16
Material transport

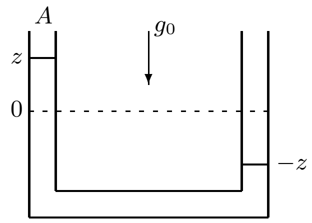
Momentum, angular momentum, and kinetic energy, are purely mechanical quantities that like mass are carried along with the movement of material. The force from a jet of water hitting a wall should for example be calculable from the momentum that the jet carries towards the wall. Besides the mechanical quantities, there are thermodynamic quantities such as internal energy and entropy, that are also transported with the material.

The mechanical and thermodynamic quantities mentioned here are all *extensive*, meaning that the amount in a composite body is the sum of the amounts in the parts. For continuous matter, where body parts may be of any size down to material particles, an extensive quantity is always described by a density field, and the amount of the quantity in a volume is calculated as the integral over the density. Other so-called *intensive* quantities, for example pressure and temperature, are not additive like the extensive quantities.

We have already seen that mass is transported without actual loss or gain, as expressed by the local and global equations for mass conservation. Extensive quantities are generally not conserved, but have *sources* that create and destroy them. Forces create momentum, work creates energy. Local and global transport equations for extensive quantities basically state the obvious: the amount of a quantity produced by the sources in a volume is either accumulated in the volume or leaves the volume through the surface.

Global transport equations often provide the first coarse estimate of the behavior of fluids. Many ordinary fluids are effectively incompressible and so close to ideal that internal friction (*i.e.* viscosity) can be disregarded in the first approximation. Balancing the budgets for mass, momentum, energy, and angular momentum in such systems is relatively easy and yields valuable information about their dynamics without having to solve the field equations explicitly.

16.1 Connected tubes



When the water is at rest, the water level will be the same in connected vertical tubes. The tube may have any shape between the two vertical sections.

We shall begin with a simple example of the use of energy conservation in a fluid system to derive how water moves in one of the most basic experiments in fluid dynamics. Consider a long straight tube with cross section A which is bent through 180° somewhere in the middle and placed with the open sections vertically upwards. Water is filled into the system, and as everybody knows, gravity will eventually make the levels of water equal in the two vertical tubes. Before reaching equilibrium the water sloshes back and forth with diminishing amplitude. Basic physics knowledge tells us that the energy originally given to the water is oscillating between being kinetic and potential, while slowly draining away because of viscosity which causes friction between the water and the tube.

Even if we do not know the exact solution to the fluid flow problem, we are nevertheless able to make a reasonable quantitative estimate of the behavior of the water. When the water level in one vertical tube is raised by z relative to the equilibrium level, mass conservation tells us that it is lowered by the same amount in the other vertical tube. The average velocity of the water in one tube is taken to be $v = dz/dt$, and the full length of the water column L. The total mass of the water column is $\rho_0 AL$, so that the kinetic energy becomes $\frac{1}{2}\rho_0 ALv^2$. Since a small water column of height z and weight $g_0\rho_0 Az$ effectively has been moved from one vertical tube to the other and thereby raised by z relative to equilibrium, the potential energy becomes $g_0\rho_0 Az^2$. Adding these contributions we get the following estimate for the total energy,

$$\mathcal{E} = \frac{1}{2}\rho_0 ALv^2 + g_0\rho_0 Az^2 \ , \tag{16-1}$$

where ρ_0 is the constant density of water.

For simplicity we assume that there is no friction that may eat away energy. The total energy must therefore be conserved, so that its time-derivative has to vanish,

$$\frac{d\mathcal{E}}{dt} = \rho_0 ALv\frac{dv}{dt} + 2g_0\rho_0 Az\frac{dz}{dt} = 0 \ . \tag{16-2}$$

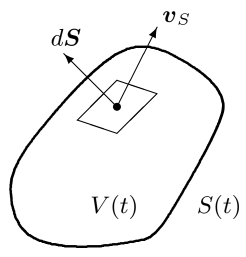
Using that $v = dz/dt$, we find the differential equation for the harmonic oscillator,

$$\frac{d^2z}{dt^2} = -\frac{2g_0}{L}z. \tag{16-3}$$

Had we included friction, there would also have been a damping term on the right hand side. The solution to the harmonic equation is of the form

$$z = h\cos\omega t \tag{16-4}$$

with amplitude h and angular frequency $\omega = \sqrt{2g_0/L}$. This is the frequency of a mathematical pendulum with length $L/2$, but contrary to the pendulum, the motion of the water is purely harmonic.

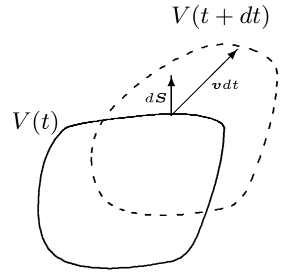


Every surface element $d\boldsymbol{S}$ *may move with a different velocity* $\boldsymbol{v}_S(\boldsymbol{x}, t)$.

16.2 Moving control volume

In continuum physics, the basic "body" is the *control volume*. For matter in motion, it is both natural and necessary to extend the notion of control volume to an arbitrary time dependent volume, $V(t)$, moving around in any way we wish. The moving volume is completely specified by its bounding surface $S(t)$, and the velocity $\boldsymbol{v}_S(\boldsymbol{x}, t)$ of a surface element $d\boldsymbol{S}$ containing point $\boldsymbol{x}$ at time t. Notice that this velocity can be chosen freely and needs only be defined on the surface itself and not all over space. It is not a field in the usual sense of the word.

