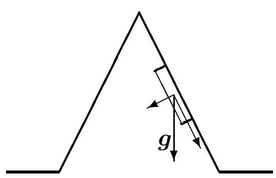
7 Hydrostatic shapes

It is primarily the interplay between gravity and contact forces that shapes the macroscopic world around us. The seas, the air, planets and stars all owe their shape to gravity, and even our own bodies bear witness to the strength of gravity at the surface of our massive planet.

What physics principles determine the shape of the surface of the sea? The sea is obviously horizontal at short distances, but bends below the horizon at larger distances following the planet's curvature. The Earth as a whole is spherical and so is the sea, but that is only the first approximation. The Moon's gravity tugs at the water in the seas and raises tides, and even the massive Earth itself is flattened by the centrifugal forces of its own rotation.

Disregarding surface tension, the simple answer is that in hydrostatic equilibrium with gravity, an interface between two fluids of different densities, for example the sea and the atmosphere, must coincide with a surface of constant potential, an equipotential surface. Otherwise, if an interface crosses an equipotential surface, there will arise a tangential component of gravity which can only be balanced by shear contact forces that a fluid at rest is unable to supply. An iceberg rising out of the sea does not obey this principle because it is solid, not fluid. But if you try to build a little local "waterberg", it quickly subsides back into the sea again, conforming to an equipotential surface.



A little "waterberg" in the sea. The tangential component of gravity requires a shear contact force to balance it, whereas the normal component can be balanced by the pressure force.

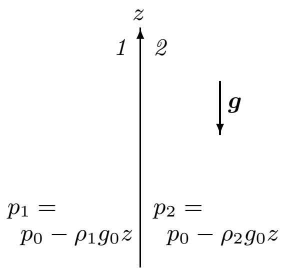
Hydrostatic balance in a gravitational field also implies that surfaces of constant pressure, isobars, must coincide with the equipotential surfaces. When surface tension plays a role, as it does for a drop of water hanging at the tip of an icicle, this is no more true, and the shape bears little relation to equipotential surfaces. Likewise for fluids in motion. Waves in the sea are "waterbergs" that normally move along the surface, but under special circumstances they are able stay in one place, as for example in a river flowing past a big stone.

 Draft 7.4, August 16, 2002

7.1 Fluid interfaces in hydrostatic equilibrium

The intuitive argument about the impossibility of hydrostatic "waterbergs" must in fact follow from the equations of hydrostatic equilibrium. We shall now show that hydrostatic equilibrium implies that the interface between two fluids with different densities ρ_1 and ρ_2 must be an equipotential surface.

Since the gravitational field is the same on both sides of the interface, hydrostatic balance $\boldsymbol{\nabla} p = \rho \boldsymbol{g}$ implies that there is a jump in the pressure gradient across the interface, because on one side $(\boldsymbol{\nabla} p)_1 = \rho_1 \boldsymbol{g}$ and on the other $(\boldsymbol{\nabla} p)_2 = \rho_2 \boldsymbol{g}$. If the field of gravity has a component tangential to the interface, there will consequently be a jump in the tangential pressure gradient. The difference in tangential gradients on the two sides of the interface implies that even if the pressures are equal in one point, they must be different a little distance away along the surface. Newton's third law, however, requires pressure to be continuous everywhere, also across an interface (as long as there is no surface tension), so this problem can only be avoided if the tangential component of gravity vanishes everywhere at the interface, implying that it is an equipotential surface.



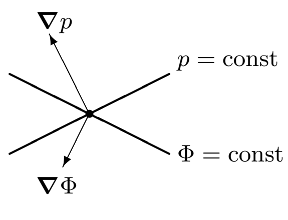
An impossible vertical interface between two fluids at rest with different densities. Even if the hydrostatic pressures on the two sides are the same for $z = 0$ they will be different everywhere else.

If, on the other hand, the fluid densities are the exactly same on both sides of the interface but the fluids themselves are different, the interface is not forced to follow an equipotential surface. This is, however, an unusual and highly unstable situation. The smallest fluctuation in density will call gravity in to make the interface horizontal. Stable vertical interfaces between fluids are simply not seen.

At an interface between fluids that are not in hydrostatic equilibrium — think of a river running down a mountain — internal friction in the water provides the shear forces necessary for establishing a non-horizontal open water surface.

Isobars and equipotential surfaces

Surfaces of constant pressure, satisfying $p(\boldsymbol{x}) = p_0$, are called *isobars*. Through every point of space runs one and only one isobar, namely the one corresponding to the pressure in that point. The gradient of the pressure is everywhere normal to the local isobar. Gravity, $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, is likewise everywhere normal to the local equipotential surface, defined by $\Phi(\boldsymbol{x}) = \Phi_0$.



If isobars and equipotential surfaces cross, hydrostatic balance $\boldsymbol{\nabla} p + \rho\boldsymbol{\nabla}\Phi = 0$ becomes impossible.

Local hydrostatic equilibrium, $\boldsymbol{\nabla} p = \rho\boldsymbol{g}$, tells us that the normal to the isobar is everywhere parallel with the normal to the equipotential surface. This can only be the case if *isobars coincide with equipotential surfaces in hydrostatic equilibrium*. For if an isobar crossed an equipotential surface anywhere at a finite angle the two normals could not be parallel.

Pressure and head

In a fluid with constant density, ρ_0, the hydrostatic equation $\boldsymbol{\nabla} p + \rho_0\boldsymbol{\nabla}\Phi = \boldsymbol{\nabla}(p + \rho_0\Phi) = 0$ implies that $p + \rho_0\Phi$ is a constant independent of $\boldsymbol{x}$. It is customary (and useful) to divide with the density and define

$$\boxed{H = \frac{p}{\rho_0} + \Phi} \ , \tag{7-1}$$

a quantity that will appear in many places in the following. Since Φ is the potential energy of a unit mass, H is of the same kind. In constant gravity where $\Phi = g_0 z$, and the quantity $H/g_0 = z + p/\rho_0 g_0$ becomes the sum of the position height z and the height of the equivalent column of fluid, $h = p/\rho_0 g_0$, which would give rise to the pressure p. In the technical literature this height is called the *pressure head* and H/g_0 is called the *total head*. In thermodynamic context H corresponds to the *total enthalpy* (see refs. [29] and [9]).

