2
Space and time

In classical Newtonian physics space is absolute and eternal, obeying the rules of Euclidean geometry everywhere. It is the perfect stage on which all physical phenomena play out. Time is equally uniform and absolute all over space from the beginning to the end, and matter has no influence on the properties of space and time. Rulers to measure length never stretch, clocks to measure time never lose a tick, and both can be put to work anywhere from the deepest levels of matter to the farthest reaches of outer space. It is a clockwork universe, orderly, rigorous, and deterministic.

This semblance of perfection was shattered in the beginning of the twentieth century by the theories of relativity and quantum mechanics. Space and time became totally intertwined with each other and with matter, and the determinism of classical physics was replaced by the still-disturbing quantum indeterminism.

Relativity and quantum mechanics are both theories of extremes. Although they in principle apply to the bulk of all physical phenomena, their special features become dominant only at velocities approaching the velocity of light in the case of relativity, or length scales approaching the size of atoms in the case of quantum mechanics. Newtonian space and time remains a valid, if not "true", conceptual framework over the vast ranges of length and velocity scales covered by classical continuum physics.

In this chapter the basic ideas behind space and time are introduced in a way which emphasizes the operational aspects of physical concepts. Care is exerted in order that the concepts defined here should remain valid in more advanced theories. A certain familiarity with Euclidean geometry in Cartesian coordinates is assumed, and the chapter serves in most respects to define the mathematical notation to be used in the remainder of the book. It may be sampled at leisure, as the need arises.

2.1 Reference frames

Physics is a quantitative discipline using mathematics to relate measurable quantities expressed in terms of real numbers. In formulating the laws of nature, undefined mathematical primitives — for example the points, lines and circles of Euclidean geometry — are not particularly useful, and such concepts have for this reason been eliminated and replaced by numerical representations everywhere in physics.

This necessitates a specification of the practical procedures by which these numbers are obtained in an experiment, for example, what units are being used. Behind every law of nature and every formula in physics, there is a framework of procedural descriptions, a *reference frame*, supplying an operational meaning to all quantities. Part of the art of doing physics lies in comprehending this — often tacitly understood — infrastructure to the mathematical formalism.

The reference frame always involves physical objects — balances to measure mass, clocks to measure time, and rulers to measure length — that are not directly a part of the mathematical formalism. Precisely because they *are* physical objects, they can at least in principle be handed over or copied, and thereby shared among experimenters. This is what is really meant by the objectivity of physics.

The system of units, the *Système Internationale (SI)*, is today fixed by international agreement. But even if our common frame of reference is thus defined by social convention, physics is nevertheless objective. In principle our frames of reference could be shared with any other being in the universe.

> The unit of mass, the kilogram, is defined by a prototype stored by the *International Bureau of Weights and Measures* near Paris, France. Copies of this prototype and balances for weighing them can be made to a precision of one part in 10^9 [1].

2.2 Time

Time is the number you read on your *clock*. There is no better definition. Clocks are physical objects which may be shared, compared, copied, and synchronized to create an objective meaning of *time*. Most clocks, whether they are grandfather clocks based on a swinging pendulum or oscillating quartz crystals, are based on periodic physical systems that return to the same state again and again, and time intervals are simply measured by counting periods. There are also aperiodic clocks, for example hour glasses, and clocks based on radioactive elements. It is especially the latter that allow time to be measured on geological scales.

Like all macroscopic physical systems, clocks are subject to small fluctuations in the way they run. The most stable clocks are those that keep time best with respect to copies of themselves as well as with clocks built on other principles. Grandfather clocks are much less stable than maritime chronometers that in turn are less stable than quartz clocks. The international frame of reference for time is always based on the currently most stable clocks.

> Formerly the unit of time, the *second*, was defined as 1/86,400 of a mean solar day, but the Earth's rotation is not that stable, and since 1966 the second has been defined by international agreement as the duration of 9,192,631,770 oscillations of the microwave radiation absorbed in a certain hyperfine transition in the cesium-133 atom. A beam of cesium-133 atoms is used to stabilize a quartz oscillator at the right frequency by a resonance method, so what we call an atomic clock is really an atomically stabilized quartz clock. The intrinsic relative precision in this time standard is about 4×10^{-14}, or about one second in a million years [1].

In the extreme mathematical limit, time may be assumed to be a real number, say t, and in Newtonian physics its value is assumed to be universally known.

2.3 Space

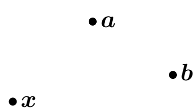
It is a mysterious and so far unexplained fact that physical *space* has three dimensions, which means that it takes exactly three real numbers — say x_1, x_2 and x_3 — to locate a point in space. These numbers are called the *coordinates* of the point, and the reference frame for coordinates is called the *coordinate system*. It must contain all the operational specifications for locating a point given the coordinates, and conversely obtaining the coordinates given the location. In this way we have relegated all philosophical questions regarding the *real* nature of points and of space to the operational procedures contained in the reference frame.

> On Earth everybody navigates by means of a geographical system, in which a point is characterized by latitude, longitude and height. The geographical coordinate system is based on agreed-upon fixed points on Earth: the north pole, Greenwich near London, and the average sea level. The modern Global Positioning System uses fixed points in the sky in the form of satellites, and the coordinates of any point on Earth is determined from differences in the time-of-flight of radio signals.

It is convenient to collect the coordinates x_1, x_2, and x_3 of a point in a single object, a triplet of real numbers

$$\boldsymbol{x} = (x_1, x_2, x_3) \ , \tag{2-1}$$

called the *position* of the point in the coordinate system [1]. The triplet notation is just a notational convenience, so a function of the position $f(\boldsymbol{x})$ is completely equivalent to a function of the three coordinates $f(x_1, x_2, x_3)$.



Points may be visualized as dots on a piece of paper.

[1] In almost all modern textbooks it is customary to use boldface notation for triplets of real numbers ("vectors") and we shall also do so here. In calculations with pencil on paper many different notations are used to distinguish such a symbol from other uses, for example a bar ($\overline{x}$), an arrow ($\vec{x}$), or underlining ($\underline{x}$).

There is nothing sacred about the names of the coordinates. In physics and especially in practical calculations, the coordinate variables are often renamed $x_1 \to x$, $x_2 \to y$ and $x_3 \to z$, so that the general point becomes $\boldsymbol{x} \to (x, y, z)$. It is also customary to write $\boldsymbol{a} = (a_x, a_y, a_z)$ for a general triplet, with the coordinate symbols used as indices instead of 1, 2 and 3. It is of course of no importance whether the range of indices is x, y, z or 1, 2, or 3 or something else, as long as there are three of them.

