25
Rotating fluids

WARNING: unfinished

The conductor of a carousel knows about fictitious forces. Moving from horse to horse while collecting tickets, he not only has to fight the centrifugal force trying to kick him off, but also has to deal with the dizzying sideways Coriolis force. On a typical carousel with a five meter radius and turning once every six seconds, the centrifugal force is strongest at the rim where it amounts to about 50% of gravity. Walking across the carousel at a normal speed of one meter per second, the conductor experiences a Coriolis force of about 20% of gravity.

Provided the carousel turns to the right, counterclockwise, as most carousels seem to do, the Coriolis force always pulls the conductor off his course to the right. The conductor seems to prefer to move from horse to horse against the rotation, and this is quite understandable, since the Coriolis force then counteracts the centrifugal force.

The whole world is a carousel, and not only in the metaphorical sense. The centrifugal force on Earth acts like a cylindrical antigravity field, reducing gravity at the equator by 1/3%. This is hardly a worry, unless you have to adjust Olympic records for geographic latitude. The Coriolis force is even less noticeable at Olympic speeds. You have to move as fast as a jet aircraft for it to amount to 1/3 of a percent of gravity.

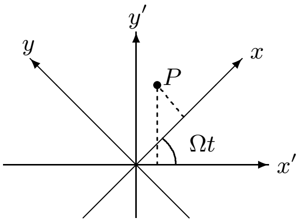
Weather systems and sea currents are so huge that the weak Coriolis force has time to become a major player in their dynamics. It is the Coriolis force which guarantees that weather cyclones on the northern half of the globe always turn anticlockwise around low pressure regions. For small scale flows, like that of a bathtub drain, its influence is tiny and cannot, as everybody can confirm in their own tub, systematically determine the sense of the flow — unless of course you happen to take your bath on a carousel with onboard facilities!

25.1 Fictitious forces

When dealing with a moving object, for example a rotating planet, it is often convenient to give up the inertial coordinate system and instead attach a fixed coordinate system to the moving object. If the object is neither accelerating nor rotating, the attached coordinate system is itself inertial, and the relativity of Newtonian mechanics then tells us that all physical laws take the same form. There is simply no way of determining the absolute state of inertial motion in Newtonian mechanics. As mentioned before, this consequence of Newton's laws was extended to a fundamental principle for all of physics by Einstein in his special theory of relativity.

Transforming to non-inertial coordinate systems, the Newtonian principle of relativity ceases to be valid, and the laws of mechanics do change their form. The general time-dependent coordinate transformation between an inertial system and a moving system changes the form of the acceleration term on the left hand side of Newton's second law. Extra terms appear in the moving system, which when shifted over to the right hand side may be interpreted as forces. Since these forces apparently have no objective cause, in contrast to forces caused by other particles, they are called *fictitious*. A better name might be *inertial* forces.

Steady rotation



Transformation of coordinates of a point particle P from a rotating to an inertial coordinate system.

In a coordinate system (x, y, z) rotating with constant angular velocity Ω around the z-axis relative to an inertial system (x', y', z'), the coordinates of a point particle are related by the transformation

$$x' = x\cos\Omega t - y\sin\Omega t \ , \tag{25-1a}$$
$$y' = x\sin\Omega t + y\cos\Omega t \ , \tag{25-1b}$$
$$z' = z \ . \tag{25-1c}$$

The transformation expresses that the position of a point particle has a geometrically invariant meaning.

Similarly, if the force acting on the point particle is $\boldsymbol{F}'$ in the inertial system, we relate it to the force $\boldsymbol{F}$ in the rotating system in the same way as above

$$F'_x = F_x\cos\Omega t - F_y\sin\Omega t \ , \tag{25-2a}$$
$$F'_y = F_x\sin\Omega t + F_y\cos\Omega t \ , \tag{25-2b}$$
$$F'_z = F_z \ . \tag{25-2c}$$

This relation expresses that force is a vector quantity. It is the only sensible way to define what we mean by force in the rotating system.

Newton's second law relates acceleration and force in the inertial system,

$md^2\boldsymbol{x}'/dt^2 = \boldsymbol{F}'$. Differentiating (25-1) twice after t we find

$$\frac{d^2x'}{dt^2} = \left(\frac{d^2x}{dt^2} - \Omega^2 x - 2\Omega\frac{dy}{dt}\right)\cos\Omega t - \left(\frac{d^2y}{dt^2} - \Omega^2 y + 2\Omega\frac{dx}{dt}\right)\sin\Omega t \ ,$$
$$\frac{d^2y'}{dt^2} = \left(\frac{d^2x}{dt^2} - \Omega^2 x - 2\Omega\frac{dy}{dt}\right)\sin\Omega t + \left(\frac{d^2y}{dt^2} - \Omega^2 y + 2\Omega\frac{dx}{dt}\right)\cos\Omega t \ ,$$
$$\frac{d^2z'}{dt^2} = \frac{d^2z}{dt^2} \ .$$