Example 7.1.1: In the incompressible sea, pressure is given by (4-2) and we find

$$H = \frac{p_0}{\rho_0} \ , \tag{7-2}$$

where p_0 is atmospheric pressure. The total head is indeed constant (*i.e.* independent of z) and equals the equivalent height of a water column that would provide atmospheric pressure ($H/g_0 \approx 10$ m).

Pressure function for compressible fluids

Even for compressible fluids we may integrate the equation of hydrostatic equilibrium, provided the material obeys a barotropic equation of state of the form $\rho = \rho(p)$. Defining the *pressure function*

$$w(p) = \int \frac{dp}{\rho(p)} \ , \tag{7-3}$$

its gradient becomes

$$\nabla w(p) = \nabla p \, \frac{dw(p)}{dp} = \frac{\nabla p}{\rho(p)} \ .$$

Since $\nabla p = -\rho \nabla \Phi$ in hydrostatic equilibrium, it follows that

$$\boxed{H = w(p) + \Phi} \tag{7-4}$$

satisfies $\nabla H = 0$ and is thus a constant independent of $\boldsymbol{x}$. The constancy of H also confirms the previous conclusion that equipotential surfaces (constant Φ) must coincide with isobars (constant p).

Example 7.1.2: The ideal gas law (4-20) leads to the pressure function

$$w = \int \frac{RT_0}{M_{\rm mol}} \frac{dp}{p} = \frac{RT_0}{M_{\rm mol}} \log \frac{p}{p_0} \ , \tag{7-5}$$

where p_0 is an undetermined constant.

In constant gravity, the pressure is the solution to the equation

$$w(p_0) = w(p) + g_0 z \; . \tag{7-6}$$

Here the constant $H = w(p_0)$ has been expressed in terms of the pressure p_0 at $z = 0$. In agreement with our conclusion on page 57, the pressure can only depend on z, and one may verify that solving this equation indeed leads to the expressions found in chapter 4 for the pressure in the sea and the atmosphere under various conditions(problem 7.2).

∗ The Münchausen effect

The question of what happens to buoyancy when the gravitational field of a body itself is taken into account, can now be answered. One may think of an Earth-sized planet floating in the atmosphere of Jupiter. Let the body's own gravitational potential be $\Phi_1(\boldsymbol{x})$ and let for simplicity the surrounding fluid be in equilibrium with a constant external gravitational field, g_0, and a pressure $p(z)$ which is the solution to (7-6).

The presence of the body will generate a pressure change, $\Delta p = \Delta p(\boldsymbol{x})$, satisfying

$$w(p_0) = w(p + \Delta p) + g_0 z + \Phi_1(\boldsymbol{x}) \; . \tag{7-7}$$

Since Φ_1 vanishes for $|\boldsymbol{x}| \to \infty$, the pressure p_0 is the pressure at $z = 0$ at great distance from the body. Assuming that the pressure change Δp is small relative to p and using that $dw/dp = 1/\rho$, we may expand to first order in Δp and obtain

$$w(p_0) = w(p) + \frac{\Delta p}{\rho} + g_0 z + \Phi_1(\boldsymbol{x})$$

Using (7-6) we get

$$\Delta p(\boldsymbol{x}) = -\rho(z)\Phi_1(\boldsymbol{x}) \; , \tag{7-8}$$

where $\rho(z)$ is the density of the fluid in the absence of the body. Since $\Phi_1 < 0$, the pressure change is positive everywhere in the vicinity of the body. This is also expected for physical reasons, because the body's gravity pulls in and compresses the surrounding fluid. The net change in buoyancy may, however, be of both signs, depending on the body's internal mass distribution (which determines Φ_1).

If the body is a uniform sphere with radius a, the potential is constant at the surface, $\Phi_1 = -g_1 a$, where g_1 is the surface gravity. Then $\Delta p = \rho(z) g_1 a$, and since $\rho(z)$ is larger for negative z than for positive because the fluid is compressed by the ambient gravity, g_0, the pressure change is larger below the sphere than above it. The net effect is therefore an increase in buoyancy. The body in a sense "lifts itself by its bootstraps" by means of its own gravity, and it is quite appropriate to call this phenomenon the Münchausen effect.

Freiherr Karl Friedrich Hieronymus von Münchhausen (1720-1797). *German (Hanoveran) soldier, hunter, nobleman, and delightful story-teller. The stories of his travels to Russia were retold and further embroidered by others and published as "The Adventures of Baron Munchausen" in 1793. In one of these, he lifts himself (and his horse) out of deep snow by his bootstraps. Incidentally, this story is also the origin of the expression "bootstrapping" (or more recently just "booting") a computer.*

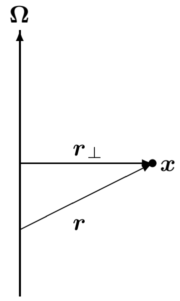
7.2 Shape of rotating fluids

Newton's second law of motion is only valid in *inertial* coordinate systems, where free particles move on straight lines with constant velocity. We shall not go into the delicate questions about what is really meant by a "free" particle and the related concept of force.

In rotating, or otherwise accelerated, non-inertial coordinate systems, one may formally write the equation of motion in their usual form, but the price to be paid is the inclusion of certain force-like terms that do not have any obvious connection with material bodies, but derive from the overall motion of the coordinate system (see chapter 25 for a more detailed analysis). Such terms are called *fictitious forces*, although they are by no means fictional, as one becomes painfully aware when standing up in a bus that suddenly stops. A more reasonable name might be *inertial* forces, since they arise as a consequence of the inertia of massive bodies.