Coordinate transformations

Having located a point by a set of coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ in one coordinate system, the coordinates $\boldsymbol{x}' = (x'_1, x'_2, x'_3)$ of the exact same point in another coordinate system must be calculable from the first

$$
\begin{aligned}
x'_1 &= f_1(x_1, x_2, x_3) \ , \\
x'_2 &= f_2(x_1, x_2, x_3) \ , \\
x'_3 &= f_3(x_1, x_2, x_3) \ ,
\end{aligned}
$$

$\bullet\, \boldsymbol{a} \leftrightarrow \boldsymbol{a}'$

$\bullet\, \boldsymbol{b} \leftrightarrow \boldsymbol{b}'$

$\bullet\, \boldsymbol{x} \leftrightarrow \boldsymbol{x}'$

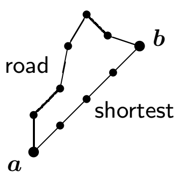
In different coordinate systems the same points have different coordinates.

In triplet notation this is written

$$
\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}) \ . \tag{2-2}
$$

This postulate reflects that physical reality is unique and that different coordinate systems are just different ways of representing the same physical space in terms of real numbers. Conversely, every such one-to-one mapping of the coordinates defines another coordinate system. The study of *coordinate transformations* is central to analytic geometry and permits characterization of geometric quantities by the way they transform rather than in abstract terms (see section 2.5).

Length

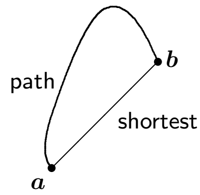


The length of the road between $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is measured by counting steps along the road. The distance is the length of the shortest road.*

From the earliest times humans have measured the *length* of a road between two points, say $\boldsymbol{a}$ and $\boldsymbol{b}$, by counting the number of steps it takes to walk along this road. In order to communicate to others the length of a road, the count of steps must be accompanied by a clear definition of a step, for example in terms of an agreed-upon unit of length.

Originally the units of length — inch, foot, span, and fathom — were directly related to the human body, but increasing precision in technology demanded better-defined units. In 1793 the meter was introduced as a ten millionth of the distance from equator to pole on Earth, and until far into the twentieth century a unique "normal meter" was stored in Paris. Later the meter became defined as a certain number of wave lengths of a certain spectral line in krypton-86, an isotope of a noble gas which can be found anywhere on Earth. Since 1983 the meter has been defined by international convention to be the distance traveled by light in exactly 1/299,792,458 of a second [1]. The problem of measuring lengths has thus been transferred to the problem of measuring time, which makes sense because the precision of the time standard is at least a thousand times better than any proper length standard.

This method for determining length may be refined to any desired practical precision by using very short steps. In the extreme mathematical limit, the steps become infinitesimally small, and the road becomes a continuous path. The shortest such path is called a *geodesic* and represents the "straightest line" between the points. Airplanes and ships travel along geodesics, *i.e.* great circles on the spherical surface of the Earth.



In the mathematical limit the shortest continuous path connecting $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is called the geodesic ("straight line").*

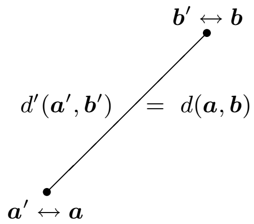
Distance

The *distance* between to points is defined to be the length of the shortest path between the points. Since the points are completely defined by their coordinates, $\boldsymbol{a}$ and $\boldsymbol{b}$ (relative to the coordinate system), the distance must be simply a real function $d(\boldsymbol{a}, \boldsymbol{b})$ of the two sets of coordinates. This function is not completely general; the definition of length in terms of numbers of steps implies that certain inequalities are fulfilled by the distance function.

From the definition it is clear that the distance between two points must be the same in all coordinate systems, because it can in principle be determined by laying out rulers between points without any reference to coordinate systems. The actual distance function $d'(\boldsymbol{a}', \boldsymbol{b}')$ in a new coordinate system may be different from the old, $d(\boldsymbol{a}, \boldsymbol{b})$, but the numerical value must be the same

$$d'(\boldsymbol{a}', \boldsymbol{b}') = d(\boldsymbol{a}, \boldsymbol{b}) \ , \qquad (2\text{-}3)$$

where $\boldsymbol{a}' = \boldsymbol{f}(\boldsymbol{a})$ and $\boldsymbol{b}' = \boldsymbol{f}(\boldsymbol{b})$ are calculated by the coordinate transformation (2-2). Knowing the distance function $d(\boldsymbol{a}, \boldsymbol{b})$ in one coordinate system, it may be calculated in any other coordinate system by means of the appropriate coordinate transformation.



Invariance of the distance.

This expresses the *invariance of the distance* under all coordinate transformations. In the same way as (2-2) may be viewed mathematically as a definition of what is meant by "another coordinate system", the equation (2-3) may be viewed mathematically as a definition of what is meant by distance in another coordinate system.

Cartesian coordinate systems

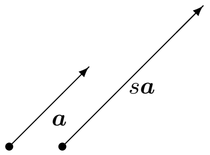
René Descartes or Renatus Cartesius (1596–1650). French scientist and philosopher, father of analytic geometry.

It is another fundamental physical fact that it is possible (within limited regions of space and time) to construct coordinate systems, in which the distance between any two points, $\boldsymbol{a}$ and $\boldsymbol{b}$, is given by the expression

$$d(\boldsymbol{a}, \boldsymbol{b}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2} \ . \qquad (2\text{-}4)$$

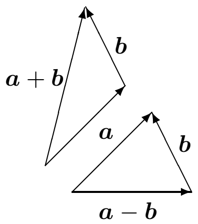
Such coordinate systems are called *Cartesian* and were first analyzed by Descartes. The distance function implies that space is *Euclidean* and therefore has all the properties one learns about in elementary geometry.

2.4 Vector algebra



Scaling of a vector.

Triplets of real numbers derived from Cartesian coordinates play a central role in everything that follows, and it is convenient immediately to introduce a set of algebraic operations on such triplets. One will readily recognize these operations as the familiar vector operations, and they are mainly listed here in order to fix notation. In the problem section at the end of this chapter a number of consequences of these definitions are derived. Visualization on paper is of course as useful as ever, so we shall as usual visualize triplets by means of arrows.



Addition and subtraction of vector.

Linear operations

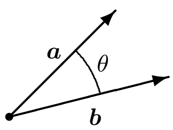
$$\boldsymbol{s a} = (sa_1, sa_2, sa_3) \qquad \text{(scaling)} \qquad (2\text{-}5)$$

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \qquad \text{(addition)} \qquad (2\text{-}6)$$

$$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3) \qquad \text{(subtraction)} \qquad (2\text{-}7)$$

These are the familiar operations which tell us that the set of all triplets, also called $\mathbb{R}^3$, mathematically is a 3-dimensional *vector space*.

Bilinear products

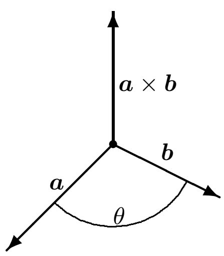


The dotproduct of two vectors is $|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$ where θ is the angle between them.

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \text{(dot product)} \qquad (2\text{-}8)$$

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \qquad \text{(cross product)} \qquad (2\text{-}9)$$

$$\boldsymbol{ab} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1, b_2, b_3) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix} \qquad \text{(tensor product)} \qquad (2\text{-}10)$$



The cross product of two vectors is a vector orthogonal to both, of length $|\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta$.