Multiplying with m and comparing with (25-2) we find the equations of motion in the rotating system

$$m\left(\frac{d^2x}{dt^2} - \Omega^2 x - 2\Omega\frac{dy}{dt}\right) = F_x \ ,$$
$$m\left(\frac{d^2y}{dt^2} - \Omega^2 y + 2\Omega\frac{dx}{dt}\right) = F_y \ ,$$
$$m\frac{d^2z}{dt^2} = F_x \ ,$$

which rearranged and put into vector notation may be written equivalently in the form

$$\boxed{m\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{F} - m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x}) - 2m\boldsymbol{\Omega}\times\frac{d\boldsymbol{x}}{dt}} \ . \tag{25-3}$$

The terms correcting the acceleration for rotation have been moved to the right hand side and now take the form of fictitious forces: the *centrifugal force* $-m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x})$ and *Coriolis force* $-2m\boldsymbol{\Omega}\times d\boldsymbol{x}/dt$.

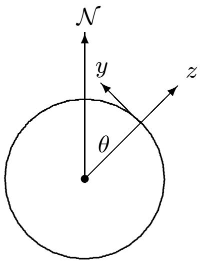
In an unsteadily moving coordinate system there are two further fictitious forces, the linear acceleration force $-m\boldsymbol{a}(t)$, where $\boldsymbol{a}(t)$ is the acceleration of the origin of the coordinate system, and the angular acceleration force $-md\boldsymbol{\Omega}/dt\times\boldsymbol{x}$. These forces are of no importance here.

Fictitious forces on Earth

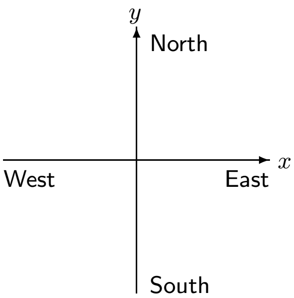
Notice that all the fictitious forces are proportional to the mass of the particle and simulate in this respect gravitational forces. Instead of fictitious force we shall in the following often speak of the equivalent fictitious or pseudo-gravitational acceleration.

The effect of the centrifugal force on Earth is primarily to flatten the spherical shape in such a way that it conforms to an equipotential surface (see chapter 7). The direction of the combined effect of gravity and centrifugal force is always orthogonal to the equipotential surface, *i.e.* vertical, everywhere on Earth. Since centrifugal acceleration is similar to gravitational acceleration, we experience only the resultant of the two.

The Coriolis force is different. It acts along different directions in different places of the Earth, because the true rotation axis is not vertical, except at the



Local coordinate system on Earth.



Flat-earth local coordinate system in a point on the surface.

poles. In a local flat-earth coordinate system tangential to the surface in a given point with the x-axis towards the east and the y-axis towards the north, the rotation vector is $\mathbf{\Omega} = \Omega\,(0, \sin\theta, \cos\theta)$ where θ is the polar angle. The components of the Coriolis acceleration $-2\mathbf{\Omega} \times \boldsymbol{v}$, expressed in terms of the velocity $\boldsymbol{v} = d\boldsymbol{x}/dt$, become

$$g_x = \quad 2\Omega\cos\theta\, v_y - 2\Omega\sin\theta\, v_z \ , \tag{25-4a}$$
$$g_y = -2\Omega\cos\theta\, v_x \ , \tag{25-4b}$$
$$g_z = \quad 2\Omega\sin\theta\, v_x \ . \tag{25-4c}$$

A good number to remember is that $2\Omega\cos\theta \approx 10^{-4}$ per second at middle latitudes ($\theta = 45°$).

The vertical Coriolis force g_z is very small compared to gravity. Even for a jet fighter plane flying on an east-west course at middle latitudes at close to sound velocity, it amounts to only about 1/3% of gravity, which accidentally is of the same order of magnitude as the centrifugal force. So we shall happily ignore the vertical Coriolis force along with the centrifugal force.

We shall also ignore the Coriolis force due to the vertical component of velocity v_z. The jet plane from before experiences again only a Coriolis acceleration in westerly direction of magnitude 1/3% of gravity during fast upwards vertical flight. For large scale systems, like weather cyclones, the horizontal motion is also by far the most important, and here the Coriolis force may over time exert a major influence, as we shall soon see.