Antigravity of rotation

A material particle at rest in a coordinate system rotating with constant angular velocity Ω in relation to an inertial system will experience only one fictitious force, the *centrifugal force*. We all know it from carroussels. It is directed perpendicularly outwards from the axis of rotation and of magnitude $dm\, r_\perp \Omega^2$, where $r_\perp$ is the shortest distance to the axis.



The geometry of a rotating system is characterized by a rotation vector along the axis of rotation with magnitude equal to the angular rate of rotation. The vector $\boldsymbol{r}_\perp$ is the radius vector directed orthogonally out from the axis to a point $\boldsymbol{x}$.

Let $\boldsymbol{r}$ be the radius vector from a point on the axis of rotation to the point in question, and let $\boldsymbol{r}_\perp$ be the shortest of such vectors to the point. In a rotating coordinate system placed with its origin on the rotation axis, and z-axis along it, we have $\boldsymbol{r}_\perp = (x, y, 0)$. The centrifugal force is proportional to the mass of the particle and thus mimics a gravitational field

$$\boldsymbol{g}_{\text{centrifugal}}(\boldsymbol{r}) = \boldsymbol{r}_\perp \Omega^2 = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \; . \tag{7-9}$$

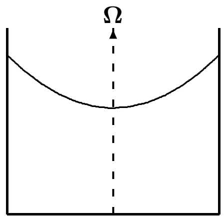
The last expression follows from the simple vector identity $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{r}\Omega^2 - \boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{\Omega})$. This fictitious gravitational field may be derived from a (fictitious) potential

$$\Phi_{\text{centrifugal}}(\boldsymbol{r}) = -\frac{1}{2} r_\perp^2 \Omega^2 = -\frac{1}{2}(\boldsymbol{\Omega} \times \boldsymbol{r})^2 = -\frac{1}{2}\Omega^2(x^2 + y^2) \; . \tag{7-10}$$

Since the centrifugal field is directed away from the axis of rotation the centrifugal field is a kind of *antigravity* field, which will try to split things apart and lift objects off a rotating planet. The antigravity field of rotation is, however, cylindrical in shape rather than spherical and has consequently the greatest influence at the equator of Earth. If our planet rotated once in a little less than $1\frac{1}{2}$ hours, people at the equator could actually levitate!

Newton's bucket

A bucket of water on a rotating plate is an example going right back to Newton himself. Internal friction (viscosity) in the water will after some time bring it to rest relative to the bucket and plate, and the whole thing will end up rotating as a solid body. In a rotating coordinate system with z-axis along the axis of rotation, the total gravitational field becomes $\boldsymbol{g} = (\Omega^2 x, \Omega^2 y, -g_0)$, including both "real" gravity and the "fictitious" centrifugal force. Correspondingly, the total gravitational potential becomes



Water surface in rotating bucket

$$\Phi = g_0 z - \frac{1}{2}\Omega^2(x^2 + y^2) \ . \tag{7-11}$$

The equipotential surfaces are in this case rotation paraboloids

$$z = z_0 + \frac{\Omega^2}{2g_0}(x^2 + y^2) \ . \tag{7-12}$$

In a bucket of diameter 20 cm at one revolution per second the water stands 2 cm higher at the rim than in the center. The same phenomenon may be observed when one stirs a pot of soup with a ladle. There internal friction in the fluid in combination with the stationary walls of the pot will of course slow down the soup and bring it to rest with horizontal surface once the stirring has stopped.

The pressure at the bottom ($z = 0$) becomes

$$p = p_0 - \rho_0 \Phi = p_0 + \frac{1}{2}\rho_0 \Omega^2 \left(x^2 + y^2\right) \tag{7-13}$$

where p_0 is the bottom pressure at the rotation axis. It grows towards the rim, reflecting everywhere the change in height of the water column.

The centrifuge

In a centrifuge the speed of rotation is so large that ordinary gravity may be completely disregarded. The fluid will then form a layer on the inside of the container parallel with the axis of rotation and, for an incompressible fluid, a hydrostatic pressure growing quadratically with the distance from the axis.

Example 7.2.1:
In an ultracentrifuge of radius 10 cm containing water and rotating at 60,000 rpm, the centrifugal acceleration becomes $400,000\ g_0$ and the maximal pressure 2 kBar, which is the double of the pressure at the bottom of the deepest abyss in the sea. At such pressures, the change in water density is about 10%.

∗ Stability of rotating bodies

From the fundamental field equation (6-5) we find by including the centrifugal field (7-9) that the divergence of the total acceleration field $\boldsymbol{g} = \boldsymbol{g}_{\text{gravity}} + \boldsymbol{g}_{\text{centrifugal}}$ becomes

$$\mathsf{div}\, \boldsymbol{g} = -4\pi G\rho + 2\Omega^2 \ . \tag{7-14}$$

Effectively, centrifugal forces create a negative mass density $-\Omega^2/2\pi G$. This is, of course, a purely formal result, but it nevertheless confirms the "antigravity" aspect of centrifugal forces.

For a body with constant mass density, ρ_0, the divergence becomes everywhere positive for $\Omega^2 > 2\pi G\rho_0$, which tells us that the effective gravitational field has become repulsive. Unless the material of the body is sufficiently cohesive, it will fly apart. The stability requirement $\Omega^2/2\pi G\rho_0 < 1$ is an absolute upper limit for planets and stars. For particular bodies, much stricter stability conditions may be derived. For the case of a spherical planet, the centrifugal force at the equator must be smaller than gravity, or $\Omega^2 a < GM_0/a^2$, which is equivalent to the stricter condition $\Omega^2/2\pi G\rho_0 < 2/3$. At the end of section 7.4 the influence of the deformation due to rotation is also taken into account.

Johannes Kepler, German mathematician and astronomer (1580–1635). Discovered that planets move in elliptical orbits and that their motion obeys mathematical laws.