The two first are familiar from ordinary vector calculus. Notice however that the cross product is defined entirely in terms of the coordinates, and that we do not in the rule itself distinguish between left-handed and right-handed coordinate systems.

The last one, the *tensor product*, is unusual in that it produces a 3×3 matrix from two triplets, but otherwise it is perfectly well-defined and useful to have around. As shown, it is nothing but an ordinary matrix product of a column-matrix and a row-matrix. It is also called the *direct product*, and sometimes in the older literature the dyadic product. In chapter 8 we shall introduce more general geometric objects, called tensors, of which the simplest are matrices of this kind. The tensor product, and tensors in general, cannot be given a simple visualization.

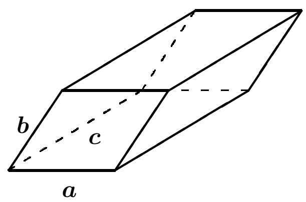
Volume product

The trilinear product of three vectors obtained by combining the cross product and the dot product,

$$\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \ , \qquad \text{(2-11)}$$

is equal to the determinant of the matrix constructed from the three vectors.

$$\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ , \qquad \text{(2-12)}$$



Three vectors spanning a parallelepiped.

The determinant shows that the volume product is antisymmetric in the three vectors. Its value is the (signed) volume of the parallelepiped spanned by the vectors.

Square and norm

The square and the norm are standard definitions

$$\boldsymbol{a}^2 = \boldsymbol{a} \cdot \boldsymbol{a} = a_1^2 + a_2^2 + a_3^3 \qquad \text{(square)} \qquad \text{(2-13)}$$

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a}^2} = \sqrt{a_1^2 + a_2^2 + a_3^3} \qquad \text{(norm or length)} \qquad \text{(2-14)}$$

This definition of the norm is of course closely related to the form of the Cartesian distance (2-4).

Derivatives

Various types of derivatives of vectors may be defined

$$\frac{\partial \boldsymbol{a}}{\partial s} = \left(\frac{\partial a_1}{\partial s}, \frac{\partial a_2}{\partial s}, \frac{\partial a_3}{\partial s} \right) \qquad \text{(scalar derivative)} \qquad \text{(2-15)}$$

$$\frac{\partial}{\partial \boldsymbol{a}} = \left(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3} \right) \qquad \text{(vector derivative)} \qquad \text{(2-16)}$$

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \qquad \text{(nabla)} \qquad \text{(2-17)}$$

In the first line, the derivative of a triplet after a parameter is defined. In the second line, a symbolic notation is introduced for the three derivatives after a triplet's coordinates (see problem 2.8 for simple uses of this notation). The vector gradient operator $\boldsymbol{\nabla}$, called *nabla*, is often used in physics contexts to denote the derivatives after position coordinates (third line).

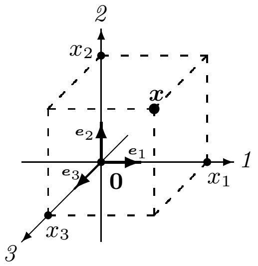
Basis vectors

The *coordinate axes* of a Cartesian coordinate system are three orthogonal lines with a common origin $\mathbf{0} = (0, 0, 0)$ and unit vectors

$$\boldsymbol{e}_1 = (1, 0, 0) \ , \tag{2-18a}$$

$$\boldsymbol{e}_2 = (0, 1, 0) \ , \tag{2-18b}$$

$$\boldsymbol{e}_3 = (0, 0, 1) \ . \tag{2-18c}$$



Visualization of the Cartesian coordinate system.

These vectors are called the *basis* vectors of the coordinate system, or just the basis, and every position may trivially be written as a linear combination, of the basis vectors,

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 \ , \tag{2-19}$$

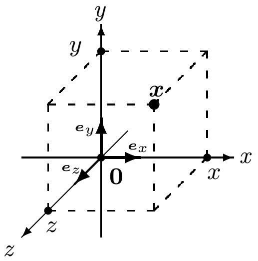
with the coordinates as coefficients.

The basis vectors are *normalized*

$$|\boldsymbol{e}_1| = |\boldsymbol{e}_2| = |\boldsymbol{e}_3| = 1 \ , \tag{2-20}$$

and mutually *orthogonal*

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0 \ , \tag{2-21}$$



A Cartesian coordinate system with axes labeled x, y and z.

Using these relations and (2-19) we find

$$x_1 = \boldsymbol{e}_1 \cdot \boldsymbol{x} \ , \tag{2-22a}$$

$$x_2 = \boldsymbol{e}_2 \cdot \boldsymbol{x} \ , \tag{2-22b}$$

$$x_3 = \boldsymbol{e}_3 \cdot \boldsymbol{x} \ , \tag{2-22c}$$

showing that the coordinates of a point may be understood as the normal projections of the point on the axes of the coordinate system.

Combining (2-19) with (2-22) we obtain the identity

$$\boldsymbol{e}_1(\boldsymbol{e}_1 \cdot \boldsymbol{x}) + \boldsymbol{e}_2(\boldsymbol{e}_2 \cdot \boldsymbol{x}) + \boldsymbol{e}_3(\boldsymbol{e}_3 \cdot \boldsymbol{x}) = \boldsymbol{x} \ ,$$

valid for all $\boldsymbol{x}$. Since this is a linear identity, we may remove $\boldsymbol{x}$ and express this *completeness* relation in a compact form by means of the tensor product (2-10)

$$\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3 = \mathbf{1} \ , \tag{2-23}$$

where on the right hand side the symbol $\mathbf{1}$ stands for the 3×3 unit matrix[2].

We emphasize that the handedness of the coordinate system has not entered the formalism. Correspondingly, the volume of the unit cube

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 \times \boldsymbol{e}_3 = +1 \ , \tag{2-24}$$

is always positive, independently of whether you call the hand you write with the left or the right.

[2] In order to distinguish a matrix from a vector, the matrix symbol will be written in heavy unslanted boldface. The distinction is not particularly visible, but matrices do not occur so often that it leads to serious confusion. With pencil on paper, 3×3 matrices are sometimes marked with a double bar ($\overline{\overline{1}}$) or a double arrow ($\overleftrightarrow{1}$).

Index notation

Triplet notation for vectors is sufficient in most areas of physics, because physical quantities are mostly scalars (*i.e.* single numbers like mass) or vectors, but sometimes it is necessary to use a more powerful and transparent notation which generalizes better to more complex expressions and quantities. It is called *index notation* or *tensor notation*, and consists in all simplicity in writing the coordinate indices explicitly wherever they occur. Instead of thinking of a position as a triplet $\boldsymbol{x}$, we think of it as the set of coordinates x_i with the index i running implicitly over the labels, say $i = 1, 2, 3$ or $i = x, y, z$, without having to state it every time.

