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Sound in fluids

Humans wouldn't be humans were it not for sound transmission in air. We talk, shout, and sing all day long, and so do our radios, telephones, and soon our computers, too. It is a severe handicap to be deaf or mute or both in any human society at any time in history. Air is not the only medium for sound. Almost all materials, be they solid or fluid, may be used for communication. Whereas sound in air is vital to humans, sound in water is vital to whales.

There are many qualitatively different types of sound. The intuitive notion of a tone seems to demand a sustained vibration pattern lasting a certain time. Bangs, thunderclaps, and rumbles would not normally be characterized as tones. Neither would we characterize various types of noise as tones, such as the sound of breakers hitting a beach, although in this case there are musically gifted people who claim to hear tones. But then again there are types of music which many people would characterize as noise.

In isotropic Newtonian fluids there will essentially only be longitudinal waves, maintained by almost reversible pressure and density changes. Transverse waves may be created, but they die out over a fraction of a wave length, and hardly deserve to be called waves. Longitudinal waves are also attenuated by the action of bulk as well as shear viscosity, and must eventually die out. Although, the energy density in normal sound is tiny, dissipation is unavoidable, and all the talking, shouting and singing always ends up as almost unmeasurable heat in the end. On the other hand, this dying out opens up the possibility of determining the bulk viscosity by measuring the attenuation of longitudinal sound.

In this chapter only the most basic formalism is presented for sound propagation far away from open surfaces. Real acoustics is a huge subject in itself which falls outside the scope of this book. The special subject of surface waves will be dealt with in a later chapter.

20.1 Time-dependent planar flow

Transverse waves do not propagate in fluids. This may be illustrated by an exactly solvable planar problem in which the velocity of a bounding plates depend on time.

Referring to the planar geometry of section 18.2 we consider a situation in which the lower plate at $y = 0$ is given a time-dependent velocity $U(t)$ in the x-direction and the upper plate is held fixed at infinity. Guessing that there may be a solution of the form $\boldsymbol{v} = (v_x(y,t), 0, 0)$, the Navier-Stokes equation (for an incompressible fluid) reduces to

$$\boxed{\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}} , \tag{20-1}$$

where $\nu = \eta/\rho_0$ is the kinematic viscosity, and the boundary conditions are $v_x(0,t) = U(t)$ and $v_x(\infty, t) = 0$.

This equation is of exactly the same form as the well-known *heat equation* in one dimension and describes like the heat equation a diffusion process, where velocity is put into the system at the moving boundary and diffuses away from it because of shear viscosity, with the kinematic viscosity playing the role of the diffusion constant.

Harmonic solution

We shall for simplicity assume that the lower plate is oscillating with circular frequency ω and amplitude U_0, so that $U(t) = U_0 \cos \omega t$. The linearity of the heat equation then guarantees that the velocity of the fluid everywhere will undergo forced oscillations at frequency ω. As usual in harmonic problems the solution may be taken to be the real part of the complex wave (for the justification of this point see problem 20.1)

$$v_x(y,t) = v_x(y) e^{-i\omega t} . \tag{20-2}$$

The heat equation (20-1) then requires that the coefficient function satisfies

$$-i\omega v_x(y) = \nu \frac{d^2 v_x(y)}{dy^2} , \tag{20-3}$$

for which the general solution is of the form $v_x(y) = U_0 e^{iky}$ where the 'wave vector' k must satisfy $i\omega = \nu k^2$. Consequently we obtain

$$k = \pm(1+i)\sqrt{\frac{\omega}{2\nu}} . \tag{20-4}$$

The positive sign corresponds to damped waves with exponentially decreasing amplitude for $y \to \infty$ in agreement with the boundary conditions, whereas the