7.3 The Earth, the Moon and the tides

Kepler thought that the Moon would influence the waters of Earth and raise tides, but Galilei found this notion of Kepler's completely crazy and compared it to common superstition. After Newton we know that the Moon's gravity acts on everything on Earth, in particular on the water in the sea, and attempts to pull it out of shape, thereby creating the tides. Since high tides occur roughly at the same time at antipodal points of the Earth, the explanation is not simply that the Moon lifts the sea towards itself but a little more sophisticated.

Galileo Galilei (1564–1642). Italian natural philosopher, astronomer, and mathematician. Considered the father of the modern scientific method. He wrote about Kepler: "But among all the great men who have philosophized about this remarkable effect, I am more astonished at Kepler than at any other. Despite his open and acute mind, and though he has at his fingertips the motions attributed to the earth, he has nevertheless lent his ear and his assent to the moon's dominion over the waters, and to occult properties, and to such puerilities." (see [24, p. 145])

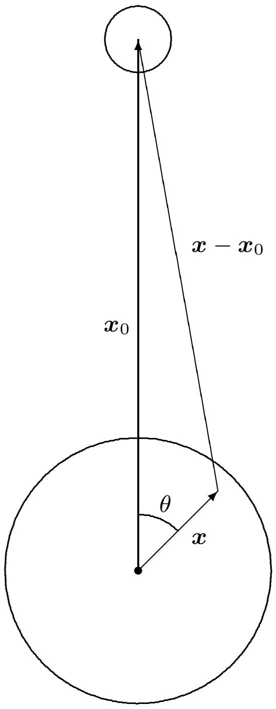
The best natural scientists and mathematicians of the eighteenth and nineteenth centuries worked on the dynamics of the tides, but here we shall only consider the simplest possible case of a quasistatic Moon. For a full discussion, including the dynamics of tidal waves, see for example Sir Horace Lamb's classical book [8] or [23] for a modern account .

The Earth

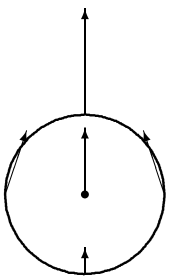
We shall limit ourselves to study the Moon's influence on a liquid surface layer of the Earth. The solid parts of the Earth will of course also react to the Moon's field, but the effects are somewhat smaller and are due to elastic deformation rather than flow. This deformation has been indirectly measured to a precision of a few percent in the daily 0.1 ppm variations in the strength of gravity [18]. There are also tidal effects in the atmosphere, but they are dominated by other atmospheric motions.

We shall furthermore disregard the changes to the Earth's own gravitational potential due to the shifting waters of the tides themselves, as well as the centrifugal antigravity of Earth's rotation causing it to deviate from a perfect sphere (which increases the tidal range by slightly more than 10 %, see section 7.4). Under all these assumptions the gravitational potential at a height h over the surface of the Earth is to first order in h given by

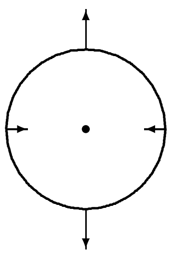
$$\boxed{\Phi_{\text{Earth}} = g_0 h} \,, \qquad (7\text{-}15)$$



Geometry of the Earth and the Moon (not to scale)



How the Moon's gravity varies over the Earth (exaggerated).



Moon's gravity with common acceleration cancelled. This also explains why the variations in tidal height have a semi-diurnal period.

where g_0 is the magnitude of the surface gravity.

The Moon

The Moon is not quite spherical, but nevertheless so small and far away that we may approximate its potential across the Earth with that of a point particle $-Gm/|\boldsymbol{x} - \boldsymbol{x}_0|$ situated at the Moon's position $\boldsymbol{x}_0$ with the Moon's mass m. Choosing a coordinate system with the origin at the center of the Earth and the z-axis in the direction of the Moon, we have $\boldsymbol{x}_0 = (0, 0, D)$ where $D = |\boldsymbol{x}_0|$ is the Moon's distance. Since the Moon is approximately 60 Earth radii (a) away, *i.e.* $D \approx 60a$, the Moon's potential across the Earth (for $r = |\boldsymbol{x}| \leq a$) may conveniently be expanded in powers of $\boldsymbol{x}/D$, and we find to second order

$$\begin{aligned}
\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|} &= \frac{1}{\sqrt{x^2 + y^2 + (z - D)^2}} = \frac{1}{\sqrt{D^2 - 2zD + r^2}} \\
&= \frac{1}{D} \frac{1}{\sqrt{1 - \dfrac{2z}{D} + \dfrac{r^2}{D^2}}} \\
&\approx \frac{1}{D} \left(1 - \frac{1}{2} \left(-\frac{2z}{D} + \frac{r^2}{D^2} \right) + \frac{3}{8} \left(-\frac{2z}{D} \right)^2 \right) \\
&= \frac{1}{D} \left(1 + \frac{z}{D} + \frac{3z^2 - r^2}{2D^2} \right) .
\end{aligned}$$

The first term in this expression leads to a constant potential $-Gm/D$, which may of course be ignored. The second term corresponds to a constant gravitational field in the direction towards the moon $g_z = Gm/D^2 \approx 30\ \mu\text{m/s}^2$, which is precisely cancelled by the centrifugal force due to the Earth's motion around the common center-of-mass of the Earth-Moon system (an effect we shall return to below). Spaceship Earth is therefore completely unaware of the two leading terms in the Moon's potential, and these terms cannot raise the tides. Galilei was right to leading non-trivial order, and that's actually not so bad.

Tidal effects come from the *variation* in the gravitational field across the Earth, to leading order given by the third term in the expansion of the potential. Introducing the angle θ between the direction to the Moon and the observation point on Earth, we have $z = r\cos\theta$, and the Moon's potential becomes (after dropping the two first terms)

$$\boxed{\Phi_{\text{Moon}} = -\frac{1}{2}(3\cos^2\theta - 1)\left(\frac{r}{D}\right)^2 \frac{Gm}{D}} \ . \tag{7-16}$$

This expansion may of course be continued indefinitely to higher powers of r/D. The coefficients $P_n(\cos\theta)$ are called Legendre polynomials (here $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$).