Triplet and index notations coexist quite peacefully as witnessed by the linear operations

$$(s\boldsymbol{a})_i = s a_i \ , \tag{2-25}$$
$$(\boldsymbol{a}+\boldsymbol{b})_i = a_i + b_i \ , \tag{2-26}$$
$$(\boldsymbol{a}-\boldsymbol{b})_i = a_i - b_i \ . \tag{2-27}$$

For the scalar product we write a sum that implicitly ranges over the coordinate labels

$$\boldsymbol{a}\cdot\boldsymbol{b} = \sum_i a_i b_i \ . \tag{2-28}$$

In full-fledged tensor calculus the summation symbol is left out and understood as implicitly present for all indices that occur precisely twice in a term, but we shall refrain from doing so here.

The nine scalar products of basis vectors has two indices that each run implicitly over the three coordinate labels, and is written

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \ . \tag{2-29}$$

The expression δ_{ij} is nothing but the unit matrix in index notation,

Leopold Kronecker (1823–1891). German mathematician, contributed to the theory of elliptic functions, algebraic equations, and algebraic numbers.

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} \end{cases} \ , \tag{2-30}$$

also called the *Kronecker delta.* This is the first example of a true *tensor* of rank 2. Another is the tensor product (2-10), which takes the form

$$(\boldsymbol{a}\boldsymbol{b})_{ij} = a_i b_j \ . \tag{2-31}$$

It is, however, not enough for a tensor just to have two (or more) indices, but it suffices for now. In section 2.6 we shall see what really characterizes tensors.

The volume product (2-11) must be written as a triple sum over indices

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c} = \sum_{ijk} \epsilon_{ijk}\, a_i b_j c_k \ , \tag{2-32}$$

where ϵ_{ijk} is the following set of 27 coefficients

$$\begin{aligned}
&\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1 \ , \\
&\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \ , \\
&\epsilon_{ijk} = 0 \ , \qquad \text{otherwise} \ .
\end{aligned} \tag{2-33}$$

Tullio Levi-Civita (1873–1941). Italian mathematician, contributed to differential calculus, relavitivy, and founded (with Ricci) tensor analysis in curved space.

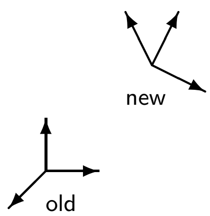
The symbol ϵ_{ijk} is in fact a tensor of third rank, called the *Levi-Civita symbol.*

Finally, using this symbol the cross product (2-9) may be written as a double sum over two indices of the form

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \sum_{jk} \epsilon_{ijk} a_j b_k \ . \tag{2-34}$$

Mostly, we can avoid this complicated notation for the cross product, but it does come in handy in some cases.

2.5 Cartesian coordinate transformations



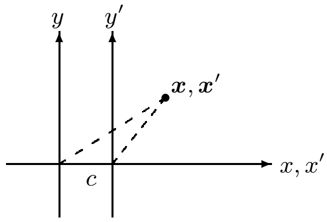
The old and the new Cartesian systems.

The same Euclidean world may be described geometrically by different observers with different reference frames. Each observer constructs his own preferred Cartesian coordinate system and determines all positions relative to that. Every observer thinks that his basis vectors have the simple form (2-18) and satisfy the same orthogonality and completeness relations. Every observer believes he is right-handed. How can they ever agree on anything with such a self-centered view of the world?

The answer is — as explained in section 2.3 — that the two descriptions are related by a coordinate transformation (2-2). Since the distance between any two points is independent of the coordinate system, the shortest paths must coincide, and straight lines must be mapped onto straight lines by any coordinate transformation. Seen from one Cartesian coordinate system, which we shall call the "old", the axes of another coordinate system, called the "new", will therefore also appear to be straight lines with a common origin. Furthermore, since the scalar product of two vectors can be expressed in terms of the norm (see eq. (2-68a)), which like distance must be independent of the specific coordinate system, the new axes will also appear to be orthogonal in the geometry of the old coordinate system. Different observers will thus agree that their respective coordinate systems are indeed Cartesian.

Simple transformations

We begin the analysis of coordinate transformations with the familiar elementary transformations: translation, rotation, and reflection. These transformations are of the general form (2-2), expressing the coordinates of a geometrical point in the new system as a function of the coordinates of the same point in the old. The simple transformations are related to a special choice of coordinate axes, and similar simple transformations may be defined for other choices.

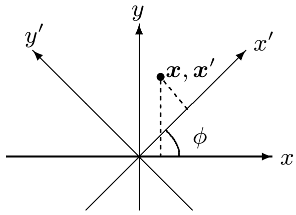


Simple translation of the coordinate system by c along the x-axis.

Simple translation: A simple *translation* of the origin of coordinates along the x-axis by a constant amount c is given by

$$x' = x - c \; , \quad y' = y \; , \quad z' = z \; . \tag{2-35}$$

The axes of the new coordinate system are in this case parallel with the axes of the old.



Simple rotation of the coordinate system around the z-axis (pointing out of the paper) through an angle ϕ.

Simple rotation: A simple *rotation* of the coordinate system through an angle ϕ around the z-axis corresponds to the transformation

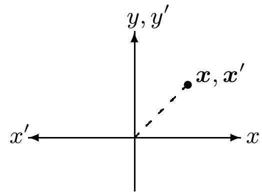
$$x' = \quad x \cos\phi + y \sin\phi \; , \tag{2-36a}$$
$$y' = -x \sin\phi + y \cos\phi \; , \tag{2-36b}$$
$$z' = \quad z \; . \tag{2-36c}$$

The signs of the terms in the first two lines may be verified by ordinary geometrical considerations.

Simple reflection: A simple *reflection* in the yz-plane is described by

$$x' = -x \; , \quad y' = y \; , \quad z' = z \; . \tag{2-37}$$

A simple reflection always transforms a right-handed coordinate system into a left-handed one, and conversely, independently of which hand you may claim to be the right one.



A simple reflection in the yz-plane.

General transformation

Let the new Cartesian coordinate system be characterized (in the old) by the origin $\boldsymbol{c}$ and the three orthogonal and normalized basis vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$, satisfying the usual relations (see fig. 2.1)

$$\left.\begin{array}{c} |\boldsymbol{a}_1| = |\boldsymbol{a}_2| = |\boldsymbol{a}_3| = 1 \\ \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 = \boldsymbol{a}_3 \cdot \boldsymbol{a}_1 = 0 \end{array}\right\} . \tag{2-38}$$

A position $\boldsymbol{x}' = (x'_1, x'_2, x'_3)$ in the new coordinate system must then correspond to the position

$$\boldsymbol{x} = \boldsymbol{c} + x'_1\boldsymbol{a}_1 + x'_2\boldsymbol{a}_2 + x'_3\boldsymbol{a}_3 \tag{2-39}$$