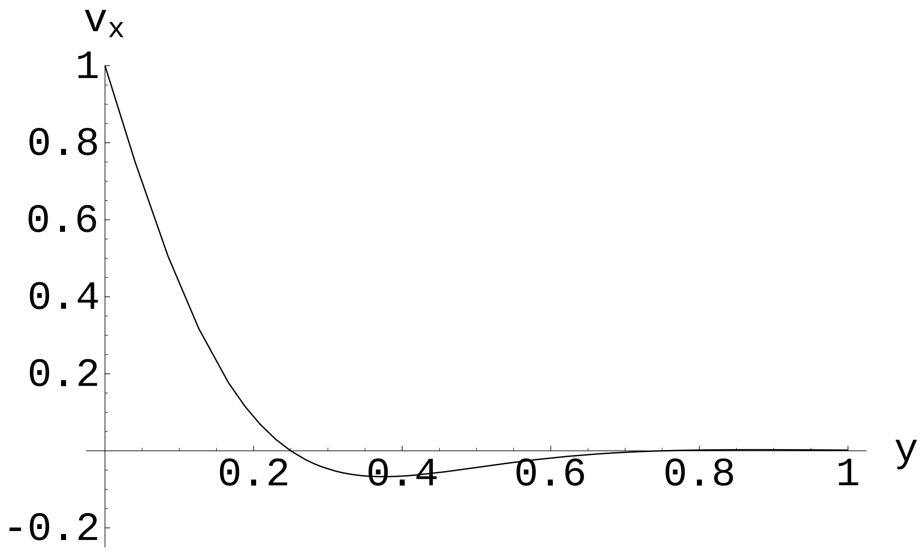

Figure 20.1: *Plot of the transverse wave amplitude* $\exp(-2\pi y)\cos 2\pi y$ *versus* y *for* $0 \leq y \leq 1$. *Notice that it is so strongly damped that only the first period is visible.*

negative sign corresponds to exponentially increasing amplitude and cannot be allowed. Putting everything together we obtain the solution to our problem

$$v_x(y,t) = \mathcal{R}e\left(U_0 e^{i(ky-\omega t)}\right) = U_0 e^{-y\sqrt{\omega/2\nu}} \cos\left(y\sqrt{\frac{\omega}{2\nu}} - \omega t\right) . \tag{20-5}$$

Since the velocity oscillations take place in the x-direction whereas the wave propagates in the y-direction this must be characterized as a transverse wave.

Transverse wave propagation

This solution demonstrates that transverse waves in fluids have phase velocity $c_T = \sqrt{2\nu\omega}$, wave length $\lambda = 2\pi\sqrt{2\nu/\omega}$, and decays over a typical distance of $\xi = \sqrt{2\nu/\omega}$, also called the *penetration depth*. The ratio between wave length and penetration depth is $\lambda/\xi = 2\pi$ so the amplitude diminishes by a factor of $e^{-2\pi} \approx 2 \times 10^{-3}$ over a single wave length. Transverse waves are so strongly damped that they barely propagate (see fig. 20.1).

> For a typical sound wave in air with frequency 1000 Hz the transverse phase velocity is only $c_T = 0.4$ m/s and the penetration depth is $\xi = 71$ μm. In water $c_T = 0.1$ m/s and $\xi = 17$ μm. Communication using transverse waves in air or water or any other fluid fails miserably.

The physical reason is of course that the shear viscosity is heavily involved in transverse wave motion and quickly drains the wave of energy. This is not the case for longitudinal waves that involve compression and therefore depend on the compressibility of the fluid rather than its viscosity.

20.2 Sound as small velocity perturbations

In elastic solids the fundamental field is the displacement field $\boldsymbol{u}(\boldsymbol{x}, t)$ which keeps track of the whereabouts of all the material particles. Since the molecules of a solid are supposed to retain their neighboring relations, sound in solids is nothing but small-amplitude oscillations of the displacement field away from a reference state.