The flat earth approximation

The result of this analysis is that only the first term in g_x and in g_y need to be considered. The Coriolis acceleration now becomes purely two-dimensional

$$g_x = +2\Omega_\perp v_y \ , \tag{25-5a}$$
$$g_y = -2\Omega_\perp v_x \ , \tag{25-5b}$$
$$g_z = 0 \ . \tag{25-5c}$$

But then the Coriolis force in a local flat-earth coordinate system looks for all practical purposes as if the Earth were indeed flat and rotated around the local vertical with a reduced effective angular velocity

$$\boxed{\Omega_\perp = \Omega\cos\theta} \ . \tag{25-6}$$

The variation of the effective rotation speed with polar angle or latitude, which can be experimentally verified by means of a Foucault pendulum, should on the other hand impress the members of the Flat Earth Society with the fallacy of their convictions.

The Equivalence Principle and Mach's principle

The similarity of fictitious and gravitational forces is part of a much deeper equivalence between gravity and accelerated motion. Einstein raised it to a *Principle of Equivalence* stating that gravity locally, *i.e.* in sufficiently small regions of space and time, in all respects is indistinguishable from accelerated motion.

But if that is the case, Mach asked, why distinguish at all? Could inertial forces also be due to subtle gravitational effects of the mass distribution of distant fixed stars? Then inertial systems would be defined to be moving with constant velocity relative to the average motion of this distribution. This would in turn imply that inertial forces on Earth could depend on the presence huge masses in our cosmic neighborhood, for example the center of the Milky Way, violating thereby the strict Equivalence Principle. No such effects have ever been observed, but the jury is still out [13].

25.2 Flow in a rotating system

The Navier-Stokes equations in a steadily rotating coordinate system are unchanged, except for inclusion of the fictitious contributions to the gravitational field. The centrifugal force resembles the Newtonian field due to mass distributions and may be written as the gradient of a potential (7-10). For an incompressible fluid it may be included together with the normal Newtonian potential in the effective pressure (15-13).

The Navier-Stokes equations for an incompressible fluid then become

$$\boxed{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}\frac{p^*}{\rho_0} + \nu\boldsymbol{\nabla}^2\boldsymbol{v} - 2\boldsymbol{\Omega}\times\boldsymbol{v}} \tag{25-7}$$

with the only significant change being the presence of the Coriolis acceleration term.

Estimates

Let us again characterize the flow by a length scale L and a velocity scale U. For nearly ideal flow with large Reynolds number $\mathsf{Re} = UL/\nu \gg 1$ the inertial term dominates the viscous term (except near boundaries), and the interesting quantity is the ratio between the inertia term and the Coriolis acceleration

$$\mathsf{Ro} = \frac{|(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}|}{|2\boldsymbol{\Omega}\times\boldsymbol{v}|} \approx \frac{U^2/L}{2\Omega U} = \frac{U}{2\Omega L} \ , \tag{25-8}$$

which is called the Rossby number. For nearly ideal flow the Coriolis force is significant only if $\mathsf{Ro} \lesssim 1$, or $U \lesssim 2\Omega L$. The flow velocity U must be of same magnitude as or smaller than the differences in rotation velocity $\pm\Omega L$ across the system.

When you swim with a speed of $U \approx 1$ m/s in a pool which is stationary on the rotating Earth, the Rossby number becomes $\mathsf{Ro} \approx 10^4$, and the Coriolis force can be completely neglected. Swimming in a pool on a carousel rotating with $\Omega \approx 1$/s, one finds $\mathsf{Ro} \approx 1$, and you probably become quite dizzy.

Similarly, the ratio between the viscosity term and the Coriolis acceleration becomes

$$\mathsf{Ek} = \frac{|\nu \boldsymbol{\nabla}^2 \boldsymbol{v}|}{|2\boldsymbol{\Omega} \times \boldsymbol{v}|} \approx \frac{\nu U/L^2}{2\Omega U} = \frac{\nu}{2\Omega L^2} = \frac{\mathsf{Ro}}{\mathsf{Re}} , \tag{25-9}$$

called the *Ekman number*. In natural systems, such as the sea and the atmosphere, with huge Reynolds numbers, the Ekman number is usually very small and has no influence on the flow, except inside boundary layers (see section 25.5) where viscosity comes to dominate over inertia.

25.3 Geostrophic flow

Ocean currents and weather cyclones are relatively steady phenomena, strongly influenced by Earth's rotation. Typical speeds are meters per second for the ocean currents, and tens of meters per second for the winds, over distances of the order of thousand kilometers. The Reynolds number comes to $\mathsf{Re} \approx 10^{12}$ in both cases. With the effective angular velocity $2\Omega \approx 10^{-4}$ per second at middle latitudes, one gets a Rossby number $\mathsf{Ro} \approx 0.01$ for ocean currents and 0.1 for weather cyclones. This is so small that one may safely disregard inertia (and of course also viscosity) in the leading approximation.