The gravitational field of the Moon is found from the gradient of the potential. It is simplest to convert to Cartesian coordinates, writing $(3\cos^2\theta - 1)r^2 =$

$2z^2 - x^2 - y^2$, before calculating the gradient. In the xz-plane we get at the surface $r = a$

$$\boldsymbol{g}_{\text{Moon}} = (-\sin\theta, 0, 2\cos\theta)\frac{aGm}{D^3} .$$

Projecting on the local normal $\boldsymbol{e}_r = (\sin\theta, 0, \cos\theta)$ and tangent $\boldsymbol{e}_\theta = (\cos\theta, 0, -\sin\theta)$ to the Earth's surface, we finally obtain the vertical and horizontal components of the gravitational field of the Moon at any point of the Earth's surface

$$g_{\text{Moon}}^{\text{vert}} = \boldsymbol{g}_{\text{Moon}} \cdot \boldsymbol{e}_r = (3\cos^2\theta - 1)\,\frac{aGm}{D^3} , \tag{7-17}$$

$$g_{\text{Moon}}^{\text{horiz}} = \boldsymbol{g}_{\text{Moon}} \cdot \boldsymbol{e}_\theta = -\sin 2\theta\,\frac{3aGm}{2D^3} . \tag{7-18}$$

The magnitude of the horizontal component is maximal for $\theta = 45^o$ (and of course also 135^o because of symmetry).

Concluding, we repeat that tide-generating forces arise from variations in the Moon's gravity across the Earth. As we have just seen, the force is generally not vertical, but has a horizontal component of the same magnitude. From the sign and shape of the potential as a function of angle, we see that effectively the Moon lowers the gravitational potential just below its position, and at the antipodal point on the opposite side of the Earth, exactly as if there were shallow "valleys" at these places. Sometimes these places are called the Moon and anti-Moon positions.

And the tides

If the Earth did not rotate and the Moon stood still above a particular spot, water would rush in to fill up these "valleys", and the sea would come to equilibrium with its open surface at constant total gravitational potential. The total potential near the surface of the Earth is

$$\Phi = \Phi_{\text{Earth}} + \Phi_{\text{Moon}} = g_0 h - \frac{1}{2}(3\cos^2\theta - 1)\left(\frac{a}{D}\right)^2 \frac{Gm}{D} , \tag{7-19}$$

Requiring this potential to be constant we find the tidal height

$$\boxed{h = h_0 + \frac{1}{2}(3\cos^2\theta - 1)\left(\frac{a}{D}\right)^2 \frac{Gm}{g_0 D}} , \tag{7-20}$$

where h_0 is a constant, equal to the average water depth. This follows because the average over the sphere of $3\cos^2\theta - 1$ vanishes

$$\frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} \sin\theta\, d\phi\,(3\cos^2\theta - 1) = \frac{1}{2}\int_{-1}^{+1}(3z^2 - 1)dz = 0 .$$

The maximal difference between high and low tides, called the tidal range, occurs between the extreme positions at $\theta = 0$ and $\theta = 90°$,

$$\boxed{H_0 = \frac{3}{2}\left(\frac{a}{D}\right)^2 \frac{Gm}{g_0 D} = \frac{3}{2} a \frac{m}{M}\left(\frac{a}{D}\right)^3}, \qquad (7\text{-}21)$$

where the last equation is obtained using $g_0 = GM/a^2$ with M being the Earth's mass.

Inserting the values for the Moon we get $H_0 \approx 54$ cm. Interestingly, the range of the tides due to the Sun turns out to be half as large, about 25 cm. This makes spring tides when the Sun and the Moon cooperate almost three times higher than neap tides when they don't.

In order for the tides to reach full height, water must move in from huge areas of the Earth as is evident from the shallow shape of the potential. Where this is not possible, for example in lakes and enclosed seas, the tidal range becomes much smaller than in the open oceans. Local geography may also influence tides. In bays and river bores, for example, funnelling can cause tides to build up to huge values. Spring tides in the range of 15 meters have been measured in the Bay of Fundy in Canada.

∗ Quasistatic tidal cycles

The rotation of the Earth cannot be neglected. If the Earth did not rotate, or if the Moon were in a geostationary orbit, it would be much harder to observe the tides, although they would of course be there (problem 7.9). It is, after all, the cyclic variation in the water level observed at the coasts of seas and large lakes, which makes the tides observable. Since the axis of rotation of the Earth is neither aligned with the direction to the Moon nor orthogonal to it, the tidal forces acquire a diurnal cycle superimposed on the 'natural' semidiurnal one (see Fig. 7.1).

For a fixed position on the surface of the Earth, the dominant variation in the lunar zenith angle θ is due to Earth's diurnal rotation with angular rate $\Omega = 2\pi/24$ hours $\approx 7 \times 10^{-5}$ radians per second. On top of that, there are many other sources of periodic variations in the lunar angle [23], which we shall ignore here.

The dominant such source is the lunar orbital period of a little less than a month. Furthermore, the orbital plane of the Moon inclines about 5° with respect to the ecliptic (the orbital plane of the Earth around the Sun), and precesses with this inclination around the ecliptic in a little less than 19 years. The Earth's equator is itself inclined about 23° to the ecliptic, and precesses around it in about 25,000 years. Due to lunar orbit precession, the angle between the equatorial plane of the Earth and the plane of the lunar orbit will range over $23 \pm 5°$, *i.e.* between 18° and 28°, in about 9 years.