If the surface of the control volume is constant in time, $\boldsymbol{v}_S = \boldsymbol{0}$, the volume is said to be *fixed*. If the volume always follows the flow of matter, $\boldsymbol{v}_S = \boldsymbol{v}$, it is said to be *comoving* or *material*.



The surface of a comoving volume follows the flow of matter.

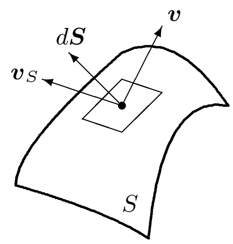
Reynolds' transport theorem

Let now $q(\boldsymbol{x}, t)$ be the density of some extensive quantity, for example mass ρ, or a component of momentum ρv_i. The amount of the quantity in a moving volume $V(t)$ is

$$Q(t) = \int_{V(t)} q(\boldsymbol{x}, t)\, dV \ . \tag{16-5}$$

This amount may change with time for two reasons, either because the density itself changes or because the volume moves. Suppressing as usual the explicit time dependence, the rate of change becomes

$$\frac{dQ}{dt} = \int_V \frac{\partial q}{\partial t}\, dV + \oint_S q\, \boldsymbol{v}_S \cdot d\boldsymbol{S} \ . \tag{16-6}$$



Matter moves with velocity $\boldsymbol{v}$ *through the surface element* $d\boldsymbol{S}$ *which itself moves with velocity* $\boldsymbol{v}_S$. *The velocity relative to the moving surface is* $\boldsymbol{v} - \boldsymbol{v}_S$.

because each surface element $d\boldsymbol{S}$ of the moving surface scoops up an amount $q\, \boldsymbol{v}_S dt \cdot d\boldsymbol{S}$ in a small time interval dt (see also problem 16.10).

Had the control volume instead been comoving, $\boldsymbol{v}_S = \boldsymbol{v}$, this would have been the *material* or *comoving* rate of change of the quantity, which we denote by

$$\frac{\delta Q}{\delta t} = \int_V \frac{\partial q}{\partial t}\, dV + \oint_S q\, \boldsymbol{v} \cdot d\boldsymbol{S} \ . \tag{16-7}$$

Eliminating the first term by means of (16-6), we may write

$$\boxed{\frac{\delta Q}{\delta t} = \frac{dQ}{dt} + \oint_S q\, (\boldsymbol{v} - \boldsymbol{v}_S) \cdot d\boldsymbol{S}} \ . \tag{16-8}$$

Since the material moves with velocity $\boldsymbol{v} - \boldsymbol{v}_S$ relative to the control surface, the equation simply states that *the material rate of change of a quantity equals the rate of change of the quantity in the given control volume plus its rate of loss through the surface*. The above equation goes by the name of *Reynolds' transport theorem* (1903) and has a fairly clear intuitive content. It is the foundation for all global considerations in continuum dynamics.

Osborne Reynolds (1842-1912). British engineer and physicist. Contributed to fluid mechanics in general, and to the understanding of lubrication, turbulence, and tidal motion, in particular.

Global balance

Let now the field $s(\boldsymbol{x},t)$ represent the *local source* of the quantity q, *i.e.* the amount produced per unit of volume and time. The corresponding *global source* is the integral,

$$\mathcal{S}(t) = \int_{V(t)} s(\boldsymbol{x},t)\, dV \ , \tag{16-9}$$

and it represents the total rate of production of the quantity q in the control volume. In a comoving control volume, nothing leaves through the surface, so that we must have

$$\boxed{\frac{\delta Q}{\delta t} = \mathcal{S}} \ . \tag{16-10}$$

This is the *global equation of balance.* Reading from right to left and using Reynolds' transport theorem (16-8), this equation states for a general control volume that *the quantity produced in the control volume is either accumulated in the volume or lost through its surface.*

Local equation of balance

In order to derive a *local equation of balance*, we use Gauss' theorem on the surface integral in (16-7) and find from the equation of global balance, which must be valid for arbitrary volumes V

$$\boxed{\frac{\partial q}{\partial t} + \mathsf{div}\,(q\boldsymbol{v}) = s} \ . \tag{16-11}$$

One notes the resemblance between this equation and the equation of continuity (14-25). It would, however, be a misnomer to call this the continuity equation for the quantity q, because of the historical meaning of "continuity" in this context being "constancy of mass".

The amount of a quantity per unit of mass $\widetilde{q} = dQ/dM = q/\rho$ is called the *specific* quantity. Inserting $q = \rho\widetilde{q}$ into local balance (16-11), we obtain

$$\begin{aligned} s &= \frac{\partial(\rho\widetilde{q})}{\partial t} + \mathsf{div}\,(\rho\widetilde{q}\boldsymbol{v}) = \rho\frac{\partial\widetilde{q}}{\partial t} + \widetilde{q}\frac{\partial\rho}{\partial t} + \widetilde{q}\mathsf{div}\,(\rho\boldsymbol{v}) + \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}\widetilde{q} \\ &= \widetilde{q}\left(\frac{\partial\rho}{\partial t} + \mathsf{div}\,(\rho\boldsymbol{v})\right) + \rho\left(\frac{\partial\widetilde{q}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\widetilde{q}\right) \ . \end{aligned}$$

The first parenthesis vanishes because of the equation of continuity (14-25), and the second is the material derivative (14-28) of the specific quantity. Consequently we have,

$$\boxed{\rho\frac{\delta\widetilde{q}}{\delta t} = s} \ , \tag{16-12}$$

which is an alternative formulation of local balance.

