radial distance from the axis, but is constant on the streamlines, as Bernoulli's theorem tells us. H is furthermore independent of z, and we shall see below that this also follows from general rules.

Asymptotically uniform flow

It is often possible to relate the values of H for different streamlines. A general and frequently occurring example is a body moving with constant velocity through a fluid otherwise at rest, were it not for the disturbance created by the body. The relativity of motion in Newtonian mechanics tells us that this unsteady flow is physically equivalent to a steady flow around a stationary body in a fluid which far away from the body moves with constant velocity under constant pressure. We tacitly used this equivalence in the discussion of the water scoop above.



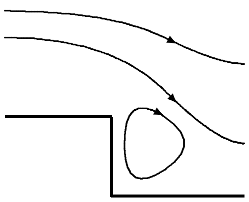
A body moving at constant speed through a fluid is physically equivalent to the same body being at rest in a steady flow which is asymptotically uniform.

At great distances from the body the flow will have the same velocity $\boldsymbol{U}$ and (effective) pressure P for all streamlines. Consequently, the Bernoulli field must take the same value

$$H_0 = \frac{1}{2}\boldsymbol{U}^2 + \frac{P}{\rho_0} \tag{15-26}$$

for all streamlines coming in from afar. If one can be sure that all streamlines around the body originate infinitely far away, then H must take the same value all over space, *i.e.* $H(\boldsymbol{x}) = H_0$.

Vorticity field



Closed streamlines may appear when a fluid flows past an edge.

The simple result that H is spatially constant for asymptotically uniform flow, is spoiled by the possibility that there may be streamlines forming closed curves unconnected with the flow at infinity. This was the case for Newton's bucket, where the streamlines were concentric circles, but experience also tells us that such circulating flow may occur in the wake of the disturbance created by the body. We are thus naturally led to the study of local circulation in a fluid, and we shall see that H in fact *is* globally constant for flow completely without local circulation anywhere.

The general conditions for H being constant may be calculated from its gradient, using Euler's equation for steady incompressible flow (15-10)

$$\begin{aligned}\nabla_i H &= \frac{1}{2}\nabla_i \boldsymbol{v}^2 + \frac{1}{\rho_0}\nabla_i p + \nabla_i \Phi \\ &= \boldsymbol{v}\cdot\nabla_i \boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{\nabla} v_i \\ &= (\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v}))_i \ .\end{aligned}$$

This result, which is also valid for barotropic compressible fluids, allows us to write

$$\boxed{\boldsymbol{\nabla} H = \boldsymbol{v}\times\boldsymbol{\omega}} \ , \tag{15-27}$$

where we have defined a new field, the *vorticity field*

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v} \ , \tag{15-28}$$

also called the *curl* of the velocity field[2] The vorticity field also goes back to Cauchy (1841) and is a quantitative measure of the local circulation in the fluid.

Example 15.6.1: The curl of the field $\boldsymbol{v} = (x^2, 2xy, 0)$ is $\boldsymbol{\omega} = (2y, 0, 0)$, whereas the curl of $\boldsymbol{v} = (y^2, 2xy, 0)$ vanishes.

In any region where the vorticity field vanishes, we have $\boldsymbol{\nabla} H = 0$, so that the Bernoulli field must take one value only in that region, *i.e.* $H(\boldsymbol{x}) = H_0$. Flow completely free of vorticity everywhere is called *irrotational* flow and is particularly simple (see the following section).

[2]In the older literature $\boldsymbol{\nabla}\times\boldsymbol{v}$ is often denoted $\mathsf{curl}\,\boldsymbol{v}$ or $\mathsf{rot}\,\boldsymbol{v}$.