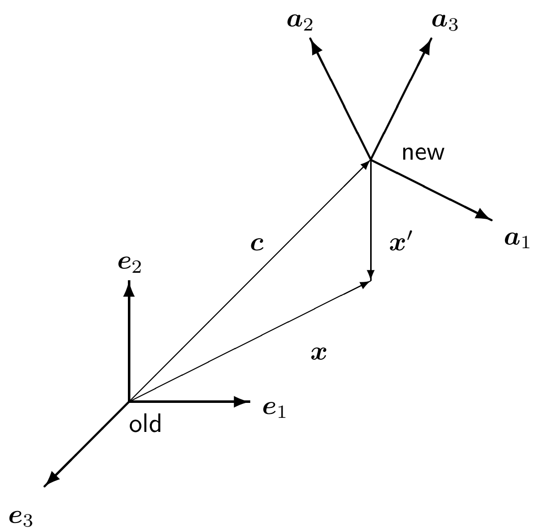


Figure 2.1: *Relations between the old and the new Cartesian systems.*

in the old, and the new coordinates are obtained by multiplying from the left with the new basis vectors and using orthonormality (2-38)

$$
\begin{aligned}
x_1' &= \boldsymbol{a}_1 \cdot (\boldsymbol{x} - \boldsymbol{c}) = a_{11}(x_1 - c_1) + a_{12}(x_2 - c_2) + a_{13}(x_3 - c_3) \ , \\
x_2' &= \boldsymbol{a}_2 \cdot (\boldsymbol{x} - \boldsymbol{c}) = a_{21}(x_1 - c_1) + a_{22}(x_2 - c_2) + a_{23}(x_3 - c_3) \ , \\
x_3' &= \boldsymbol{a}_3 \cdot (\boldsymbol{x} - \boldsymbol{c}) = a_{31}(x_1 - c_1) + a_{32}(x_2 - c_2) + a_{33}(x_3 - c_3) \ .
\end{aligned}
$$

where $a_{ij} = (\boldsymbol{a}_i)_j$ are the coordinates of the new basis vectors. In index notation this may be written more compactly as

$$x_i' = \sum_j a_{ij}(x_j - c_j) \ . \tag{2-40}$$

This is the most general transformation between two Cartesian coordinate systems and is characterized by a *translation vector* $\boldsymbol{c}$ and a *transformation matrix* $\mathbf{A} = \{a_{ij}\}$ having the new basis vectors as rows. It is not very difficult to show that the general transformation may be composed from a sequence of simple transformations (problem 2.18).

The transformation matrix for a simple translation along the x-axis is just the unit matrix, $\mathbf{A} = \mathbf{1}$. For a simple rotation around the z-axis, we obtain

$$\mathbf{A} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{2-41}$$

and for a simple reflection in the yz-plane it is a diagonal matrix with $(-1, 1, 1)$ along the diagonal.

Orthogonality

The orthogonality of the new basis vectors implies that

$$\boldsymbol{a}_i \cdot \boldsymbol{a}_j = \sum_k a_{ik} a_{jk} = \delta_{ij} \ . \tag{2-42}$$

On matrix form this is written

$$\mathbf{A} \cdot \mathbf{A}^\top = \mathbf{1} \tag{2-43}$$

where $\mathbf{A}^\top = \{a_{ji}\}$ is the transposed matrix having the new basis vectors as columns, and $\mathbf{1}$ is as before the 3×3 unit matrix.

Since the transposed matrix has the same determinant and the determinant of a product of matrices is the product of the determinants, we obtain by taking the determinant of (2-43) that $(\det \mathbf{A})^2 = 1$, or

$$\det \mathbf{A} = \pm 1 \ . \tag{2-44}$$

The transformation matrices are thus divided into two completely separated classes, those with determinant $+1$, called *rotations* or sometimes *proper rotations*, and those with determinant -1, called reflections. Since the simple reflection (2-37) has determinant -1, all reflections may be composed of a simple reflection followed by a rotation.

2.6 Scalars, vectors, and tensors

Various *geometric quantities*, scalars, vectors and tensors, may now be classified according to their behavior under rotations. Further subclassification according to the behavior under reflection and translation is also possible.

Classification under rotation

Scalars: A single quantity S is called a *scalar*, if it is invariant under rotations

$$\boxed{S' = S} \tag{2-45}$$

Thus the distance, the norm and the dotproduct are scalars. In physics the natural constants, material constants, as well as mass and charge of particles are scalars.

Vectors: Any triplet of quantities, $\boldsymbol{V}$, is called a *vector*, if it transforms under rotations according to

$$\boxed{V_i' = \sum_j a_{ij} V_j} \ . \tag{2-46}$$

In physics, velocity, acceleration, momentum, force, and many other quantities are vectors in this sense. The coordinates $\boldsymbol{x}$ of a point may also be called a vector according to this definition, but that is only correct in linear coordinate systems, in particular the Cartesian ones, and would be very wrong in curvilinear coordinates or curved spaces.

Tensors: Using the vector rule, the tensor product of two vectors $\boldsymbol{V}\boldsymbol{W}$, is found to transform according to the rule,

$$(\boldsymbol{V}'\boldsymbol{W}')_{ij} = V_i'W_j' = \left(\sum_k a_{ik}V_k\right)\left(\sum_l a_{jl}W_l\right) = \sum_{kl} a_{ik}a_{jl}V_kW_l \ ,$$

where we in the end have reordered the sums into a convenient form.

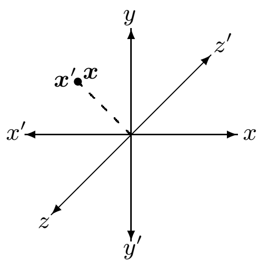
More generally, any set of nine quantities arranged in a matrix

$$\mathbf{T} = \{T_{ij}\} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \tag{2-47}$$

is called a *tensor* of rank 2, provided it obeys the transformation law

$$T'_{ij} = \sum_{kl} a_{ik}a_{jl}T_{kl} \tag{2-48}$$

In physics, the moment of inertia of an extended body and the quadrupole moment of a charge distribution are well-known tensors of second rank. Similarly, tensors of rank r have r indices and are collections of 3^r quantities that transform as the direct product of r vectors.



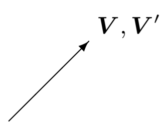
A complete reflection of the coordinate system in the origin. A rotation through π around the x-axis converts this to a simple reflection in the yz-plane.

∗ Classification under reflection

Instead of a simple reflection in the yz-plane, we shall classify quantities according to their behavior under a complete reflection of the coordinate system through its origin

$$\boldsymbol{x}' = -\boldsymbol{x} \ . \tag{2-49}$$

Geometrically, this transformation may be viewed as a simple reflection of a coordinate axis followed by a simple rotation through π around the same axis.



A polar vector retains its geometric placement under a reflection of the coordinate system.

Polar vectors: A vector which obeys the usual transformation equation (2-46) under rotations as well as reflections is called a *polar* vector. Under a reflection in the origin, the coordinates of a polar vector change sign just like the coordinates of a point, *i.e.* $\boldsymbol{V}' = -\boldsymbol{V}$. Since the coordinate axes all reverse direction, the geometrical position in space of a polar vector is unchanged by a reflection of the coordinate system, and the vector may faithfully be represented by an arrow, also under reflections. In physics, acceleration, force, velocity, and momentum are all polar vectors.