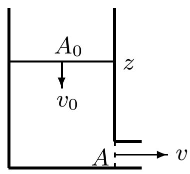
16.3 Mass balance

The equation of continuity (14-25) tells us that mass has no sources, so the global equation of balance for mass becomes,

$$\boxed{\frac{\delta M}{\delta t} = 0} \; . \tag{16-13}$$

It states the obvious fact that a comoving volume has constant mass. For a general control volume, it expresses through the Reynolds transport theorem (16-8) that a change of mass in the volume will always be balanced by the (signed) loss of mass through its surface.

Example 16.3.1: If you choose a control volume equal to your own body surface, it is to a good approximation comoving and your mass is constant, whichever way you move around while holding your breath. When you breathe, the mass of the control volume will change slightly in tune with the flux of air through the surface, where it covers your open mouth.



The water leaves the container through the drain.

Example 16.3.2: Suppose a cistern filled with water is flushed through a drain pipe in the bottom of the container (for example be the wine barrel of section 15.5). For simplicity we assume that the drain has cross section A, and that the cistern is vertical with cross section A_0. We choose a fixed control volume encompassing all the water in the cistern, so that if the water level is z, the total mass of the water is $M = \rho_0 A_0 z$. The flux through the drain is $\rho_0 A v$, where v is the average velocity of the water in the drain, and global mass balance (16-13) becomes

$$\frac{\delta M}{\delta t} = \frac{dM}{dt} + \rho_0 A v = \rho_0 A_0 \frac{dz}{dt} + \rho_0 A v = 0 \; . \tag{16-14}$$

But since $dz/dt = -v_0$ where v_0 is the average downwards velocity of the water in the cistern, this becomes $A_0 v_0 = Av$ which is (of course) Leonardo's law (14-13).

16.4 Momentum balance

The total momentum in a control volume is

$$\boldsymbol{\mathcal{P}} = \int_V \rho \boldsymbol{v} \, dV \; . \tag{16-15}$$

Although momentum strictly speaking is conserved for truly isolated systems, this is not so interesting because the systems we are dealing with, the control volumes, are subject to external forces that produce momentum.

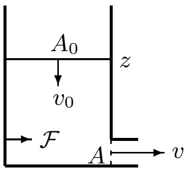
The specific momentum is $\boldsymbol{v}$, and comparing the basic equation of continuum dynamics (14-32) with the specific local balance (16-12), we see that the local source of momentum is the effective density of force, $\boldsymbol{f}^*$, and this is in fact not so surprising. The corresponding *global equation of momentum balance* then becomes,

$$\boxed{\frac{\delta \boldsymbol{\mathcal{P}}}{\delta t} = \boldsymbol{\mathcal{F}}}\,, \tag{16-16}$$

where

$$\boldsymbol{\mathcal{F}} = \int_V \boldsymbol{f}^* \, dV \tag{16-17}$$

is the total force acting on the material in the control volume. On the left hand side, the material time derivative again indicates that this is the comoving rate of change of momentum, which for an arbitrary moving control volume may be calculated from Reynolds' theorem (16-8) with q replaced by the momentum density, $\rho\boldsymbol{v}$. From right to left one may therefore read the equation of momentum balance (16-16) as saying that *the momentum produced by the force is either accumulated in the control volume or discharged through its surface.*



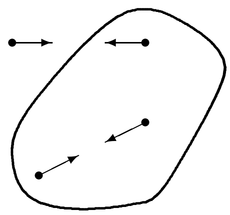
The loss of momentum from the water leaving the container through the drain requires a reaction force from the wall.

Example 16.4.1: The loss of momentum from the water jet emerging horizontally from the cistern (example 16.3.2) requires a force from the cistern which may be calculated from global momentum balance (16-16). Under steady (or rather quasi-steady) flow, the total momentum in the control volume is constant, so that the left hand side is entirely given by the flux of horizontal momentum through the drain. This flux may be estimated as the product of the density of momentum $\rho_0 v$ and the volume rate of change Av, or

$$\frac{\delta \mathcal{P}}{\delta t} = \rho_0 A v^2 = \mathcal{F} \tag{16-18}$$

With the parameters of the wine barrel (section 15.5), *i.e.* $A \approx 20\ \mathsf{cm}^2$ and $v \approx 6\ \mathsf{m/s}$, one finds $\mathcal{F} \approx 72\ \mathsf{N}$. Provided the cistern does not move, this force can only come from the walls of the cistern (see also section 16.4).

Internal and external forces

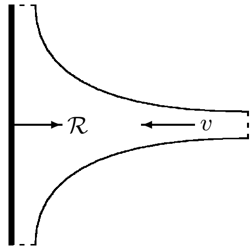


Gravitational forces between internal material particles cancel against each other, whereas gravitational forces between internal and external particles do not.