In a fluid the fundamental field is the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. In fluids molecules do not retain their neighboring relations, but move more or less freely among each other, albeit limited somewhat by the pressure forces and braking action of viscosity. The description of sound in fluids must therefore begin by considering small-amplitude oscillations in the velocity field. This approximation gives us a hope of simplicity, whereas large-amplitude oscillations can be as complicated as any time-dependent flow.

The background or reference state on which small-amplitude sound phenomena play out must be assumed to possess a high degree of regularity. Otherwise it becomes hard to calculate. It is, for example, well-known that sound velocity in the atmosphere changes with altitude. Describing the vertical propagation of a sound wave throughout the atmosphere is bound to be more difficult than describing the propagation over small distances. Likewise, it must be exceedingly hard to describe the propagation of sound through a complex background velocity field, such as that of a tornado!

The 'doldrums' reference state

In discussing the properties of sound in fluids it is therefore important that the region under study be so small that the background density ρ_0 and (effective) pressure p_0 may be viewed as being constant in both space and time. Since longitudinal sound waves are carried by pressure and density variations, one cannot of course assume that the fluid is perfectly incompressible.

We shall furthermore assume that the velocity changes are so rapid that no heat exchange can take place between the regions of the wave that are momentarily compressed and those that are expanded. This means that the equation of state $p_0 = p_0(\rho_0)$ should express the isentropic relation between pressure and density, for example the polytrope (4-34).

We shall in the same vein assume that the region of interest is so small that the background velocity field — the ambient 'wind' — likewise may be viewed as constant in both space and time, a constant which may be made to vanish by choosing a suitable inertial system. In other words, we assume that the coordinate system is chosen such that there is no ambient wind, or in maritime terms that we only consider sound in the 'doldrums'.

Finally we shall assume that the region is far from any boundaries which could interfere with the free sound propagation as we saw it did for surface waves in solids in section 13.3.

Equations of motion

In the reference state the homogeneous, isotropic fluid at rest must have constant density ρ_0, and pressure, p_0. This is obviously a solution to the Navier-Stokes equations for $\boldsymbol{v} = 0$ and $\boldsymbol{f} = 0$. Perturbing the equilibrium state ever so slightly with a small time-dependent velocity field $\boldsymbol{v}$ will change the density to $\rho = \rho_0 + \rho_1$ and the pressure to $p = p_0 + p_1$ where the corrections are of first order in the perturbing velocity.

To this approximation the equation of continuity (14-25) becomes

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \operatorname{div} \boldsymbol{v} \; . \tag{20-6}$$

Likewise from the Navier-Stokes equation (17-12) we find to first order in the velocity

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla} p_1 + \eta \boldsymbol{\nabla}^2 \boldsymbol{v} + \frac{\eta}{3} \boldsymbol{\nabla} \operatorname{div} \boldsymbol{v} \; . \tag{20-7}$$

The non-linear inertia term does not enter this equation, because it is of second order in the velocity.

There are two contributions to p_1, one from the change of density through the equation of state for the fluid, and one from the second viscosity given by (17-11). To first order in the velocity

$$p_1 = \frac{dp_0}{d\rho_0} \rho_1 - \zeta \operatorname{div} \boldsymbol{v} \; . \tag{20-8}$$

The quantity $dp_0/d\rho_0$ has the dimension of velocity squared, so that we may write

$$c_0^2 = \frac{dp_0}{d\rho_0} = \frac{K}{\rho_0} \; . \tag{20-9}$$

In the last equation we have expressed it in terms of the bulk modulus or compressibility K defined in eq. (4-26). We shall soon see that c_0^2 is indeed the phase velocity of longitudinal sound waves.