Vorticity and local rotation

The primary example with non-vanishing vorticity is a steadily rotating rigid body, for example Newton's bucket. If the, constant angular velocity vector is $\boldsymbol{\Omega}$, the velocity field becomes $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{x}$, and the vorticity

$$\begin{aligned}\boldsymbol{\omega} &= \boldsymbol{\nabla} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) \\ &= \boldsymbol{\Omega}(\boldsymbol{\nabla} \cdot \boldsymbol{x}) - (\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{x} \\ &= 3\boldsymbol{\Omega} - \boldsymbol{\Omega} = 2\boldsymbol{\Omega} \ .\end{aligned}$$

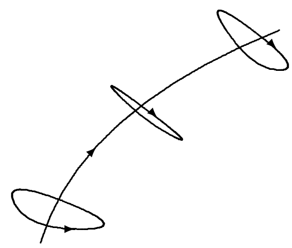
The vorticity is in this case constant and equal to twice the rotation vector. The factor of 2 seems a bit strange but is a general result which we shall return to later. It may be verified by calculating the gradient of H for Newton's bucket, and we find using (15-27)

$$\boldsymbol{\nabla} H = (\boldsymbol{\Omega} \times \boldsymbol{x}) \times 2\boldsymbol{\Omega} = 2\Omega^2(x, y, 0) \ ,$$

which agrees with the gradient of the Bernoulli field calculated from (15-25).

Vortex lines

The field lines of the vorticity field are called *vortex lines*, and are defined as curves that everywhere are tangent to the vorticity field. They are solutions to the ordinary differential equation

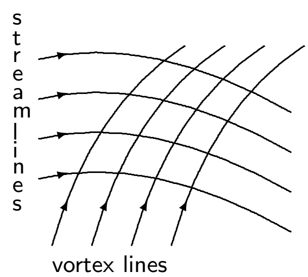


Around a vortex line there is local circulation of fluid.

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{\omega}(\boldsymbol{x}, t_0) \ . \tag{15-29}$$

In a steady flow these lines are fixed curves in space, just like the streamlines.

Bernoulli's theorem, $(\boldsymbol{v} \cdot \boldsymbol{\nabla})H = 0$, follows immediately from (15-27) by multiplying with $\boldsymbol{v}$. Similarly, by multiplying with $\boldsymbol{\omega}$ we obtain



The Bernoulli field is constant on surfaces made from vortex lines and streamlines.

$$\boxed{(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})H = 0} \ . \tag{15-30}$$

The Bernoulli field is therefore also constant along vortex lines.

Together these results imply that *the Bernoulli field is constant on the two-dimensional surfaces formed by combining vortex lines and streamlines.* Since streamlines for circulating fluid tend to form closed curves, these surfaces will form long tubes, called *vortex tubes.* For Newton's bucket the vortex tubes are cylinders concentric with the axis of rotation, and this explains that the Bernoulli field cannot depend on z, as we also noticed earlier.

∗ Equation of motion for vorticity

The vorticity field is derived from the velocity field. The equation of motion for the vorticity field must therefore follow from the equation of motion for the velocity field, (15-1). Retracing the calculation leading to (15-27), Euler's equation (for barotropic fluids) may be written,

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{v} \times \boldsymbol{\omega} - \boldsymbol{\nabla} H \ . \tag{15-31}$$

For steady flow, $\partial \boldsymbol{v}/\partial t = \boldsymbol{0}$, we recover of course (15-27). But since the curl of a gradient vanishes, we get from the curl of the above equation

$$\boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega})}$$

The main lesson to draw from this equation is that if the vorticity vanishes everywhere inside a region of space at one instant of time, then the equation guarantees that $\partial \boldsymbol{\omega}/\partial t = \boldsymbol{0}$ in that region, and it will therefore vanish forever. In the absence of viscosity, *vorticity cannot be generated by the flow of the fluid, but must be present from the outset.* Thus if you accelerate a body from rest in a truly ideal fluid, the flow will forever remain without vorticity, because $\boldsymbol{\omega} = \boldsymbol{0}$ in the beginning. The whirling air that is left by a speeding car or an airplane must somehow be caused by viscous forces, independently of how tiny the viscosity is.