The total force on the control volume is composed from internal forces arising from the material inside the control volume itself and external forces arising from the environment (see page 278 for further details),

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_{\text{int}} + \boldsymbol{\mathcal{F}}_{\text{ext}} \ . \tag{16-19}$$

Internal volume forces, for example gravity, normally obey Newton's third law and therefore cancel against each other, so that $\boldsymbol{\mathcal{F}}_{\text{int}} = \mathbf{0}$. If they did not cancel, an isolated body would start to move all by itself. Contact forces likewise cancel against each other on internal boundaries. Consequently, internal forces can in general be ignored, and *the total force acting on the material in a control volume may be calculated from the external volume forces alone plus external contact forces, say pressure, acting on its surface.*



A jet of water hitting a wall. The control volume is limited by the jets surface and by the dashed lines.

Jet on a wall

A steady jet of water hitting a fixed wall without back-splash will transfer all its orthogonal momentum to the wall and this determines the reaction force from the wall. Disregarding gravity, the only external force acting on the water is the normal reaction force $\mathcal{R}$ from the wall. Choosing a control volume cutting through the jet and the radial outflow sufficiently far away, the momentum in the control volume will be constant, so that the only contribution to the material rate of change of the normal momentum is due to the influx of momentum from the jet. The momentum density in the jet (in the direction of the normal) may be estimated to be $-\rho_0 v$, where v is the average velocity at the entrance to the control volume. Multiplying with the volume rate $-Av$, where A is the jet's cross section, we find the rate of momentum lost to the control volume

$$\frac{\delta \mathcal{P}}{\delta t} = \rho_0 A v^2 . \tag{16-20}$$

Momentum balance (16-16) equates this to the reaction force,

$$\mathcal{R} = \rho_0 A v^2 . \tag{16-21}$$

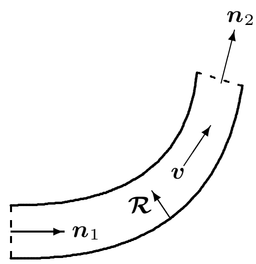
The quadratical growth of the reaction force is typical of all cases where a fluid is diverted away from an object and thereby gives up momentum, such as for example the drag on a car at high speed.

> **Example 16.4.2:** Police use water jets to control crowds. If the jet diameter is 10 cm, and the water speed 10 m/s, the force on a person becomes about $\mathcal{R} = 785$ N, corresponding to the weight of nearly 80 kg. You don't stay on your feet for long after being hit by this jet!

Around the bend

Fluids flowing through tubes, pipes, ducts, and conduits are important components in many of our machines and a large part of industrial processing. Whenever fluid flow changes direction, reaction forces from the sides of the conduits must deliver the necessary change in momentum. In the water pipes in our homes, the reaction forces are normally taken up by the fixations of the pipes to the walls.

Consider, for example, a section of a water pipe with a bend, which turns water flowing steadily with speed v from the direction $\boldsymbol{n}_1$ into the direction $\boldsymbol{n}_2$. The rate at which momentum enters the pipe is the momentum density $\rho_0 v \boldsymbol{n}_1$ times the volume rate of change Av, where A is the area of the cross section of the pipe. Similarly, the rate at which momentum exits the pipe is $\rho_0 v \boldsymbol{n}_2$ times Av. Since no momentum is accumulated in the pipe when the flow is steady, the left hand side of the momentum balance equation (16-16) for all the water in the pipe section becomes simply the total momentum flux out of the volume (using Reynolds' theorem (16-8)),



The sides of the pipe exert a reaction force $\boldsymbol{\mathcal{R}}$ on the water flowing around a bend.

$$\frac{\delta \boldsymbol{\mathcal{P}}}{\delta t} = \rho_0 v^2 A(\boldsymbol{n}_2 - \boldsymbol{n}_2) \tag{16-22}$$

The total external force on the water consists of the unknown reaction force, $\boldsymbol{\mathcal{R}}$, from the sides of the pipe plus the pressure forces acting on the entry and exit surfaces. Bernoulli's theorem tells us, however, that there must be the same pressure $p_1 = p_2$ at both ends of the pipe, because the entry velocity is the same as the exit velocity (assuming also that the bend is horizontal, so that there is no gravity at play). If the whole system is bathed in another fluid, say the atmosphere or the sea, with ambient pressure p_0, only the excess pressure $\Delta p = p_1 - p_0$ evokes a reaction force from the pipe. Consequently, the total force is

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{R}} - \Delta p A(\boldsymbol{n}_2 - \boldsymbol{n}_1) \ .$$

Momentum balance (16-16) allows us to solve for the reaction force

$$\boxed{\boldsymbol{\mathcal{R}} = A(\rho_0 v^2 + \Delta p)(\boldsymbol{n}_2 - \boldsymbol{n}_1)} \ . \tag{16-23}$$

If the pipe bends through an angle α, we have $|\boldsymbol{n}_2 - \boldsymbol{n}_1| = 2\sin(\alpha/2)$, so that the magnitude of the reaction force becomes

$$|\boldsymbol{\mathcal{R}}| = 2 \sin\frac{\alpha}{2} A(\rho_0 v^2 + \Delta p) \tag{16-24}$$

The reaction force has a dynamic part due to the water's motion and a static part due to the excess pressure in the water, which is there even when the water is not moving.