Steady flow in a rotating fluid without inertia and viscosity is called *geostrophic* flow, and obeys the field equation

$$\boxed{-\frac{1}{\rho_0}\boldsymbol{\nabla} p^* - 2\boldsymbol{\Omega} \times \boldsymbol{v} = \boldsymbol{0}} . \tag{25-10}$$

Inserting the effective pressure (15-13) this may be written

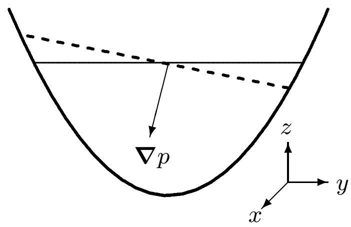
$$-\frac{1}{\rho_0}\boldsymbol{\nabla} p + \boldsymbol{g} - 2\boldsymbol{\Omega} \times \boldsymbol{v} = \boldsymbol{0} , \tag{25-11}$$

which is almost identical to the hydrostatic equation (4-8), but now includes the pseudo-gravitational Coriolis field.

Water level in an open canal

Let a constant water current with velocity U flow through a canal of width d. If the canal runs along the x-direction and the angular rotation is Ω around the z-direction, we find

$$\frac{1}{\rho_0}\frac{\partial p^*}{\partial y} = -2\Omega U . \tag{25-12}$$



The water level is tilted by the Coriolis force, so that the surface stays orthogonal to the gradient of the true pressure. Here the Earth's rotation is positive around z and water flows out of the picture in the positive x-direction.

For positive Ω and U, the Coriolis force wants to turn the water towards the right, creating an increasing effective pressure in the negative y-direction. Intuitively this seems to indicate that the Coriolis force will raise the water level for negative y and lower it for positive.

In order to check our intuition, we solve the above equation, making use of (15-13) with a gravitational potential $\Phi = g_0 z$, and get

$$\frac{p^*}{\rho_0} = \frac{p}{\rho_0} + g_0 z = -2\Omega U y \ , \tag{25-13}$$

apart from an unimportant constant. The water level $z = h(y)$ is determined from the requirement that the true pressure p should be constant at the free surface, so we get

$$h(y) = -\frac{2\Omega U}{g_0} y \ , \tag{25-14}$$

confirming our intuition. If the canal and the current run towards the north, the water level will be highest at the eastern bank.

For a 10 km wide strait and a current velocity of 1 m/s, the difference in water level at the two sides of the strait due to the Coriolis force is about 10 cm. Although $\mathsf{Ro} = U/2\Omega d \approx 1$, the use of the geostrophic equation (25-10) is nevertheless fully justified, because the inertial term vanishes for a constant flow in the x-direction (like for Poiseuille flow).

Isobaric flow and weather maps

An immediate consequence of the geostrophic equation (25-10) is that $\boldsymbol{v} \cdot \boldsymbol{\nabla} p^* = 0$ which means that the effective pressure is constant along stream lines. In horizontal motion the gravitational potential plays no role and the effective pressure is the same as the hydrostatic pressure. This means that *streamlines and isobars coincide in geostrophic flow*, as is well-known from weather maps where wind directions can be read off from the isobars. To get the correct sign for the wind direction one must also use that the winds on the northern hemisphere turn anti-clockwise around low pressure regions (cyclones) and clockwise around regions with high pressure (anti-cyclones).

Quantitatively we we may invert the geostrophic equation (25-10) to calculate the wind velocities from the pressure gradients. Using that $\boldsymbol{\Omega} \cdot \boldsymbol{v} = 0$ implies $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{v}) = -\Omega^2 \boldsymbol{v}$ we find from (25-10)

$$\boldsymbol{v} = \frac{\boldsymbol{\Omega} \times \boldsymbol{\nabla} p^*}{2\Omega^2 \rho_0} \ . \tag{25-15}$$

Were it not for the Coriolis force, air masses would according the Navier-Stokes equation (25-7) stream along the negative pressure gradient, from high pressure towards low pressure regions. On the northern hemisphere, the Coriolis force makes the wind veer to the right, tending to become orthogonal to the pressure gradient, thereby generating cyclones and anticyclones.

25.4 Two-dimensionality of geostrophic flow

The geostrophic equation (25-10) is an extremely serious constraint on the flow. Forming the scalar product with $\boldsymbol{\Omega}$ we get

$$(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})p^* = \Omega \frac{\partial p^*}{\partial z} = 0 \ . \tag{25-16}$$

The effective pressure is constant along the axis of rotation. In a constant gravitational field g_0 anti-parallel with the axis of rotation, we may use (15-13) to obtain the hydrostatic pressure

$$p = p^*(x, y) - \rho_0 g_0 z \ , \tag{25-17}$$

as we did in the case of the open canal.