Calculating the divergence of (20-7) we obtain the equation of motion for the density disturbances

$$\boxed{\frac{\partial^2 \rho_1}{\partial t^2} = c_0^2 \boldsymbol{\nabla}^2 \rho_1 + \frac{\zeta + \frac{4}{3}\eta}{\rho_0} \boldsymbol{\nabla}^2 \frac{\partial \rho_1}{\partial t}} \; . \tag{20-10}$$

Differentiating (20-7) after time we obtain the equation for the velocity field

$$\boxed{\frac{\partial^2 \boldsymbol{v}}{\partial t^2} = c_0^2 \boldsymbol{\nabla} \operatorname{div} \boldsymbol{v} + \frac{\zeta + \frac{1}{3}\eta}{\rho_0} \boldsymbol{\nabla} \operatorname{div} \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\eta}{\rho_0} \boldsymbol{\nabla}^2 \frac{\partial \boldsymbol{v}}{\partial t}} \; . \tag{20-11}$$

Although they look complicated, these equations are in fact rather simple due to the linearity and constant coefficients.

20.3 Plane waves

The solution of linear, partial differential equations with constant coefficients is most conveniently carried out in terms of free plane waves as we did in section 13.2. A plane velocity wave takes the form

$$\boldsymbol{v} = \boldsymbol{v}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t} , \tag{20-12}$$

where $\boldsymbol{v}_0$ is the amplitude, $\boldsymbol{k}$ the wave vector and ω the frequency. We have here used complex notation for convenience. The true velocity field is obtained as the real part of the complex wave (see problem 20.1). Elementary waves of this kind may be superposed to form all possible sound waves.

Inserting this into (20-11) we obtain

$$\omega^2 \boldsymbol{v}_0 = c_0^2 \boldsymbol{k}\,\boldsymbol{k}\cdot\boldsymbol{v}_0 - i\omega\frac{\zeta + \frac{1}{3}\eta}{\rho_0}\boldsymbol{k}\,\boldsymbol{k}\cdot\boldsymbol{v}_0 - i\omega\frac{\eta}{\rho_0}\boldsymbol{k}^2\boldsymbol{v}_0 \ .$$

There are as for sound in solids two qualitatively different solutions to this equation, depending on whether the amplitude $\boldsymbol{v}_0$ is parallel with the wave vector $\boldsymbol{k}$, or orthogonal to it.

The transverse case $\boldsymbol{v}_0 \cdot \boldsymbol{k} = 0$ is quickly dispensed with since $\omega = -i\nu\boldsymbol{k}^2$. Solving for k we find the same transverse wave vector (20-4) as we did in section 20.1 and conclude that transverse waves in fluids immediately lose their energy over a fraction of a wave length.

Longitudinal waves

For $\boldsymbol{v}_0 \sim \boldsymbol{k}$ we find the dispersion relation for longitudinal waves

$$\omega^2 = c_0^2 k^2 - i\omega k^2 \frac{\zeta + \frac{4}{3}\eta}{\rho_0} ,$$

which could also have been obtained from the density equation (20-10). Solving for k we get

$$k = \pm\frac{\omega}{c_0\sqrt{1 - i\omega\frac{\zeta+\frac{4}{3}\eta}{K}}} \tag{20-13}$$

where we inside the square root have gone back to the compressibility $K = \rho_0 c_0^2$. For sufficiently small frequency this expands to

$$k = k_R + ik_I = \pm\frac{\omega}{c_0}\left(1 + i\frac{\omega}{2K}\left(\zeta + \frac{4}{3}\eta\right)\right) \tag{20-14}$$

The real part $k_R = \pm\omega/c_0$ shows that the longitudinal waves indeed have phase velocity c_0 whereas the imaginary part k_I expresses the rate of attenuation. Notice that the overall sign is always such that a wave running in a given direction is also attenuated along that direction.