15.7 Potential flow

Flow with vanishing vorticity field is called *irrotational* and obeys a much simpler formalism than flow with vorticity, in particular when it is also incompressible. Although irrotational flow allows for analytic solutions in many geometries, it is unfortunately not as useful as might be thought at first, because vorticity is always generated in the unavoidable viscous boundary layers and will from there spread to the rest of the fluid.

The flow potential

The vanishing of the vorticity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v} = 0$, everywhere in the fluid implies (see problem 15.5) that the velocity is the gradient of a scalar field, called the *flow potential* or the *velocity potential*,

$$\boxed{\boldsymbol{v} = \boldsymbol{\nabla} \psi} \ . \tag{15-32}$$

A uniform flow with velocity $\boldsymbol{U}$ has the flow potential $\psi = \boldsymbol{U} \cdot \boldsymbol{x}$, apart from an unimportant constant. If the fluid is furthermore incompressible, the vanishing of the divergence of the velocity field implies that the flow potential must satisfy Laplace's equation

$$\boxed{\boldsymbol{\nabla}^2 \psi = 0} \ . \tag{15-33}$$

The flow potential is obtained by solving this equation with suitable boundary conditions, for example that the the normal derivative $\boldsymbol{n} \cdot \boldsymbol{v} = \boldsymbol{n} \cdot \boldsymbol{\nabla} \psi$ must vanish

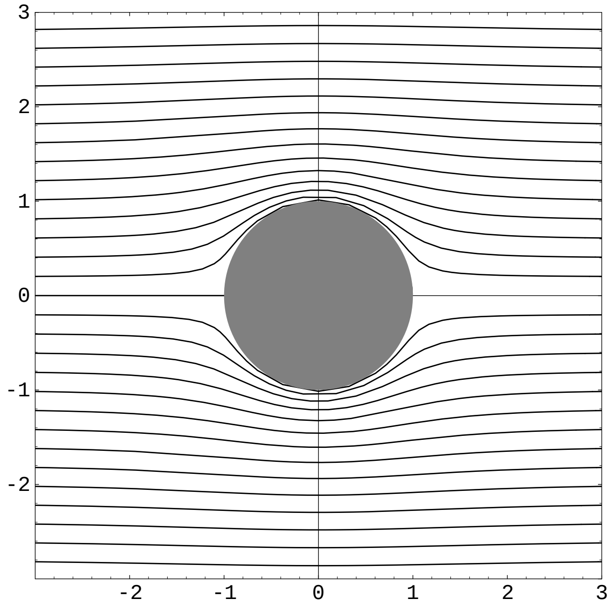


Figure 15.1: *Potential flow around the unit sphere. The streamlines have been obtained by numeric integration of the differential equation for streamlines (14-15) with the velocity field given by the solution (15-36).*

at all impermeable container walls, and that the velocity asymptotically at great spatial distances should approach a uniform field $\boldsymbol{U}$.

The Bernoulli field is a constant, $H(\boldsymbol{x}) = H_0$, in any simply connected region of space. This means that having obtained the flow potential from Laplace's equation and the boundary conditions, the effective pressure may be calculated from Bernoulli's theorem (15-14)

$$\boxed{p^* = p_0 - \frac{1}{2}\rho_0 \boldsymbol{v}^2}\,, \tag{15-34}$$

where $p_0 = \rho_0 H_0$ is a constant.

Potential flow around a sphere

Suppose a solid sphere of radius a is inserted into an ideal fluid flowing uniformly with velocity $\boldsymbol{U}$. The flow is of course disturbed by the presence of the sphere, but we expect that the velocity field at large distances $r \gg a$ from the sphere approaches the uniform field $\boldsymbol{U}$. Since the original uniform flow is manifestly irrotational, it must remain so after the insertion of the sphere.