The invariance under translations along the axis of rotation actually extends to the whole flow. To verify this, we calculate the curl of the geostrophic equation (25-10), using that the curl of a gradient vanishes.

$$0 = \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{v}) = \boldsymbol{\Omega}\, \mathsf{div}\, \boldsymbol{v} - (\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\Omega \frac{\partial \boldsymbol{v}}{\partial z} \ .$$

The flow field $\boldsymbol{v}$ is consequently a function of x and y only. When the z-component v_z vanishes, the flow becomes entirely two-dimensional. This result is the *Taylor-Proudman theorem* (Proudman 1916).

To the extent that weather cyclones satisfy the conditions for steady geostrophic flow, one may conclude that the same wind patterns are found all way up through the atmosphere, but here one should really take into account the change in atmospheric density with height [16].

Taylor columns

The Taylor-Proudman theorem is a strange result, which predicts that if one disturbs the flow of a rotating fluid at, say, $z = 0$, then the pattern of the disturbance will, after all time-dependence has died away, have become copied to all other values of z. This is also true if the disturbance is caused by a three-dimensional object with finite extent in the z-direction. A so-called Taylor column (of disturbed flow) is created in the rotating fluid.

Taylor columns are also called Proudman pillars.

The strangeness is however only in the mind, for many experiments beginning with Taylor's in 1923 have amply verified the existence of Taylor columns. Even a body moving along the axis of rotation, for example a falling sphere, will push a long column of fluid in front of itself, and trail another behind. The mechanics underlying the formation of Taylor columns results from a complicated interplay between the inertial and viscous terms left out in the geostrophic equation (25-10), and it is not easy to give an explanation in simple physical terms (see for example [17]).

25.5 The Ekman layer

Boundary layers arise around a body in nearly ideal flow, because viscous forces must necessarily get involved in securing the no-slip boundary condition. Steady boundary layers are asymmetric with respect to the direction of the slip-flow outside the layer. They are generally thinnest at the leading edge of a body and thicken towards the rear, and may even separate from the body and create a completely new flow pattern in the fluid at large.

Downstream thickening of a steady boundary layer, even at an otherwise featureless body surface, may be understood as a cumulative effect of the slowing down of the fluid by the contact with the boundary. Downstream from a given point the fluid has been under the influence of shear forces from the boundary for a longer time than upstream and their effect have spread farther into the fluid. The accumulation of slower fluid may however be avoided by sucking fluid through the body surface at the same constant rate everywhere (see problem 24.1).

A rotating fluid in geostrophic flow must also form boundary layers around the bodies immersed in it. For small bodies the effect of the rotation is negligible, but for larger bodies at the same scale as the geostrophic flow itself, $L \gtrsim U/\Omega$, the Coriolis force comes to play a major role in the formation of boundary layers. As the flow velocity rises from zero at the boundary to its terminal value in the geostrophic slip-flow outside the boundary, the Coriolis force becomes progressively stronger, making the flow veer more and more to the right for anti-clockwise rotation. The geostrophic cross-wind effectively 'blows away' accumulated fluid and prevents the downstream growth of the boundary layer. Such a boundary layer, confined to a finite thickness by the Coriolis force is called an *Ekman layer*.

Estimates

One may estimate the thickness of the Ekman layer by comparing the Coriolis acceleration $|2\boldsymbol{\Omega}\times\boldsymbol{v}| \approx 2\Omega U$ with the viscous acceleration $|\nu\boldsymbol{\nabla}^2\boldsymbol{v}| \approx \nu U/\delta^2$ across the Ekman layer. Since they must be approximately equal in magnitude we get (apart from a numeric factor)

$$\boxed{\delta \approx \sqrt{\frac{\nu}{\Omega}}} \,. \tag{25-18}$$

Notice that the estimate is independent of the velocity of the flow. For air at middle latitudes this leads to a thickness of a little less than a meter, and this disagrees with the measured thickness of the Ekman layer in the atmosphere which is about a thousand meters. The reason is that atmospheric flow tends not to be laminar, but turbulent with an effective viscosity that can be up to a million times larger than the diffusive viscosity. We shall not go further into this question here.

Field equations

Outside the Ekman layer we assume that there is a steady geostrophic flow in the x-direction with velocity U, accompanied by a pressure gradient $-2\Omega U$ in the y-direction as in eq. (25-12).

In looking for a solution which interpolates between the static boundary and the geostrophic flow, we again exploit the symmetry of the problem. As long as the centrifugal acceleration can be ignored, the equations of motion as well as the boundary conditions are independent of the exact position in x and y, *i.e.* invariant under arbitrary translations in these coordinates. It is then natural to guess that there may be a maximally symmetric solution $\boldsymbol{v} = \boldsymbol{v}(z)$ which is also independent of x and y and only depends on the height z.