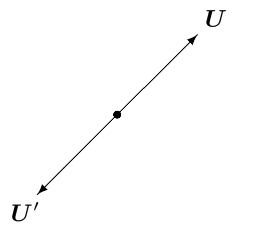
Axial vectors: There is, however, another possibility. The cross product of two polar vectors, $\boldsymbol{U} = \boldsymbol{V} \times \boldsymbol{W}$, behaves differently than a polar vector under a reflection. According to our rules for calculating the cross product, which are the same in all coordinate systems, we find

$$\boldsymbol{U}' = \boldsymbol{V}' \times \boldsymbol{W}' = (-\boldsymbol{V}) \times (-\boldsymbol{W}) = \boldsymbol{V} \times \boldsymbol{W} = \boldsymbol{U} \tag{2-50}$$

without the expected change of sign. Since $\boldsymbol{U}$ behaves normally under rotations with determinant $+1$, we conclude that the missing minus sign is associated with any transformation with determinant -1, that is with any reflection. Generalizing, we define an *axial* vector $\boldsymbol{U}$ as a set of three quantities, transforming according to the rule

$$\boxed{U_i' = \det \mathbf{A} \sum_j a_{ij} U_j} \,, \tag{2-51}$$

under a general coordinate transformation. The extra determinant will for an axial vector eliminate the sign change otherwise associated with a reflection in the origin, $\mathbf{A} = -\mathbf{1}$. In physics, angular momentum, moment of force (torque), and the magnetic field are axial vectors.



Geometrically, an axial vector has its direction reversed under a reflection of the coordinate system in the origin.

The direction of an axial vector depends on what we choose to be right and left. It is for this reason wrong to think of an axial vector as an arrow in space. Geometrically, it has magnitude and direction, but not *sense*, meaning that the positive direction of an axial vector is not a geometric property, but fixed by convention, and changes under a reflection of the coordinate system. For consistency, we may decide that one coordinate system, and all other coordinate systems that are obtained from it by proper rotation, are right-handed, whereas coordinate systems that are related to it by reflection, are left-handed.

The basis vectors of the old coordinate system are polar vectors, and get a change of sign, $\boldsymbol{e}_i' = -\boldsymbol{e}_i$, under a reflection in the origin. The basis vectors of the new coordinate system, $\boldsymbol{a}_i$, have in the new system (by definition) the same coordinates as the old in the old, *i.e.* $\boldsymbol{a}_i' = \boldsymbol{e}_i$. It would, however, be a mistake to take this to mean that the basis vectors are axial. They always transform as polar vectors under reflection.

Pseudo-scalars: The volume product of three polar vectors $P = \boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$ is a scalar quantity, which changes sign under a reflection of the coordinate system, or more generally $P' = \det \mathbf{A}\ P$, because of the cross product. Such quantities are called *pseudoscalars* to distinguish them from proper scalars, obeying (2-45). Likewise, axial vectors are called pseudovectors, and one may similarly define pseudotensors. The Levi-Civita symbol is a pseudo-tensor of third rank (problem 2.20).

The sign of a pseudoscalar is not absolute, but depends on the handedness of the coordinate system. One might think that physics had no use for such quantities, because after all physics itself does not depend on coordinate systems, only its mathematical description does. Nevertheless, magnetic charge, if it were ever found, would be pseudoscalar, and more importantly, some of the familiar elementary particles, for example the pi-mesons, are described by pseudoscalar fields.

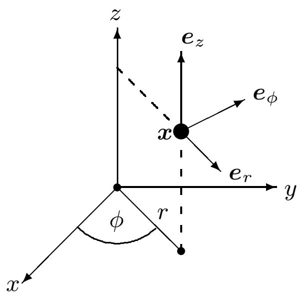
∗ Classification under translation

A vector may always be viewed as the difference between two positions $\boldsymbol{v} = \boldsymbol{b} - \boldsymbol{a}$, and is invariant under a pure translation $\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{c}$. Such vectors are called *proper*. Triplets that transform as vectors under rotations but change under translations, like the position $\boldsymbol{x}$ itself, are called *improper*.

∗ 2.7 Curvilinear coordinates

The distance between two points Euclidean space takes the simplest form (2-4) in Cartesian coordinates. The geometry of concrete physical problems may make non-Cartesian coordinates more suitable as a basis for analysis, even if the distance becomes more complicated in the new coordinates. Since the new coordinates are non-linear functions of the Cartesian coordinates, they define three sets of intersecting curves, and are for this reason called *curvilinear coordinates*.

At a deeper level, it is often the *symmetry* of a physical problem that points to the most convenient choice of coordinates. Cartesian coordinates are well suited to problems with translational invariance, cylindrical coordinates for problems that are invariant under rotations around a fixed axis, and spherical coordinates for problems that are invariant or partially invariant under arbitrary rotations. Elliptic and hyperbolic coordinates are also of importance but will not be discussed here (see [17, p. 455]).



Cylindrical coordinates and basis vectors.

Cylindrical coordinates

The relation between Cartesian coordinates x, y, z and cylindrical coordinates r, ϕ, z is given by

$$x = r\cos\phi \ , \tag{2-52a}$$
$$y = r\sin\phi \ , \tag{2-52b}$$
$$z = z \tag{2-52c}$$

with the range of variation $0 \leq r < \infty$ and $0 \leq \phi < 2\pi$. The first equations two simply define polar coordinates in the xy-plane. The last is rather trivial but included to emphasize that this is a transformation in 3-dimensional space.

Curvilinear basis: The *curvilinear basis vectors* are defined from the tangent vectors, obtained by differentiating the Cartesian position after the cylindrical coordinates,

$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\cos\phi, \sin\phi, 0) \ , \tag{2-53a}$$

$$\boldsymbol{e}_\phi = \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0) \ , \tag{2-53b}$$

$$\boldsymbol{e}_z = \frac{\partial \boldsymbol{x}}{\partial z} = (0, 0, 1) \ . \tag{2-53c}$$

As may be directly verified, they are orthogonal and normalized everywhere, a thus define a local curvilinear basis with an orientation that changes from place to place. An arbitrary vector field may therefore be resolved in this basis

$$\boldsymbol{V} = \boldsymbol{e}_r V_r + \boldsymbol{e}_\phi V_\phi + \boldsymbol{e}_z V_z \ , \tag{2-54}$$

where the vector coordinates

$$V_r = \boldsymbol{V} \cdot \boldsymbol{e}_r \ , \quad V_\phi = \boldsymbol{V} \cdot \boldsymbol{e}_\phi \ , \quad V_z = \boldsymbol{V} \cdot \boldsymbol{e}_z \ , \tag{2-55}$$

are the projections of $\boldsymbol{V}$ on the local basis vectors.