Example 16.4.3: For a typical $\alpha = 90^\circ$ horizontal bend in a 1 inch water pipe (diameter) with water running at $v = 3$ m/s, the magnitude of the reaction force due to the moving water becomes a little less than 10 N, which is the same as the weight of 1 kg.

Approximations

Although having the appearance of an exact calculation, there are approximations involved here. First of all we have assumed that the velocity field in the jet or pipe is the same all over the cross section. This could be remedied by defining v to be the average of the velocity field over the cross section (as in Leonardo's law, page 221). But in calculating the momentum flux, we need the average of the square of the velocity, which is not in general the same as square of the average. This is in fact the greatest source of error, but without solving the field equations to find the actual distribution of velocities, we cannot get a better result (see, however, chapter 18). Finally, we have implicitly assumed that there is no friction between the water and the pipe (and therefore no pressure difference between the two ends), which is also reasonable under normal circumstances.

16.5 Angular momentum balance

The angular momentum of a material particle with momentum $d\boldsymbol{\mathcal{P}}$ is

$$d\boldsymbol{\mathcal{L}} = \boldsymbol{x} \times d\boldsymbol{\mathcal{P}} = \boldsymbol{x} \times \boldsymbol{v}\, dM = \rho \boldsymbol{x} \times \boldsymbol{v}\, dV \ , \tag{16-25}$$

corresponding to an *angular momentum density* $\rho\boldsymbol{x} \times \boldsymbol{v}$ and *specific angular momentum* $\boldsymbol{x} \times \boldsymbol{v}$. Consequently,

$$\boldsymbol{\mathcal{L}} = \int_V \rho\, \boldsymbol{x} \times \boldsymbol{v}\, dV \ , \tag{16-26}$$

is the total amount of angular momentum in an arbitrary control volume.

Although this seems intuitively clear, there are still some tacit assumptions involved. The above expression contains only the angular moment of the center-of-mass motion of the material particles. The fast random (thermal) motion of the molecules of a gas could in principle also contribute, but as long as there is no correlation between the positions and velocities of the molecules making up a material particle, this contribution tends to average out (see problem 16.11). Furthermore, there might be an intrinsic angular moment density, a *spin density*, associated with the particles in addition to the above *orbital* angular momentum. We shall, however, disregard that possibility in the following.

Using the equation of local balance (16-11) on the specific angular momentum $\boldsymbol{x} \times \boldsymbol{v}$, the local source of angular momentum is found to be,

$$\rho \frac{\delta(\boldsymbol{x} \times \boldsymbol{v})}{\delta t} = \rho \frac{\delta \boldsymbol{x}}{\delta t} \times \boldsymbol{v} + \rho \boldsymbol{x} \times \frac{\delta \boldsymbol{v}}{\delta t} = \boldsymbol{x} \times \boldsymbol{f}^* \ , \tag{16-27}$$

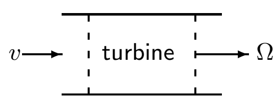
where we have used that $\delta\boldsymbol{x}/\delta t = \boldsymbol{v}$. The corresponding *global equation of angular momentum balance* takes the form,

$$\boxed{\frac{\delta \boldsymbol{\mathcal{L}}}{\delta t} = \boldsymbol{\mathcal{M}}} \ , \tag{16-28}$$

where

$$\boldsymbol{\mathcal{M}} = \int_V \boldsymbol{x} \times \boldsymbol{f}^*\, dV \tag{16-29}$$

is the total moment of the forces (or couple) acting on the material in the control volume.



Water streams through the turbine with average speed v and the turbine rotates with angular velocity Ω.

Example 16.5.1 (Water turbine): Water streams through a circular pipe with radius a at an average longitudinal velocity v and passes through a turbine rotating with angular frequency Ω. Under steady conditions, the total angular momentum $\boldsymbol{\mathcal{L}}$ of the water in the turbine will be constant (and unknown), so that the material rate of change is determined by the flux of angular momentum through the downstream end of the system. Assuming that the water emerges from the turbine close to the pipe wall with velocity $a\Omega$, the density of angular momentum is $\rho_0 \Omega a^2$. Multiplying with the volume rate $\pi a^2 v$, we obtain

$$\frac{\delta \mathcal{L}}{\delta t} \approx \pi \rho_0 \Omega a^4 v \ . \tag{16-30}$$

From angular momentum balance (16-28), we know that this must equal the moment of force (here a torque) acting on the turbine, which in terms of the mass flux through the pipe, $Q = \rho_0 \pi a^2 v$, may be written

$$\mathcal{M} \approx \Omega a^2 Q \ . \tag{16-31}$$

A number of approximations have again been made underway, so this is only a rough estimate. Suppose the turbine has radius $a = 50$ cm and rotates at $\Omega = 1200$ rpm $\approx$ 120 rad/s with a throughput of $Q = 200$ kg/s. Then $v \approx 0.25$ m/s and the couple becomes $\mathcal{M} \approx 6,000$ Nm.