With this assumption, mass conservation $\partial v_z/\partial z = 0$ implies that the vertical velocity is a constant. Since it has to vanish on the non-permeable boundary $z = 0$ it vanishes everywhere, $v_z = 0$. The flow in the transition layer is entirely two-dimensional, but varies with height z, and the Navier-Stokes equations become

$$
\begin{aligned}
0 &= -\frac{1}{\rho_0}\frac{\partial p^*}{\partial x} + \nu\frac{d^2 v_x}{dz^2} + 2\Omega v_y \ , \\
0 &= -\frac{1}{\rho_0}\frac{\partial p^*}{\partial y} + \nu\frac{d^2 v_y}{dz^2} - 2\Omega v_x \ , \\
0 &= -\frac{1}{\rho_0}\frac{\partial p^*}{\partial z} \ .
\end{aligned}
$$

Notice that the inertial term, like in Poiseuille flow, automatically vanishes for the maximally symmetric solution.

Solution

From the last equation it follows that the dynamic pressure is independent of height z and consequently must be everywhere equal to its terminal value $p^*/\rho_0 = -2\Omega U y$ in the geostrophic flow outside the boundary layer. The equations of motion now simplify to

$$
\begin{aligned}
\nu\frac{d^2 v_x}{dz^2} &= -2\Omega v_y \ , \\
\nu\frac{d^2 v_y}{dz^2} &= -2\Omega(U - v_x) \ .
\end{aligned}
$$

This is a pair of coupled homogenous differential equations for $U - v_x$ and v_y. Solving the first for v_y and inserting it into the second, we get a single fourth order equation

$$
\frac{d^4(U - v_x)}{dz^4} = -\frac{4\Omega^2}{\nu^2}(U - v_x) \ .
$$

The general solution is a linear combination of four terms of the form $\exp(rz\sqrt{\Omega/\nu})$ where $r = \pm(1 \pm i)$ are the four roots of $r^4 = -4$. The roots

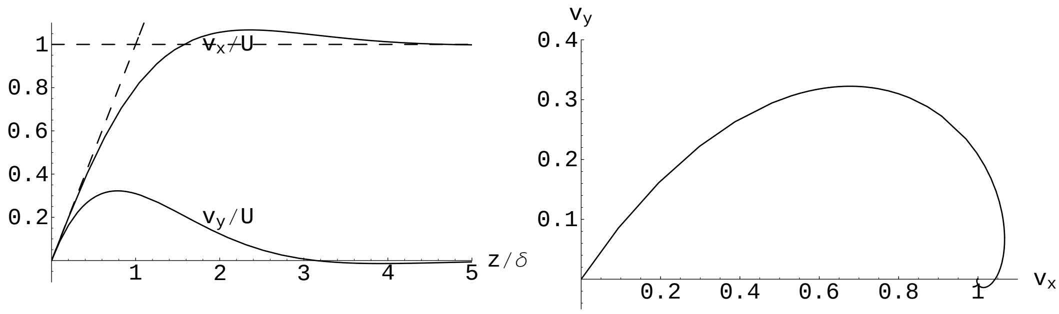


Figure 25.1: *Plot of Ekman layer velocity components. The figure on the left shows the velocity as a function of height. On the right, a parametric plot of the velocities as a function of height leads to the characteristic Ekman spiral.*

with positive real part are of no use, because the solution then would grow exponentially for $z \to \infty$, so we have

$$U - v_x = Ae^{-(1+i)z/\delta} + Be^{-(1-i)z/\delta} ,$$
$$v_y = i\left(Ae^{-(1+i)z/\delta} - Be^{-(1-i)z/\delta}\right) ,$$

where we have put

$$\boxed{\delta = \sqrt{\frac{\nu}{\Omega}}} . \tag{25-19}$$

Shape of the Ekman layer

The no-slip boundary condition, $v_x = v_y = 0$ for $z = 0$, leads to $A = B = U/2$, and the final solution becomes

$$\boxed{\begin{aligned} v_x &= U\left(1 - e^{-z/\delta}\cos z/\delta\right) \\ v_y &= Ue^{-z/\delta}\sin z/\delta \end{aligned}} \tag{25-20}$$

Near $z = 0$ we have $v_x = v_y = Uz/\delta$ so the thickness δ may as before be defined as the height of intercept of the linearly growing part of the velocity in the x-direction with the terminal velocity U.

In fig. 25.1 the velocity components are plotted as a function of height. One notices that v_x overshoots its terminal value, and then quickly returns to it. The velocity approaches the terminal velocity with damped oscillations and the y-component being 90° out of phase with the x-component. The direction of the velocity close to $z = 0$ is 45° to the left of the geostrophic flow higher up. Plotted parametrically as a function of height the velocity components create a characteristic spiral, called the *Ekman spiral.* The damping is however so strong that only the very first turn in this spiral is of any importance.