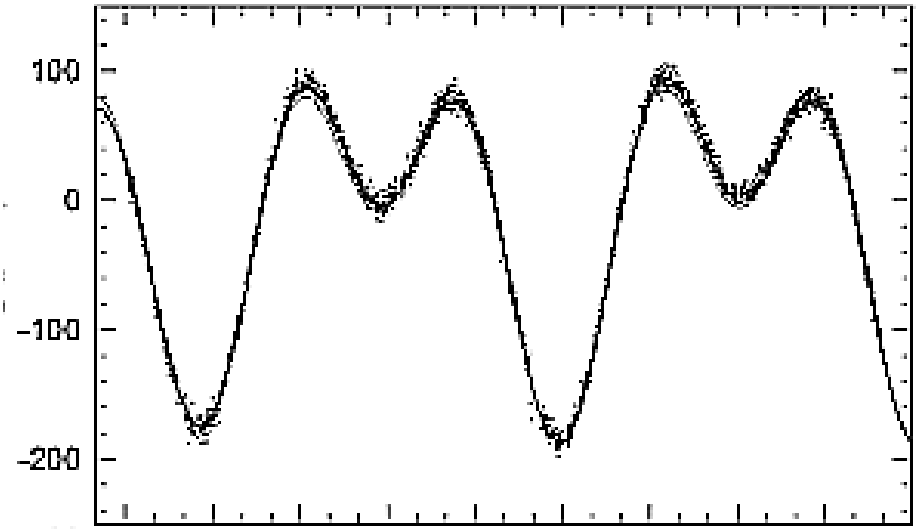
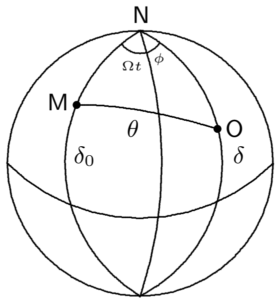


Figure 7.1: *Variation in the gravitational acceleration over a period of 56 hours in units of* $10^{-9}g_0$ *(measured in Stanford, California, on Dec. 8–9, 1996 [18], reproduced here with the permission of the authors (to be obtained)). The semidiurnal as well as diurnal tidal variations are prominently visible as dips in the curves. Modelling the Earth as a solid elastic object and taking into account the effects of ocean loading, the measured data is reproduced to within a few times* $10^{-9}g_0$.



Spherical triangle formed by Moon (M), observer (O) and north pole (N).

Let the fixed observer position at the surface of the Earth have (easterly) longitude ϕ and (northerly) latitude δ. The lunar angle θ is then calculated from the spherical triangle NMO formed by the north pole, the lunar position and the observer's position,

$$\cos\theta = \sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos(\Omega t + \phi) \ . \tag{7-22}$$

Here δ_0 is the latitude of the lunar position and the origin of time has been chosen such that the Moon at $t = 0$ is directly above the meridian $\phi = 0$.

Inserting this into the static expression for the tidal height (7-20), we obtain the quasistatic height variation with time at the observer's place, which becomes the sum of a diurnal and a semidiurnal cycle

$$\boxed{h = h_0 + \frac{1}{2}h_1\cos(\Omega t + \phi) + \frac{1}{2}h_2\cos 2(\Omega t + \phi)} \ . \tag{7-23}$$

Here h_0 is the average height, and $h_1 = H_0\sin 2\delta\sin 2\delta_0$ and $h_2 = H_0\cos^2\delta\cos^2\delta_0$ are respectively the diurnal and semidiurnal tidal amplitudes. The full tidal range is not $h_1 + h_2$, but somewhat smaller, because the two cosines cannot simultaneously take the value -1 (see problem 7.10).

To go beyond the quasistatic approximation, the full theory of fluid dynamics on a rotating planet becomes necessary. The tides will then be controlled not only by the tide-generating forces, but also by the interplay between the inertia of the moving water and friction forces opposing the motion. High tides will no more be tied to the Moon's instantaneous position, but may both be delayed and advanced relative to it.

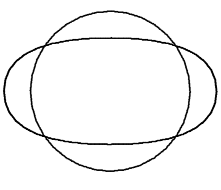
∗ Influence of the Earth-Moon orbital motion

A question is sometimes raised concerning the role of centrifugal forces from the Earth's motion around the center-of-mass of the Earth-Moon system. This point lies a distance $d = Dm/(m + M) \approx Dm/M$ from the center of the Earth, which is actually about 1600 km below the surface, and during a lunar cycle the center of the Earth moves in a circular orbit around it. Were the Earth in bound rotation so that it always turned the same side towards the Moon, one would in the corotating coordinate system, where the Moon and the Earth have fixed positions, have to add a centrifugal potential to the previously calculated potential (7-19), and the tidal range would become up to 27 times larger (problem 7.11), *i.e.* about 15 m!

Luckily, this is not the case. The Earth's own rotation is fixed with respect to the inertial system of the fixed stars (disregarding the precession of its rotation axis). A truly non-rotating Earth would, in the corotating system, rotate backwards in synchrony with the lunar cycle, cancelling the centrifugal potential. Seen from the inertial system, the circular orbital motion imparts the same centripetal acceleration $\Omega^2 d$ (along the Earth-Moon line) to all parts of the Earth. This centripetal acceleration must equal the constant gravitational attraction, Gm/D^2, coming from the linear term in the Moon's potential, and equating the two, one obtains $\Omega^2 = Gm/D^2 d = GM/D^3$, which is the usual equation relating the Moon's period of revolution to its mass and distance.

∗ 7.4 Rotating fluid planet

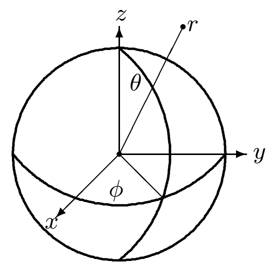
On a rotating spherical planet, centrifugal forces will add a component of "anti-gravity" to the gravitational acceleration field, making the road from the pole to equator slightly downhill. At equator the centrifugal acceleration amounts to only 1/291 of the surface gravity, so a first guess would be that there is a centrifugal "valley" at the equator with a depth of 1/291 of the Earths radius, which is about 22 km. If such a difference suddenly came to exist on a spherical Earth, all the water would like huge tides run towards the equator. Since there *is* land at equator, we may conclude that even the massive Earth must over time have flowed into the centrifugal valley. The difference between the equatorial and polar radii is in fact 21.4 km [1], and coincidentally, this is roughly the same as the difference between the highest mountain top and the deepest ocean trench on Earth.