Resolution of the gradient: The derivatives after the cylindrical coordinates are found by differentiation through the Cartesian coordinates

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \ , \\
\frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y} \ .
\end{aligned}$$

From these relations we may calculate the "projections" of the gradient operator $\boldsymbol{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ on the cylindrical basis, and we obtain

$$\nabla_r = \boldsymbol{e}_r \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial r} \ , \tag{2-56a}$$

$$\nabla_\phi = \boldsymbol{e}_\phi \cdot \boldsymbol{\nabla} = \frac{1}{r}\frac{\partial}{\partial \phi} \ , \tag{2-56b}$$

$$\nabla_z = \boldsymbol{e}_z \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial z} \ . \tag{2-56c}$$

Conversely, the gradient may be resolved on the basis

$$\boxed{\boldsymbol{\nabla} = \boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \boldsymbol{e}_z \frac{\partial}{\partial z}} \ . \tag{2-57}$$

Together with the differential relations between the basis vectors

$$\frac{\partial \boldsymbol{e}_r}{\partial \phi} = \boldsymbol{e}_\phi \ , \tag{2-58a}$$

$$\frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\boldsymbol{e}_r \ , \tag{2-58b}$$

we are now in possession of all the necessary tools for calculating in cylindric coordinates.

The Laplacian: An operator which often occurs in differential equations is the *Laplace operator* or *Laplacian*,

$$\boldsymbol{\nabla}^2 = \nabla_x^2 + \nabla_y^2 + \nabla_z^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ . \tag{2-59}$$

In cylindrical coordinates this operator takes a different form, which may be found by squaring the resolution of the gradient (2-57). Keeping track of the order of the operators and basis vectors we get

$$\begin{aligned}
\boldsymbol{\nabla}^2 &= (\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\cdot(\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z) \\
&= (\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\cdot \boldsymbol{e}_r\nabla_r \\
&+ (\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\cdot \boldsymbol{e}_\phi\nabla_\phi \\
&+ (\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\cdot \boldsymbol{e}_z\nabla_z \\
&= \nabla_r^2 + \frac{1}{r}\nabla_r + \nabla_\phi^2 + \nabla_z^2 \ .
\end{aligned}$$

In the second equation we have distributed the first factor on the terms of the second, and going to the last equation we have furthermore distributed the terms of the first factor, using the orthogonality of the basis and taking into account that differentiation after ϕ may change the basis vectors according to (2-58).

Finally, using (2-56) we arrive the cylindrical Laplacian,

$$\boxed{\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}} \ , \tag{2-60}$$

expressed in terms of the usual derivatives.

Spherical coordinates

The treatment of spherical coordinates follows much the same pattern as cylindrical coordinates. Spherical or polar coordinates consist of the radial distance r, the polar angle θ and the azimuthal angle ϕ. If the z-axis is chosen as polar axis and the x-axis as the origin for the azimuthal angle, the transformation from spherical to Cartesian coordinates becomes,

$$x = r\sin\theta\cos\phi \ , \tag{2-61a}$$
$$y = r\sin\theta\sin\phi \ , \tag{2-61b}$$
$$z = r\cos\theta \ . \tag{2-61c}$$

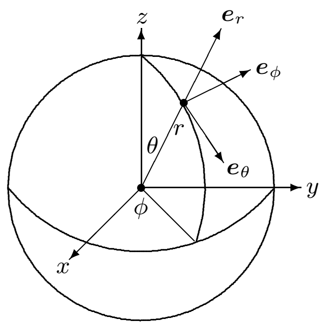
The domain of variation for the spherical coordinates is $0 \le r < \infty$, $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

Curvilinear basis: The normalized tangent vectors along the directions of the spherical coordinate are,

$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \ , \tag{2-62a}$$

$$\boldsymbol{e}_\theta = \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \ , \tag{2-62b}$$

$$\boldsymbol{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0) \ . \tag{2-62c}$$



Spherical coordinates and their basis vectors.

They are orthogonal, so that an arbitrary vector field may be resolved after these directions,

$$\boldsymbol{V} = \boldsymbol{e}_r V_r + \boldsymbol{e}_\theta V_\theta + \boldsymbol{e}_\phi V_\phi \tag{2-63}$$

with $V_a = \boldsymbol{e}_a \cdot \boldsymbol{V}$ for $a = r, \theta, \phi$.

Resolution of the gradient: The gradient operator may also be resolved on the basis,

$$\boxed{\boldsymbol{\nabla} = \boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi} \ , \tag{2-64}$$

where

$$\nabla_r = \boldsymbol{e}_r \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial r} \ , \tag{2-65a}$$

$$\nabla_\theta = \boldsymbol{e}_\theta \cdot \boldsymbol{\nabla} = \frac{1}{r}\frac{\partial}{\partial \theta} \ , \tag{2-65b}$$

$$\nabla_\phi = \boldsymbol{e}_\phi \cdot \boldsymbol{\nabla} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \ . \tag{2-65c}$$

The non-vanishing derivatives of the basis vectors are

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, \qquad \frac{\partial \boldsymbol{e}_r}{\partial \phi} = \sin\theta\, \boldsymbol{e}_\phi \ , \tag{2-66a}$$

$$\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r, \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial \phi} = \cos\theta \boldsymbol{e}_\phi \ , \tag{2-66b}$$

$$\frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\sin\theta\, \boldsymbol{e}_r - \cos\theta\, \boldsymbol{e}_\theta \ . \tag{2-66c}$$

These are all the relations necessary for calculations in spherical coordinates.

The Laplacian: The Laplacian (2-59) becomes in this case

$$\begin{aligned}
\boldsymbol{\nabla}^2 &= (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \\
&= (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot \boldsymbol{e}_r \nabla_r \\
&+ (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot \boldsymbol{e}_\theta \nabla_\theta \\
&+ (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot \boldsymbol{e}_\phi \nabla_\phi \\
&= \left(\nabla_r^2 + \frac{1}{r}\nabla_r + \frac{\sin\theta}{r\sin\theta}\nabla_r\right) + \left(\nabla_\theta^2 + \frac{\cos\theta}{r\sin\theta}\nabla_\theta\right) + \nabla_\phi^2 \ .
\end{aligned}$$

In the last step we have used the orthogonality and the derivatives (2-66). Finally, using (2-65) this becomes

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \ . \tag{2-67}$$

in standard notation.

Problems

2.1 Let $\boldsymbol{a} = (2, 3, -6)$ and $\boldsymbol{b} = (3, -4, 0)$. Calculate

a) the lengths of the vectors,

b) the dot product,

c) the cross product,

d) and the tensor product.

2.2 Are the vectors $\boldsymbol{a} = (3, 1, -2)$, $\boldsymbol{b} = (4, -1, -1)$ and $\boldsymbol{c} = (1, -2, 1)$ linearly dependent (meaning that there exists a non-trivial set of coefficients such that $\alpha\boldsymbol{a}+\beta\boldsymbol{b}+\gamma\boldsymbol{c} = \boldsymbol{0}$)?

2.3 Calculate the distance between two points on Earth in terms of longitude α, latitude β and height h over the average sea level.