External and internal moments

The total moment may be split in an internal and external part (see page 278 for further details)

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_{\text{int}} + \boldsymbol{\mathcal{M}}_{\text{ext}} \ . \tag{16-32}$$

In this case, one may ignore the internal volume forces, provided they obey Newton's third law and furthermore like gravity are *central*, *i.e.* act along the line connecting two material particles (see problem 16.9). If the stress tensor is symmetric (which it normally is), the moment of internal contact forces also cancels, such that $\boldsymbol{\mathcal{M}}_{\text{int}} = \mathbf{0}$ and *the total moment of forces may be calculated alone from the moment of the external volume forces and the moment of the contact forces acting on the external surface of the body.*

16.6 Kinetic energy balance

The kinetic energy of a material particle with mass dM is

$$d\mathcal{T} = \frac{1}{2}\boldsymbol{v}^2 \, dM = \frac{1}{2}\rho \boldsymbol{v}^2 \, dV \ , \tag{16-33}$$

corresponding to a *density of kinetic energy* $\frac{1}{2}\rho\boldsymbol{v}^2$ and a specific kinetic energy $\frac{1}{2}\boldsymbol{v}^2$. The total kinetic energy in a control volume becomes

$$\mathcal{T} = \int_V \frac{1}{2}\rho \boldsymbol{v}^2 \, dV \ . \tag{16-34}$$

A moment of reflection should, however, lead to serious reservations about this expression, because it only represents the kinetic energy associated with the overall flow of matter. There is definitely also kinetic energy, called *heat*, associated with the rapid thermal motion of the molecules relative to the flow (see problem 16.11). Since kinetic energy is positive definite, the molecular contribution cannot average out as it did for the case of angular momentum.

The proper discussion of heat in continuous systems, *i.e.* continuum *thermodynamics*, will be postponed until later chapters. Here we shall simply use the above expression (16-34) for the kinetic energy of the flow and analyze its dynamic behavior. Since friction forces, *i.e.* viscosity, are capable of converting motion into heat, we must not expect in general to be able to find a conserved energy, even for totally isolated systems. But if friction forces are absent, there will be no dissipation of energy into heat, leading to conservation of the total energy for isolated systems (see section 16.7).

Using the local equation of balance (16-12) on the specific kinetic energy $\frac{1}{2}\boldsymbol{v}^2$, we find the *local source of kinetic energy*

$$\rho \frac{\delta\left(\frac{1}{2}\boldsymbol{v}^2\right)}{\delta t} = \rho \boldsymbol{v} \cdot \frac{\delta \boldsymbol{v}}{\delta t} = \boldsymbol{v} \cdot \boldsymbol{f}^* \ . \tag{16-35}$$

The corresponding *global equation of kinetic energy balance* takes the form

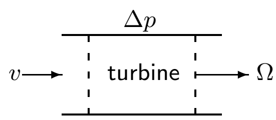
$$\boxed{\frac{\delta \mathcal{T}}{\delta t} = P} \ , \tag{16-36}$$

where

$$P = \int_V \boldsymbol{v} \cdot \boldsymbol{f}^* \, dV \tag{16-37}$$

is the total rate of work, or *power*[1], of the forces acting on the material in the control volume.

Example 16.6.1 (Water turbine, continued): Returning to the water turbine of example 16.5.1, we find that the kinetic energy $\mathcal{T}$ is constant under steady flow conditions, so that the the material rate of change is given by the flux of kinetic energy through the system. At the entry, the kinetic energy density is $\frac{1}{2}\rho_0 v^2$ whereas it is approximately $\frac{1}{2}\rho_0(v^2 + (\Omega a)^2)$ at the exit, because the speed of rotation must be added to the forward speed. Multiplying the difference by the volume rate $\pi a^2 v$ we find the material rate of change



There is a pressure drop Δp over the length of the turbine.

$$\frac{\delta \mathcal{T}}{\delta t} \approx \frac{1}{2}\pi\rho_0\Omega^2 a^4 v = \frac{1}{2}\Omega^2 a^2 Q \ , \tag{16-38}$$

where $Q = \rho_0 \pi a^2 v$ as before is the mass flux through the system. For nearly ideal and incompressible flow, the only forces that perform work on the system are the pressures at the entry and exit. Their power is $P = \pi a^2 \Delta p\, v$ where Δp is the pressure drop over the turbine. Kinetic energy balance (16-36) sets the power equal to the flux of kinetic energy, so that we get the pressure drop over the turbine,

$$\Delta p \approx \frac{1}{2}\rho_0 \Omega^2 a^2 \ , \tag{16-39}$$

[1]There is a slight clash between the symbol for total momentum $\mathcal{P}$ and for total power P, but it is resolved by momentum being the script version of the letter.

With the numbers given above, this becomes $\Delta p \approx 1.8$ MPa = 18 bar, corresponding to 180 m water head. The total power going into the turbine from the water is thus $P \approx 0.36$ MW.

Internal and external power

As before, we may also split the total power into an internal and external part (see page 279 for details)

$$P = P_{\text{int}} + P_{\text{ext}} \ . \tag{16-40}$$

Although internal volume forces, say gravity, cancel against each other in the total force, they will not in general cancel against each other in the total power, because different material particles move with different velocities. Internal contact forces also cancel against each other on internal boundaries, but may likewise perform work on compressible or viscous fluids, because neighboring material particles move with slightly different velocities.