Exaggerated sketch of the change in shape of the Earth due to rotation.

The flattening of the Earth due to rotation has like the tides been a problem attracting the best minds of the past centuries [8]. We shall here consider the simplest possible model, which nevertheless captures the main features.

Rotationally symmetric potential



Polar and azimuthal angles.

We shall assume that the centrifugal forces will deform the Earth so that its formerly spherical surface $r = a$ becomes modified to $r = a + h(\theta)$ where $h(\theta)$ is the height of the actual surface over the original sphere. This height may be both positive and negative, and may depend on the polar angle θ (complement to the latitude), but for symmetry reasons it cannot depend on the azimuthal angle ϕ (the longitude). The estimate above shows that only a small fraction of the Earth's mass is involved, so that the height will always be small compared to the radius.

Using (7-15) and (7-10) we find for the potential near the surface

$$\Phi = g_0 h(\theta) - \frac{1}{2}\Omega^2 a^2 \sin^2\theta + \Phi_1(\theta) \ , \tag{7-24}$$

where we have allowed for an extra potential at the surface $\Phi_1(\theta)$ arising from the shift of material involved in shape change itself. Rotational symmetry again dictates that this potential can only depend on θ.

Naive result

Without Φ_1 the height would have been determined entirely by the two first terms

$$h(\theta) = h_0\left(\sin^2\theta - \frac{2}{3}\right) \tag{7-25}$$

with $h_0 = \Omega^2 a^2/2g_0$. The constant term $(-2/3)$ is fixed by the requirement that the average of the change of height over the surface of the Earth should vanish. We recognize in the angular factor the second order Legendre polynomial $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ in a light disguise. Numerically, the value comes to about $h_0 = 10.8$ km, which is only half the expected result.

Self-potential

Even if at first sight one might think that the self-potential Φ_1 is too small to matter, this is actually not the case. The surface potential is in fact strongly influenced by the movement of surface material.

Let us assume that the material shifted from the poles to the equator has constant mass density ρ_1 so that it forms a thin surface layer with mass element $dM = \rho_1 dV = \rho_1 h(\theta) dS$ above a surface element $dS = a^2 \sin\theta d\theta d\phi$. The potential from this surface layer may be calculated by means of the usual expression for the potential of a mass distribution (3-27), which now becomes a surface integral

$$\Phi_1(\theta) = -G\rho_1 \oint_{|\boldsymbol{x}'|=a} \frac{h(\theta')}{|\boldsymbol{x} - \boldsymbol{x}'|} dS' \ . \tag{7-26}$$

The self-potential is seen to be of first order in the height and may for this reason competes with the first term in (7-24). But the integral cannot be evaluated unless we know $h(\theta)$, and $h(\theta)$ cannot be found unless we know the integral. Mathematically, the requirement of constant potential leads to an integral equation for $h(\theta)$ (see problem 7.13).

A trick

There are various ways to solve the integral equation. We shall (see problem 7.12) use a wonderful theorem about Legendre polynomials, which says that a mass distribution with an angular dependence given by a Legendre polynomial, creates a potential with exactly the same angular dependence. It is therefore consistent to assume that the height indeed takes the form (7-25) with an unknown constant h_0, because this leads to a self-potential of the same form

$$\Phi_1(\theta) = C\left(\sin^2\theta - \frac{2}{3}\right) . \tag{7-27}$$

The constant C may now be determined from the integral (7-26) by taking $\theta = 0$. All the difficult integrals disappear and we obtain

$$\begin{aligned}
-\frac{2}{3}C &= -G\rho_1 a h_0 2\pi \int_0^\pi \frac{\sin\theta'\left(\sin^2\theta' - \frac{2}{3}\right)}{\sqrt{2(1-\cos\theta')}} d\theta' \\
&= -G\rho_1 a h_0 2\pi \int_{-1}^1 \frac{\frac{1}{3} - u^2}{\sqrt{2(1-u)}} du \\
&= \frac{8\pi}{15}\rho_1 a G h_0 ,
\end{aligned}$$

which leads to

$$C = -\frac{3}{5} g_0 h_0 \frac{\rho_1}{\rho_0} , \tag{7-28}$$

where ρ_0 is the average density of the Earth and where we have used that $g_0 = GM_0/a^2 = \frac{4}{3}\pi G a \rho_0$.

Height of the bulge

Inserting this into (7-24) we finally obtain

$$\boxed{h_0 = \frac{\Omega^2 a^2}{2g_0\left(1 - \frac{3}{5}\frac{\rho_1}{\rho_0}\right)}} . \tag{7-29}$$

For Earth, the average density of the mantle material is $\rho_1 \approx 4.5$ g/cm^3 whereas the average density is $\rho_0 \approx 5.5$ g/cm^3. With these densities one gets $h_0 = 21.5$ km in close agreement with the quoted value [1].

In the same vein, we may now calculate the influence of the self-potential on the tidal range. Since the density of water is $\rho_1 \approx 1.0$ g/cm^3, it is seen that the tidal range is increased by a factor 1.12.

Fast rotating planet

One of the main assumptions behind the calculations in this section was that the planet should be slowly rotating, meaning that the deformation of the planet due to rotation be small, or $|h(\theta)| \ll a$. Intuitively, it is fairly obvious, that if the rate of rotation of the planet is increased, the flattening increases until it reaches a point, where the "antigravity" of rotation overcomes the "true" gravity of planetary matter as well as cohesive forces. Then the planet becomes unstable with dramatic change of shape or even breakup as a consequence.