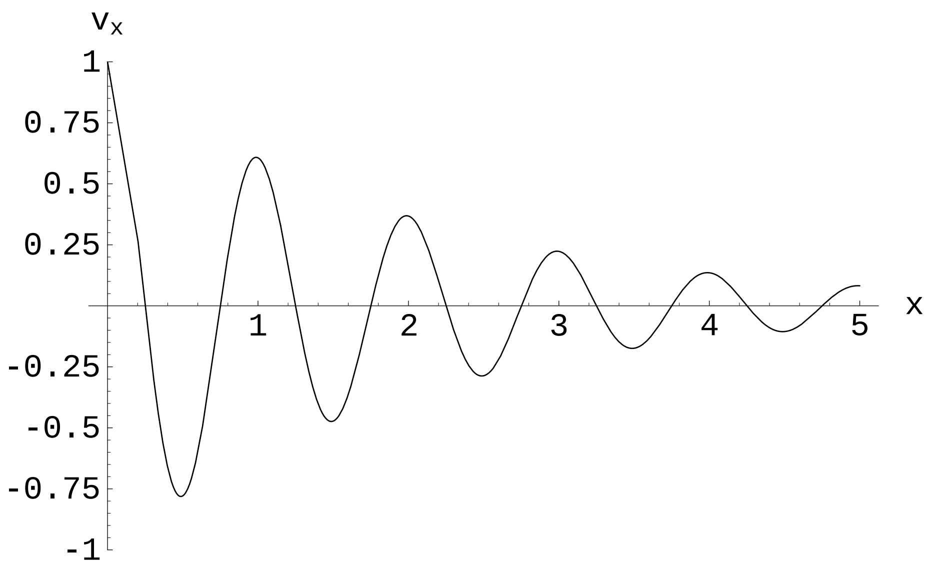

Figure 20.2: *Plot of the longitudinal wave amplitude* $\exp(-0.5y)\cos 2\pi y$ *versus* y *for* $0 \leq y \leq 5$ *corresponding to* $\lambda = 1$ *and* $\xi = 2$ *such that the energy attenuation coefficient is* $a_L = 1$.

Attenuation

Denoting the wave length by $\lambda = 2\pi/k_R$ and the attenuation length by $\xi = 1/k_I$ we find the ratio

$$\frac{\lambda}{\xi} = \frac{\pi\omega\left(\zeta + \frac{4}{3}\eta\right)}{K} = \frac{2\pi^2\left(\zeta + \frac{4}{3}\eta\right)}{\rho_0 c_0 \lambda} \tag{20-15}$$

This is the reciprocal of the number of wave lengths it takes to attenuate the sound amplitude by a factor of $e = 2.718\ldots$. The kinetic energy of the flow is actually dissipated at the double of this rate, because it depends on the square of the velocity. The *energy attenuation coefficient* is accordingly

$$\boxed{a_L = 2\frac{\lambda}{\xi} = \frac{4\pi^2\left(\zeta + \frac{4}{3}\eta\right)}{\rho_0 c_0 \lambda}} \tag{20-16}$$

for longitudinal waves (see fig. 20.2). Notice that the attenuation coefficient grows with decreasing wavelength which is the same as increasing frequency. This is the reason that bulk viscosity is best determined in the laboratory by means of ultrasound which has larger attenuation than audible sound due to the higher frequency.

For air at normal temperature and pressure we have $c_0 \approx 330$ m/s and $\eta \approx 1.8 \times 10^{-5}$ Pa s. Ignoring the bulk viscosity, the attenuation coefficient for sound at 1000 Hz comes to about 10^{-5}. It takes 100.000 wave lengths which is about 33 km to attenuate such a sound signal. Over 1 km it only attenuates by 3%.

Problems

20.1 Show that the harmonic solution to the heat equation (20-1) may be taken to be complex.

20.2 a) Show that the total volume discharge for time-dependent planar flow is

$$\frac{Q}{L} = U_0\sqrt{\frac{\nu}{\omega}}\cos\left(\omega t - \frac{\pi}{4}\right)$$

b) Discuss the physics behind the phase shift relative to the driving motion.

20.3 Derive the equations of motion for sound in a non-homogeneous background fields $\rho_0(\boldsymbol{x}, t)$ and $\boldsymbol{v}_0(\boldsymbol{x}, t)$.

20.4 Calculate the wave vector for plane sound waves without assuming that the frequency is small.