If the fluid is effectively incompressible and nearly ideal, the power of the internal contact forces can be ignored, and *the only forces that actually do perform work are the internal and external volume forces as well as the contact forces acting on the surface of the control volume.*

16.7 Energy balance

In a time-independent external field of gravity $\boldsymbol{g}_{\text{ext}}(\boldsymbol{x}) = -\boldsymbol{\nabla}\Phi_{\text{ext}}(\boldsymbol{x})$, a material particle has potential energy $\Phi_{\text{ext}}\, dM$ (see section 3.5). The energy of a material particle is as in Newtonian particle mechanics defined to be the sum of the kinetic and potential energies

$$d\mathcal{E} = \frac{1}{2}\boldsymbol{v}^2\, dM + \Phi_{\text{ext}}\, dM \ , \tag{16-41}$$

corresponding to an *energy density*,

$$\epsilon = \frac{1}{2}\rho\boldsymbol{v}^2 + \rho\Phi_{\text{ext}} \ . \tag{16-42}$$

The total energy in a control volume is as usual calculated as the integral over this density,

$$\mathcal{E} = \int_V \epsilon\, dV \tag{16-43}$$

From local balance (16-12) with the specific energy $\widetilde{\epsilon} = \frac{1}{2}\boldsymbol{v}^2 + \Phi_{\text{ext}}$, we obtain in the now familiar way the *local source of energy*,

$$\rho\frac{\delta\left(\frac{1}{2}\boldsymbol{v}^2 + \Phi_{\text{ext}}\right)}{\delta t} = \boldsymbol{v}\cdot(\boldsymbol{f}^* - \rho\boldsymbol{g}_{\text{ext}}) \ , \tag{16-44}$$

and the corresponding *global equation of energy balance* becomes

$$\boxed{\frac{\delta \mathcal{E}}{\delta t} = P'} \tag{16-45}$$

where

$$P' = \int_V (\boldsymbol{f}^* - \rho \boldsymbol{g}_{\rm ext})\, dV \tag{16-46}$$

is the power of all forces except the external gravitational forces, that are now included in the total energy on the left hand side. If this reduced power vanishes, the total energy is conserved.

Internal volume forces, for example the self-gravity of a planet or star, may likewise be included in the total energy and removed from the power. We shall later see how some contact contributions to the power may be moved into the expression of the energy. Assuming that the fluid is effectively incompressible and nearly ideal, and in the absence of other volume forces, only the power of the contact forces acting on the surface of the body will be seen to contribute to the right hand side. If furthermore the system is isolated from such forces, the total energy will be conserved.

16.8 Down the drain

In many practical cases, kinetic energy balance or energy balance, is most convenient to use, because the pressure forces acting on the sides of fluid conduits are orthogonal to the flow and perform no work. Assuming also that there is no friction (viscosity) and that the fluid is incompressible, the external power is entirely due to pressure. We shall later also study this problem for viscous fluids.

Take, for example, the flushing of a water-filled cistern through a narrow drain. For simplicity we assume that the drain is a horizontal pipe of length L and cross section A, and that the cistern is vertical with cross section $A_0 \gg A$. Initially, the water level in the cistern is h, the pipe is filled, and all the water is assumed to be at rest. Experience indicates that after the drain is opened the water is rapidly accelerated to a quasi-stationary terminal speed.

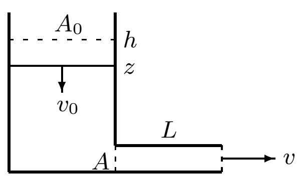
From mass conservation we know that the amount of water that at any moment runs out of the pipe equals the amount lost from the cistern. If v is the (average) horizontal velocity in the pipe and v_0 the (average) vertical velocity in the cistern, we have Leonardo's law

$$Av = A_0 v_0 \ , \tag{16-47}$$

In section 15.5 we used Bernoulli's theorem to show that the terminal speed,

$$U = \sqrt{2g_0 h} \ , \tag{16-48}$$

was the same as the water would have obtained by falling freely from the top of the water surface in the cistern to the exit of the pipe. But Bernoulli's theorem could not tell us how the water arrived at this speed.



The water accelerates in the pipe while pouring out through the exit.

Energy balance

Let now the water level be $z < h$ in the cistern. Choosing a control volume containing all the water in the cistern and pipe, the total energy becomes,

$$\mathcal{E} = \frac{1}{2}\rho_0 A_0 z v_0^2 + \frac{1}{2}\rho_0 A L v^2 + \frac{1}{2} g_0 \rho_0 A_0 z^2 \ . \tag{16-49}$$

The first two terms are the kinetic energies of the water in the cistern and the pipe, and the last is the potential energy of the water in the cistern, calculated as the height $z/2$ of the center of mass of the water in the cistern multiplied with the weight of the water, $\rho_0 A_0 z g_0$. The potential energy of the water in the pipe is constant, and need not be included. With this choice of coordinates, the water in the pipe at $z = 0$ has anyway zero potential energy.

The ratio of the kinetic energies in the cistern and the pipe (the first two terms in the energy) is

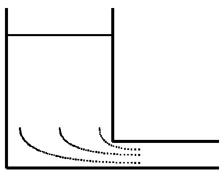
$$\frac{A_0 z v_0^2}{A L v^2} = \frac{zA}{LA_0} \ . \tag{16-50}$$

Under the reasonable condition that $hA \ll LA_0$, we may (as we shall do in the following) disregard the kinetic energy of the water in the cistern.