The natural coordinates are of course spherical (page 28) with the polar axis along the asymptotic velocity $\boldsymbol{U}$. We shall not solve the problem by deduction, but just verify that the potential,

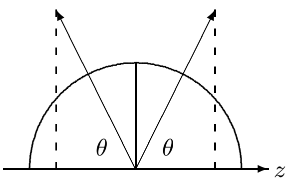
$$\psi = \left(r + \frac{a^3}{2r^2}\right) U \cos\theta \;, \tag{15-35}$$

satisfies Laplace's equation (using the spherical Laplacian (2-67)). Having done that, the velocity field may be calculated from the spherical representation of the gradient (2-64), and we find,

$$\boxed{\begin{aligned} v_r &= \left(1 - \frac{a^3}{r^3}\right) U \cos\theta \\ v_\theta &= -\left(1 + \frac{a^3}{2r^3}\right) U \sin\theta \\ v_\phi &= 0 \end{aligned}} \;. \tag{15-36}$$

To check the boundary conditions, we notice that for $r \to \infty$ the potential approaches the potential of a homogeneous field $\psi \to U r \cos\theta = \boldsymbol{U} \cdot \boldsymbol{x}$, and for $r \to a$ the radial field vanishes, as it should. The uniqueness of solutions to the Laplace equation therefore guarantees that this is the right solution. The streamlines are shown in fig. 15.1.

d'Alembert's paradox: No drag in steady potential flow



The projection of the radial pressure force on the z-axis is equal and opposite for θ and $\pi - \theta$.

The pressure on the surface of the sphere is calculated from (15-34), and becomes

$$p = p_0 + \frac{1}{2}\rho_0(\boldsymbol{U}^2 - \boldsymbol{v}^2) = p_0 + \frac{1}{2}\rho_0 U^2 \left(1 - \frac{9}{4}\sin^2\theta\right) \;. \tag{15-37}$$

Its symmetry under $\theta \to \pi - \theta$ implies that the total pressure force acting on the forward half of the sphere is equal and opposite to the pressure force acting on the rear half. A sphere in steady potential flow does not experience any resultant pressure force whatsoever, and thus no drag. The absence of drag in steady potential flow is in fact true for any other body shape (see page 457). But everyday experience tells us that a moving object is exposed to drag from the fluid that surrounds it, even if the viscosity is vanishingly small. That is d'Alembert's paradox.

The resolution of the paradox elucidates the danger in assuming potential flow. Although a tiny viscosity may not give rise to an appreciable friction force between body and fluid, it will generate vorticity close to the surface of the body. The vorticity will then spread into the fluid and produce a *trailing wake* behind the moving body, carrying a non-vanishing kinetic energy. The constant loss of kinetic energy produces a drag on the body. In potential flow around a sphere, the fluid returns to its original state with no kinetic energy after the sphere has passed, implying that there is no resultant drag. Potential flow may be a mathematically correct solution, but it misses in this case important aspects of the physics of real flow.

∗ Effective mass in unsteady potential flow

Even if no kinetic energy is lost from the fluid in steady potential flow around a sphere, there will be kinetic energy in the flow around the body. In the reference frame where the fluid is asymptotically at rest and the sphere moves with velocity $-\boldsymbol{U}$, the total kinetic energy of the sphere and fluid becomes (problem 15.7)

$$\mathcal{T} = \frac{1}{2}MU^2 + \int_{r\geq a} \frac{1}{2}\rho_0(\boldsymbol{v}-\boldsymbol{U})^2\, dV = \frac{1}{2}\left(M + \frac{2\pi}{3}a^3\rho_0\right)U^2 \ , \qquad (15\text{-}38)$$

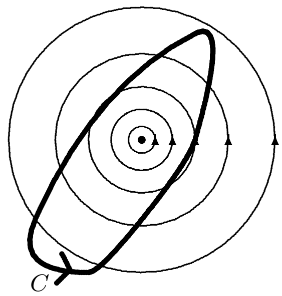
where M is the mass of the sphere. This shows that the *effective mass* of the sphere plus fluid is $M+m/2$ where $m = \frac{4\pi}{3}a^3\rho_0$ is the mass of the fluid displaced by the sphere. Apart from the factor of 1/2, Archimedes would have liked this result!