Colin MacLaurin (1698–1746). Scottish mathematician who developed and extended Newton's work on calculus and gravitation.

The study of the possible forms of rotating planets was initiated very early by Newton and in particular by MacLaurin. It was found that oblate ellipsoids of rotation are possible allowed shapes for rotating planets with constant matter density, ρ_0. An oblate ellipsoid of rotation is characterized by equal-size major axes, $a = b$ and a smaller minor axis $c < a$, about which it rotates.

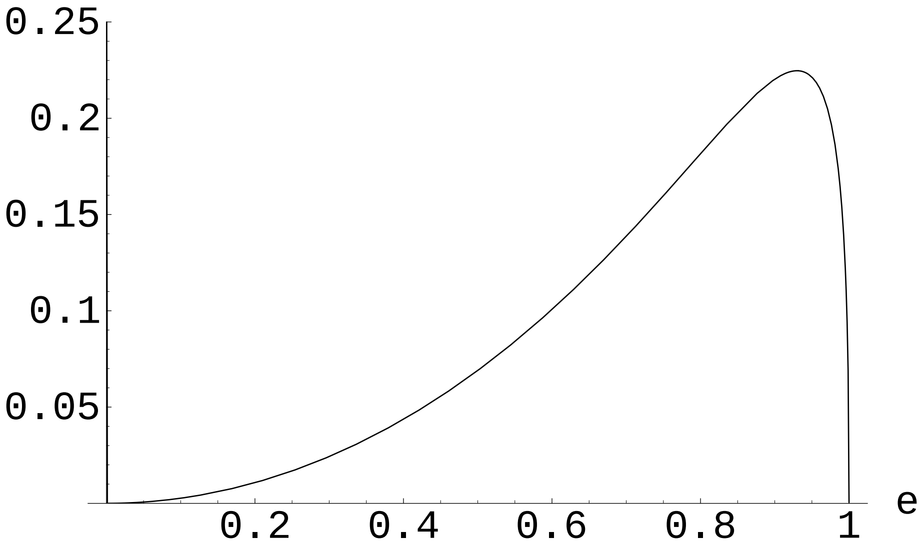


Figure 7.2: *The MacLaurin function (right hand side of eq. (7-30)). The maximum 0.225 is reached for* $e = 0.93$.

MacLaurin found that the angular rotation rate is related to the eccentricity $e = \sqrt{1 - c^2/a^2}$ through the formula

$$\frac{\Omega^2}{2\pi G\rho_0} = \frac{1}{e^3}\left(\sqrt{1-e^2}\left(3-2e^2\right)\arcsin e - 3e\left(1-e^2\right)\right) . \tag{7-30}$$

The right hand side is shown in Fig. 7.2 and has a maximum 0.225 for $e = 0.93$, implying that stability can only be maintained for $\Omega^2/2\pi G\rho_0 < 0.225$. Actually, various other shape instabilities set in at even lower values of the eccentricity (see ref. [20] for a thorough discussion of these instabilities and their astrophysical consequences). For small e, the MacLaurin(!) expansion of the right hand side of (7-30) becomes $4e^2/15$. Since $e^2 \approx 2h_0/a$, we obtain $h_0 = 15\Omega^2 a/16\pi G\rho_0 = 5\Omega^2 a^2/4g_0$, in complete agreement with (7-29) for $\rho_1 = \rho_0$.

Problems

7.1 The spaceship Rama (from the novel by Arthur C. Clarke) is a hollow cylinder hundreds of kilometers long and tens of kilometers in diameter. The ship rotates so as to create a standard pseudo-gravitational field g_0 on the inner side of the cylinder. Calculate the escape velocity to the center of the cylinder.

7.2 Find the pressure by solving (7-6) for the case of a) constant density, c) constant compressibility, b) ideal gas, and d) isentropic gas.

7.3 Calculate the increase in buoyancy due to its own gravity for a spherical body for a) constant density, b) constant compressibility, and c) isentropic gas.

7.4 a) Calculate the repulsion between two identical spheres with constant mass density ρ_1 and radii a, submerged a distance $D \gg a$ apart in a fluid obeying the equation of state $\rho = \rho(p)$. There is no other gravitational field present, the fluid pressure is p_0 in the absence of the spheres, and one may assume that the pressure corrections due to the spheres are everywhere small in comparison with p_0. b) Compare with the gravitational attraction between the spheres. c) Under which conditions will the total force between the spheres vanish.

7.5 Calculate the change in sea level if the air pressure locally rises by 20 millibars.

7.6 Calculate the changes in air pressure due to tidal motion of the atmosphere a) over sea, and b) over land?

7.7 a) Show that for an ideal gas in isentropic balance the quantity

$$H = \frac{\gamma}{\gamma - 1} \frac{RT}{M_{\text{mol}}} + \Phi \tag{7-31}$$

is constant. b) Use this to calculate the temperature gradient in an arbitrary gravitational field.

7.8 How much water is found in the tidal bulge (the water found above average height).

7.9 a) Calculate the height of a geostationary orbit, *i.e.* an orbit where a sattelite would appear to be at rest with respect to a point on the surface of the Earth. b) How heavy must a sattelite in geostationary orbit be for the tides to be of the same size as the Moon's? c) Assuming the same average density, what would be the apparent size of such a sattelite?

7.10 Calculate the tidal range in the quasistatic approximation.

7.11 Calculate the tidal range that would result if the Earth were in bound rotation around the center-of-mass of the Earth-Moon system.

∗ **7.12** Show that $\boldsymbol{\nabla}^2[f(r)(3\cos^2\theta - 1)] = g(r)(3\cos^2\theta - 1)$ and determine $g(r)$.

7.13 Show that the integral (7-26) may be written explicitly as

$$\Phi_1(\theta) = -\frac{G\rho_1 a}{\sqrt{2}} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \frac{h(\theta')}{\sqrt{1-\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\phi'}} \qquad \text{(7-32)}$$