2.4 Show that

$$|\boldsymbol{a}+\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\,\boldsymbol{a}\cdot\boldsymbol{b} \ , \tag{2-68a}$$

$$|\boldsymbol{a}\cdot\boldsymbol{b}| \leq |\boldsymbol{a}|\,|\boldsymbol{b}| \ , \tag{2-68b}$$

$$|\boldsymbol{a}+\boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}| \ , \tag{2-68c}$$

2.5 Show that the mixed product is a determinant

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ . \tag{2-69}$$

Use this to show that

$$(\boldsymbol{a}\times\boldsymbol{b}\cdot c)\ (\boldsymbol{d}\times\boldsymbol{e}\cdot f) = \begin{vmatrix} \boldsymbol{a}\cdot\boldsymbol{d} & \boldsymbol{a}\cdot\boldsymbol{e} & \boldsymbol{a}\cdot\boldsymbol{f} \\ \boldsymbol{b}\cdot\boldsymbol{d} & \boldsymbol{b}\cdot\boldsymbol{e} & \boldsymbol{b}\cdot\boldsymbol{f} \\ \boldsymbol{c}\cdot\boldsymbol{d} & \boldsymbol{c}\cdot\boldsymbol{e} & \boldsymbol{c}\cdot\boldsymbol{f} \end{vmatrix} \ . \tag{2-70}$$

2.6 Show that

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c} = \boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{a} = \boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{b} \ , \tag{2-71}$$

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{c})\,\boldsymbol{a} \ , \tag{2-72}$$

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}) \ , \tag{2-73}$$

$$|\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2\,|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2 \ . \tag{2-74}$$

2.7 Show that with the normal definition of the matrix product the following relations make sense for the tensor product

$$(\boldsymbol{a}\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c}) \tag{2-75}$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \tag{2-76}$$

2.8 Show that

$$\frac{\partial(\boldsymbol{a}\cdot\boldsymbol{b})}{\partial\boldsymbol{a}} = \boldsymbol{b} \tag{2-77}$$

and that

$$\frac{\partial\,|\boldsymbol{a}|}{\partial\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \ . \tag{2-78}$$

2.9 Show that

$$\sum_i \delta_{ii} = 3 \tag{2-79}$$

$$\sum_j \delta_{ij}\delta_{jk} = \delta_{ik} \tag{2-80}$$

2.10 Show that

$$\sum_i \nabla_i x_i = 3 \ , \tag{2-81}$$

$$\nabla_i x_j = \delta_{ij} \ , \tag{2-82}$$

$$\nabla_i\nabla_j(x_k x_l) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \ , \tag{2-83}$$

2.11 Show that the Levi-Civita symbol is completely antisymmetric in all three indices,

$$\epsilon_{ijk} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji} \tag{2-84}$$

2.12 Show that the product of two Levi-Civita symbols is (see problem 2.5)

$$\begin{aligned}\epsilon_{ijk}\,\epsilon_{lmn} &= \begin{vmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{vmatrix}\\ &= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km}\\ &\quad - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn}\end{aligned} \tag{2-85}$$

and from this

$$\sum_k \epsilon_{ijk}\,\epsilon_{lmk} = \begin{vmatrix}\delta_{il} & \delta_{im}\\ \delta_{jl} & \delta_{jm}\end{vmatrix} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \ , \tag{2-86}$$

$$\sum_{jk} \epsilon_{ijk}\,\epsilon_{ljk} = 2\delta_{ij} \ , \tag{2-87}$$

$$\sum_{ijk} \epsilon_{ijk}\,\epsilon_{ijk} = 6 \ . \tag{2-88}$$

2.13 Show that the trace (sum of the diagonal elements) of a tensor is invariant under a rotation.

2.14 Show that the Kronecker delta is a second rank tensor.

2.15 Show that if $W_i = \sum_j T_{ij} V_j$ and if it is known that $\boldsymbol{W}$ is a vector for all vectors $\boldsymbol{V}$, then T_{ij} must be a tensor.

2.16 Show that the distance $|\boldsymbol{x} - \boldsymbol{y}|$ is invariant under any transformation between Cartesian coordinate systems.

∗ **2.17** Consider two Cartesian coordinate systems and make no assumptions about the transformation $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ between them. Show that the invariance of the distance,

$$\left|\boldsymbol{x}' - \boldsymbol{y}'\right| = |\boldsymbol{x} - \boldsymbol{y}| \ , \tag{2-89}$$

implies that the transformation is of the form $\boldsymbol{x}' = \mathbf{A} \cdot \boldsymbol{x} + \boldsymbol{b}$ where $\mathbf{A}$ is an orthogonal matrix.

∗ **2.18** Show that the general Cartesian coordinate transformation may be built up from a combination of simple translations, rotations and reflections.

∗ **2.19** Show that under a simple rotation, a tensor T_{ij} transforms into

$$T'_{xx} = \cos\phi(T_{xx}\cos\phi + T_{xy}\sin\phi) + \sin\phi(T_{yx}\cos\phi + T_{yy}\sin\phi) \ , \tag{2-90a}$$
$$T'_{xy} = \cos\phi(-T_{xx}\sin\phi + T_{xy}\cos\phi) + \sin\phi(-T_{yx}\sin\phi + T_{yy}\cos\phi) \ , \tag{2-90b}$$
$$T'_{xz} = \cos\phi T_{xz} + \sin\phi T_{yz} \ , \tag{2-90c}$$
$$T'_{yx} = -\sin\phi(T_{xx}\cos\phi + T_{xy}\sin\phi) + \cos\phi(T_{yx}\cos\phi + T_{yy}\sin\phi) \ , \tag{2-90d}$$
$$T'_{yy} = -\sin\phi(-T_{xx}\sin\phi + T_{xy}\cos\phi) + \cos\phi(-T_{yx}\sin\phi + T_{yy}\cos\phi) \ , \tag{2-90e}$$
$$T'_{yz} = -\sin\phi T_{xz} + \cos\phi T_{yz} \ , \tag{2-90f}$$
$$T'_{zx} = T_{zx}\cos\phi + T_{zy}\sin\phi \ , \tag{2-90g}$$
$$T'_{zy} = -T_{zx}\sin\phi + T_{zy}\cos\phi \ . \tag{2-90h}$$
$$T'_{zz} = T_{zz} \tag{2-90i}$$

∗ **2.20** Show that

a) the Levi-Civita symbol satisfies

$$\sum_{lmn} a_{il} a_{jm} a_{kn} \epsilon_{lmn} = \det \mathbf{A}\, \epsilon_{ijk} \tag{2-91}$$

where $\mathbf{A}$ is an arbitrary matrix.

b) the Levi-Civita symbol (which by the definition of the cross product must be invariant, $\epsilon'_{ijk} = \epsilon_{ijk}$) obeys the rule

$$\epsilon'_{ijk} = \epsilon_{ijk} = \det \mathbf{A} \sum_{lmn} a_{il} a_{jm} a_{kn} \epsilon_{lmn} \tag{2-92}$$

for an arbitrary coordinate transformation (which has $\det \mathbf{A} = \pm 1$).

c) the cross product of two vectors $\boldsymbol{W} = \boldsymbol{U} \times \boldsymbol{V}$ must transform like

$$W'_i = \det \mathbf{A} \sum_j a_{ij} W_j \tag{2-93}$$