If the sphere moves steadily with constant velocity, the total kinetic energy is also constant, and no external forces need to perform any work. If the sphere is accelerated by means of an external force $\mathcal{F}$, its rate of work is $\mathcal{F}U = \dot{\mathcal{T}} = (M+m/2)U\dot{U}$. The fluid carried along with the accelerated sphere thus produces a drag of magnitude $|\mathcal{D}| = \frac{1}{2}m\left|\dot{U}\right|$.

∗ 15.8 Circulation

The vorticity field is a measure of local circulation in the fluid, and we shall now see that there also is a global measure of circulation, related to the flux of vorticity.

Streamlines may form closed curves like the circles of Newton's bucket but often they are much more complicated. To avoid the problem of streamlines we shall instead consider an arbitrary closed curve C. If it encircles a region of whirling fluid, the projection of the velocity field onto the curve will tend to be of the same sign all the way around. Formally, the *circulation* of the velocity field around a closed curve C is defined to be the integrated projection of the velocity field on the line elements of the curve,



A closed curve C encircling a whirl.

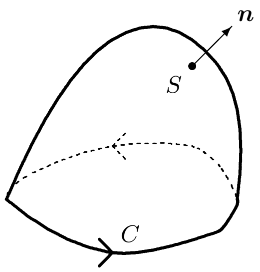
$$\boxed{\Gamma(C,t) = \oint_C \boldsymbol{v}(\boldsymbol{x},t)\cdot d\boldsymbol{\ell}} \ . \qquad (15\text{-}39)$$

Whether it is positive or negative depends on whether the curve runs with the whirling flow or against it. We emphasize that the circulation may be calculated for any curve, not just a streamline encircling a whirl, although that may be the natural thing to do.

Stokes' theorem

George Gabriel Stokes (1819–1903), British mathematician and physicist.

The most important theorem about circulation is due to Stokes. It is completely general and states that the circulation of the velocity field around a closed curve



A surface S with perimeter C. Notice how the normal to the surface is consistent with the orientation of C (here using a right hand rule).

is equal to the flux of the vorticity through any surface bounded by the curve,

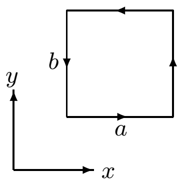
$$\boxed{\oint_C \boldsymbol{v} \cdot d\boldsymbol{\ell} = \int_S \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S}} \ . \tag{15-40}$$

It does not matter which surface S the flux is calculated for, as long as it has C as boundary, lies entirely within the fluid, and is oriented consistently with the orientation of C. Stokes' theorem is like Gauss' theorem (6-3) valid for any vector field.

Example 15.8.1: A fluid rotates like a solid body with velocity $v = \Omega r$. The circulation around a circle (also a streamline) with radius r is (in the non-rotating laboratory system) obtained by multiplying the constant velocity with the circumference of the circle,

$$\Gamma(r) = 2\pi r \ \Omega r = 2\pi\Omega r^2 = 2\Omega \ \pi r^2 \ . \tag{15-41}$$

The last expression shows that the circulation is also the product of the constant vorticity 2Ω with the area of the circle, thus confirming Stokes' theorem.

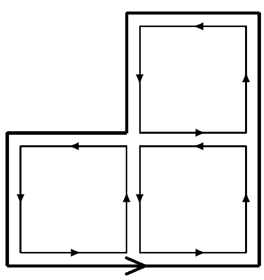


Circulation around a small rectangle of dimensions $a \times b$.