$*$ Upwelling and suction

The Ekman flow (25-20) may be rewritten in vector notation as

$$\boldsymbol{v} = \boldsymbol{U}\left(1 - e^{-z/\delta}\cos z/\delta\right) + \boldsymbol{e}_z \times \boldsymbol{U} e^{-z/\delta}\sin z/\delta \tag{25-21}$$

If the geostrophic velocity vector $\boldsymbol{U} = \boldsymbol{U}(x, y)$ changes slowly with x and y on a large scale $L \gg \delta$ this expression should still be valid. The thickness (25-19) may be taken to be constant everywhere, because it does not depend on the geostrophic flow velocity.

The slow variation of the geostrophic flow creates a slow variation of the flow in the Ekman layer. In particular, there will arise a vertical flow component, which may be calculated from mass conservation $\mathsf{div}\,\boldsymbol{v} = 0$. We find

$$\frac{\partial v_z}{\partial z} = -\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} = \left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right) e^{-z/\delta}\sin z/\delta$$

where we have used that geostrophic flow has to satisfy horizontal mass conservation $\partial U_x/\partial x + \partial U_y/\partial y = 0$. We recognize the first factor on the right hand side as the vorticity in the geostrophic flow. Integrating over z and using that $v_z = 0$ for $z = 0$ we obtain

$$v_z = \frac{1}{2}\delta\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right)\left[1 - e^{-z/\delta}(\cos z/\delta + \sin z/\delta)\right] \tag{25-22}$$

The vertical velocity is always small of magnitude $U\delta/L$, but it may have strong influence on the formation of the geostrophic flow.

Asymptotically, for $z \gg \delta$, it adds a vertical component to the geostrophic flow

$$U_z = \frac{1}{2}\delta\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right) \tag{25-23}$$

This is not at variance with the geostrophic nature of the flow, since the Taylor-Proudman theorem guarantees that it is the same at all heights, and thus cannot depend on z.

If the geostrophic vorticity is positive, *i.e.* of the same sign as the global rotation, fluid wells up from the Ekman layer. This is, for example, the case for a low-pressure cyclone, where the upwelling of fluid must be accompanied by a cross-isobaric flow inside the Ekman layer towards the center of the cyclone. Conversely if the geostrophic vorticity is negative, as in high-pressure anticyclones, fluid is sucked down into the Ekman layer from the geostrophic flow. Both these effects tend to equalize the pressure between the center and the surroundings of these vast whirls.

$*$ 25.6 Steady vortex in rotating container

In the laboratory gravity-sustained vortices may be created by letting a liquid, typically water, run freely out through a small drain-hole in the center of a slowly rotating cylindrical container (see fig. 25.2). Provided the liquid lost through the drain is constantly pumped back into the container, a steady flow will after some time become established. The dissipated kinetic energy is supplied by the pump, and the angular momentum of the rapidly spinning fluid passing through the drain is produced by the torque delivered by the motor that rotates the container. We shall refer repeatedly to the experiment shown in fig. 25.2 in the following.

Cylindrical flow in rotating frame

The physics of flow in rotating systems will be discussed in more detail in chapter 25, but we need some of the results here. In an infinite container rotating steadily with angular velocity Ω_0, we expect that the azimuthal velocity field in the laboratory far from the drain will approach a solid rotation $v_\phi \to \Omega_0 r$.

Rossby radius and Ekman flow

is of the form

$$v_\phi = \frac{C}{r} + \Omega_0 r \ , \tag{25-24}$$

in analogy with the Couette solution. There is a competition between the decreasing tail of the vortex and the rising field of solid rotation imposed by the container. The minimal velocity is found at $r = R$, where

$$R = \sqrt{\frac{C}{\Omega_0}} \tag{25-25}$$

is called the *Rossby radius*. In the experiment of fig. 25.2 it comes to $R = 2.9$ cm.

The physics of flow in rotating systems will be discussed in detail in chapter 25, but we need some of the results here. Inside the Rossby radius the flow is rather complicated, but outside it is dominated by the container's rotation, and inserting (25-24) into the azimuthal equation (21-14) we obtain $\Omega_0 v_r = 0$. This shows that a cylindrical radial inflow is simply not possible outside the Rossby radius.

But since the fluid falling out of the drain-hole must come from somewhere, it is forced (by the Coriolis force) to creep along the bottom in a thin boundary layer, called the *Ekman layer*, of thickness

$$\delta = \sqrt{\frac{\nu}{\Omega_0}} \ . \tag{25-26}$$

In the experiment of fig. 25.2, the thickness of the Ekman layer is merely $\delta \approx 0.7$ mm. The total volume of fluid passing through the Ekman layer per unit of time is calculable (see chapter 25),