Proof of Stokes' theorem: As before the relation between global and local quantities is established by calculating the global quantity for an infinitesimal geometric figure, in this case a tiny rectangle in the xy-plane with sides a and b. To first order in the sides we find the circulation (suppressing z and t)

$$\begin{aligned}
\oint_{a\times b} \boldsymbol{v} \cdot d\boldsymbol{\ell} = & \int_x^{x+a} v_x(x', y)\, dx' + \int_y^{y+b} v_y(x+a, y')\, dy' \\
& - \int_x^{x+a} v_x(x', y+b)\, dx' - \int_y^{y+b} v_y(x, y')\, dy' \\
& \approx - \int_x^{x+a} b\, \nabla_y v_x(x', y)\, dx' + \int_y^{y+b} a\, \nabla_x v_y(x, y')\, dy' \\
& \approx ab\big(\nabla_x v_y(x, y) - \nabla_y v_x(x, y)\big) \\
& = ab\, (\boldsymbol{\nabla} \times \boldsymbol{v})_z
\end{aligned}$$

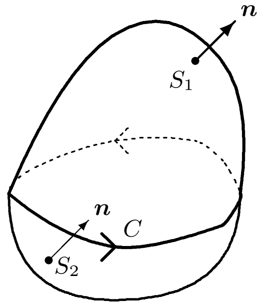
The last expression is the projection $(\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot d\boldsymbol{S}$ of the vorticity field on the small vector surface element of the rectangle, $d\boldsymbol{S} = (0, 0, ab)$. A similar result would of course have been obtained for any other orientation of the little rectangle.



Adding rectangles together, the circulation cancels along the edges where two rectangles meet. This is also valid if the rectangles bend into the other coordinate directions.

Consider now a surface built up from little rectangles of this kind. Adding together the circulation for each rectangle, the contributions from the inner common edges cancel and one is left only with the circulation around the outer perimeter of the surface. Since an arbitrary surface may be built up from infinitesimal rectangles, we conclude that the circulation around the perimeter is equal to the flux of the vorticity field through the surface, which is Stokes theorem. Since the normal to each little rectangle is oriented consistently with the way the perimeter is followed, this must also be required for the surface S.

> Stokes theorem is like Gauss' theorem a multi-dimensional version of the trivial result $\int_a^b f'(x)\,dx = f(b) - f(a)$. Where Gauss theorem was a relation between a three-dimensional volume and its surface, Stokes theorem is a relation between a two-dimensional surface and its boundary curve. Along this line one might even view the gradient integral $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\nabla} p(\boldsymbol{x}) \cdot d\boldsymbol{\ell} = p(\boldsymbol{b}) - p(\boldsymbol{a})$ as a relation between a one-dimensional line segment and its endpoints.



The flux of vorticity is the same for two surfaces with the same bounding curve. The normals to the surfaces are both consistent with the orientation of the curve.

The proof implies that the shape of the surface S does not matter. This may be explicitly verified by calculating the difference between two surfaces S_1 and S_2, both having the curve C as perimeter and oriented consistently,

$$\int_{S_1} \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S} - \int_{S_2} \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S} = \oint_S \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v})\, dV = 0 \,.$$

Here $S = S_1 - S_2$ is the closed surface formed by the two open surfaces with an extra minus sign because of the requirement that the closed surface should have an outwardly oriented normal. Gauss' theorem has been used to convert the integral over S to an integral over the volume V contained in S.

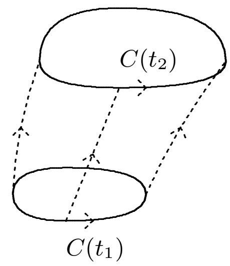
Kelvin's circulation theorem

Kelvin's famous circulation theorem from 1868 states that in an *ideal* flow the circulation around a *comoving* closed curve (also called a closed material curve) is independent of time. In other words, if $C(t)$ is a comoving closed curve, then

William Thomson (1824–1907), alias lord Kelvin, British physicist.

$$\boxed{\frac{\delta\Gamma}{\delta t} = \frac{d\Gamma(C(t), t)}{dt} = 0} \,. \tag{15-42}$$

A comoving closed curve is washed along with the fluid and may thus change shape dramatically without change in circulation. In steady flow the circulation around any fixed closed curve is like all other quantities independent of time, but the theorem concerns a curve following the material of the fluid whether the motion is steady or not.



In an ideal flow the circulation around the comoving curve $C(t)$ is the same at t_1 as at t_2.

Kelvins theorem applies only to ideal or nearly ideal flow. For a viscous fluid, the circulation will change at a rate proportional to the viscosity. Viscosity will act both as dissipator and generator of vorticity. It is as mentioned before virtually impossible to generate vorticity — or get rid of it — without the aid of viscosity.

Proof of Kelvin's theorem: The proof of the theorem is straightforward. We shall carry it through for an incompressible fluid in the absence of gravity, but it is equally valid for compressible fluids and including gravity. Let $C(t)$ be the comoving closed curve. In a small time interval dt the circulation along this curve changes by

$$
\begin{aligned}
\delta\Gamma(C(t),t) &= \Gamma(C(t+\delta t),t+\delta t) - \Gamma(C(t),t) \\
&= \oint_{C(t+\delta t)} \boldsymbol{v}(\boldsymbol{x},t+\delta t)\cdot d\boldsymbol{\ell} - \oint_{C(t)} \boldsymbol{v}(\boldsymbol{x},t)\cdot d\boldsymbol{\ell} \\
&= \oint_{C(t)} [\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{v}(\boldsymbol{x},t)\delta t, t+\delta t) - \boldsymbol{v}(\boldsymbol{x},t)]\cdot d\boldsymbol{\ell} \\
&= \delta t \oint_{C(t)} \left[\frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + (\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\boldsymbol{v}(\boldsymbol{x},t)\right]\cdot d\boldsymbol{\ell} \\
&= -\delta t \frac{1}{\rho_0} \oint_{C(t)} \boldsymbol{\nabla} p(\boldsymbol{x},t)\cdot d\boldsymbol{\ell} \\
&= 0 \ .
\end{aligned}
$$

In going to the third line we have used that the point $\boldsymbol{x}$ at time t of a comoving curve is found at $\boldsymbol{x}+\boldsymbol{v}(\boldsymbol{x},t)dt$ at time $t+dt$, and this generates the comoving derivative in the fourth line. Euler's equation (15-1) with constant density and no gravity is then used in going to the fifth line. Gravity may be included as an effective pressure (15-13) and for compressible fluids the theorem may be proven by means of the pressure function (7-3). Finally, in the last line we have used that the pressure is a single-valued function of space so that the integral of its gradient around a closed curve must vanish.

Problems

15.1 There is a small correction to the flow from the wine barrel (page 244) because the velocity of the flow does not vanish exactly on the top of the barrel. Estimate this correction from the ratio of the barrel cross section A_0 and the spout cross section A.

15.2 A wine barrel has two spouts with different cross sections A_1 and A_2. Show that under steady flow conditions the wine emerges with the same speed from the two spouts.

15.3 Assume that air is an ideal gas with constant temperature T_0. a) Calculate the relation between pressure and air velocity for a Pitot tube which is closed in one end. b) Estimate the pressure increase relative to outside pressure in the Pitot tube of a passenger jet flying at a height of 10 km with speed 250 m/s.

15.4 **Venturi tube.** Incompressible fluid runs through a tube with cross section A_1. Somewhere in the middle of the tube there is a constriction with cross section A_2. Calculate the pressure drop in the constriction.

* **15.5** Show that when $\mathrm{curl}\, \boldsymbol{v} = 0$ everywhere then there exists a potential ψ such that $\boldsymbol{v} = \boldsymbol{\nabla}\psi$.

15.6 Show that for unsteady potential flow the effective pressure becomes

$$p^* = p_0(t) - \frac{1}{2}\rho_0 \boldsymbol{v}^2 - \rho_0 \frac{\partial \psi}{\partial t} \tag{15-43}$$

where $p_0(t)$ may depend on time, but not on space.

15.7 Show that the kinetic energy of the fluid surrounding a sphere moving with velocity U in steady potential flow is $\frac{\pi}{3}\rho_0 a^3 U^2$.

15.8 Show that the vector area of a surface bounded by a closed curve C only depends on the boundary.

15.9 Show that Kelvin's theorem is valid for a compressible ideal fluid.