$$\boxed{Q = \pi C \sqrt{\frac{\nu}{\Omega_0}}} \ . \tag{25-27}$$

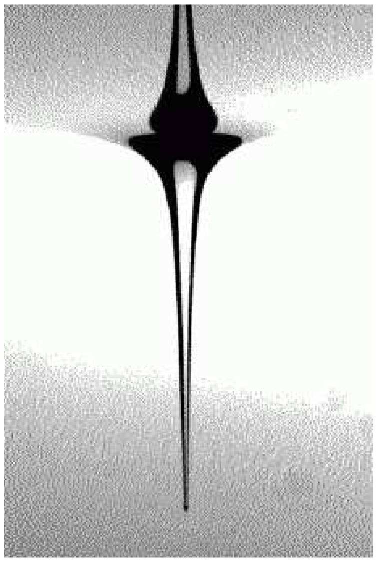


Figure 25.2: *Water vortex (and its reflection in the surface) in a container of radius* $b = 20$ cm*, rotating at* $\Omega_0 = 18$ rpm*. The radius of the drain-hole is* $a = 1$ mm *and the asymptotic water level is* $L = 11$ cm *with a central dip of* $L - h_0 = 6$ cm*, so that* $h_0 = 5$ cm*. The volume discharge through the drain is measured to be* $Q = 3.16$ cm^3 s^{-1} *corresponding to an average drain velocity of* $W = 101$ cm s^{-1} *and a Reynolds number* $\mathrm{Re}_z = 2328$*. The circulation constant is* $C = 16.0$ cm^2 s^{-1}*, and the Rossby radius* $R = 2.9$ cm*. The core rotates about 50 times per second! Picture and data from* A. Andersen, T. Bohr, B. Lautrup, J. J. Rasmussen, and B. Stenum, *Bathtub vortex flows with a free surface.*

This must be the same as the total volume discharge through the drain-hole,$Q = \pi a^2 W$, so we have now established a connection between the primary circulating flow and the secondary flow .

Conversely, we may use this relationship to determine the circulation constant C from the drain-flow, and in the experiment of fig. 25.2 we predict $C = 15$ cm^3 s^{-1} which is in decent agreement with the measured value $C = 16.0$ cm^3 s^{-1}. If at all significant, the discrepancy may be caused by a tiny secondary inflow along the open surface of the vortex.

Example 25.6.1: Although a real bathtub is normally anchored to the rotating Earth, the angular velocity of a bathtub is (at middle latitudes) $\Omega_0 = 5 \times 10^{-5}$ which is about 40,000 times smaller than the container velocity in the experiment. In a bathtub with drain radius $a = 3$ cm and water level $L = 50$ cm, the average drain velocity becomes $W = 3.1$ m s^{-1} and the drain discharge $Q \approx 9$ liter s^{-1}. Using the Ekman result we find the circulation $C = 0.02$ m^2 s^{-1} and a Rossby radius $R = 21$ m. Most bathtubs are not that big and this shows that Earth's rotation can only have little influence on a real bathtub vortex, in spite of various claims to the contrary. Unless of course the bathtub is a swimming pool of Olympic dimensions. What matters for the real bathtub vortex is, as we discussed earlier, the bather's accidental deposition of angular momentum in the water while getting out.

Modelling the Ekman upwelling

Inside the Rossby radius the slow-down in the primary flow relative to (25-24) diverts the inflow carried by the Ekman layer upwards into the bulk of the fluid (see chapter 25). The upflow from the bottom of the container turns towards the core of the vortex and is finally sucked down into the drain, on the way supplying the energy and angular momentum to sustain the vortex.

We shall model the Ekman upwelling by choosing a drain function, $w(r)$, which is a normal downwards plug-flow inside the drain combined with an upflow outside with a typical length scale R, for example

$$w(r) = \begin{cases} -W & (r < a) \\ W\dfrac{a^2}{R^2}\exp\left(-\dfrac{r^2-a^2}{R^2}\right) & (r > a) \end{cases} . \tag{25-28}$$

Other choices are also possible. The normalization of the upflow is chosen such that the total flow vanishes

$$\int_0^\infty w(r) 2\pi r \, dr = 0 \; .$$

Problems

25.1 Write the inertial coordinates in terms of the unit vectors along the axes of the moving coordinate system, $\boldsymbol{x}' = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}'_z$, and use that the time derivative of the unit vectors may always be written $\dot{\boldsymbol{e}_x} = \boldsymbol{\Omega} \times \boldsymbol{e}_x$, $\dot{\boldsymbol{e}_y} = \boldsymbol{\Omega} \times \boldsymbol{e}_y$, and $\dot{\boldsymbol{e}_z} = \boldsymbol{\Omega} \times \boldsymbol{e}_z$. Find and characterize all the fictitious forces due to rotation.

25.2 In the Danish Great Belt, which is a strait with a width of 20 km, the typical current velocity is 1 m/s, there are two layers of water, a lower and slower saline layer with a lighter brackish one on top. Assuming a density difference of about 4% and a velocity difference of about 25%, calculate the difference in water levels for the separation surface between saline and brackish water.