The only external forces that perform work on the water is the atmospheric pressure p_0 acting on the open water surfaces in the cistern and at the pipe exit. Using mass conservation (16-47), it follows immediately that that the total power of the atmospheric pressure vanishes, $P = A_0 p_0 v_0 - A p_0 v = 0$. The water leaving through the pipe exit carries energy out of the system at a rate given by the energy density $\frac{1}{2}\rho_0 v^2$ times the volume flux, Av. The energy balance (16-45) therefore takes the form

$$\frac{\delta \mathcal{E}}{\delta t} = \frac{d\mathcal{E}}{dt} + \frac{1}{2}\rho_0 A v^3 = 0 \ . \tag{16-51}$$

Inserting the expression for the energy (16-49) (dropping the tiny first term) and using that $dz/dt = -v_0$, we obtain after eliminating v_0 by means of mass conservation (16-47) the following differential equation for the velocity in the pipe,



The kinetic energy of the water in the transition region between cistern and pipe is assumed to be negligible.

$$\boxed{\frac{dv}{dt} = \frac{2g_0 z - v^2}{2L} \approx \frac{U^2 - v^2}{2L}} \ . \tag{16-52}$$

where we have also assumed that $z \approx h$ in the last step.

In deriving this equation we have as before made the approximation of equating the averages of products with the product of averages. Furthermore, we do not know the actual flow pattern in the cistern near the drain without solving Euler's equation, but we shall assume that the transition from v_0 in the cistern to v in the drain happens in a small volume of dimensions comparable to the linear size of the drain, say $\sqrt{A}$. Provided $L \gg \sqrt{A}$, it should be permissible to ignore the kinetic energy of the water in the transition region in comparison with the kinetic energy of the water in the pipe (see problem 16.5).

Acceleration time

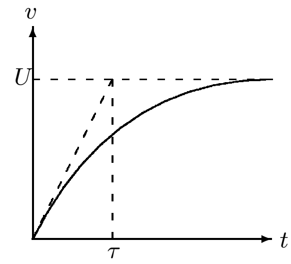
The right hand side of (16-52) vanishes for $v = U$, which is the terminal velocity. Solving the differential equation with $v = 0$ at $t = 0$, we find

$$v(t) = U \tanh \frac{t}{\tau} , \tag{16-53}$$

where

$$\tau = \frac{2L}{U} . \tag{16-54}$$

At $t = \tau$, the water has reached 76 % of terminal velocity, which is afterwards approached exponentially fast. At $t = 2\tau$, it has reached 96 % of terminal speed.



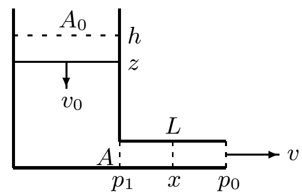
The rise of the velocity in the drain pipe.

Example 16.8.1 (Barrel of wine): For a cylindrical barrel of wine (see section 15.5) with diameter 1 m and height 2 m, emptied through a spout with diameter 5 cm and length L = 20 cm, the terminal speed is 6 m/s, and the water obtains 96 % of this speed speed after just 0.13 seconds! If you try to put back the plug in the spout after merely half a second, there will nevertheless arrive about 5 liters of wine on the floor (and on you).

Pressure drop in the pipe

Until now we have completely escaped the problem of the pressure in the cistern and in the pipe. Taking the x-axis along the pipe, and using that v does not depend on x, Euler's equation (15-1) becomes

$$\frac{dv}{dt} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \tag{16-55}$$



During acceleration there is a pressure drop between the entry and exit of the pipe.

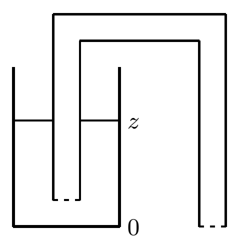
Since $v(t)$ takes the same value everywhere along the pipe, the pressure gradient is independent of x. Inserting the acceleration (16-52) and multiplying with the length L of the pipe, we obtain the pressure difference between the pipe entry and the exit

$$\Delta p = p_1 - p_0 = \frac{1}{2}\rho_0(U^2 - v^2) = \frac{\rho_0 g_0 h}{\cosh^2(t/\tau)} . \tag{16-56}$$

At $t = 0$ the entry pressure p_1 equals the hydrostatic pressure of the water in the cistern, as one would expect, and for $t \to \infty$ it approaches atmospheric pressure p_0.

Shape of the pipe

The energy (16-49) and thereby the equation of motion (16-52) does not depend on the shape of the pipe as long as it has length L and the difference in height between the water surface in the cistern and the pipe exit is z. For a sloping or curving pipe, there may be a non-zero potential energy of the water in the pipe, but since it is constant in time, that cannot influence the equation of motion. It will, however, influence the pressure distribution in the pipe.



The cistern may even be drained through a siphon. What matters is only the length of the pipe and the difference in height between the levels of water in the cistern and the pipe exit.

16.9 The Roman water-hammer pump

The Romans used a so-called *water-hammer* to raise water up above the level of a lake or reservoir (see fig. 16.1). A long aqueduct of length L runs downwards from a lake with a total vertical drop D from the surface of the lake. When the water is streaming steadily at average velocity U_1, the flow is suddenly blocked with a wedge and the momentum of the moving water now makes the water rise in a vertical pipe while its velocity decreases, until it overflows into a cistern at height h. For simplicity, the pipe is assumed to have the same cross section, A, as the aqueduct, so that the water according to Leonardo's law at any time has the same average velocity v everywhere in the system. We assume that the levels of water in the lake and the cistern stay essentially constant.

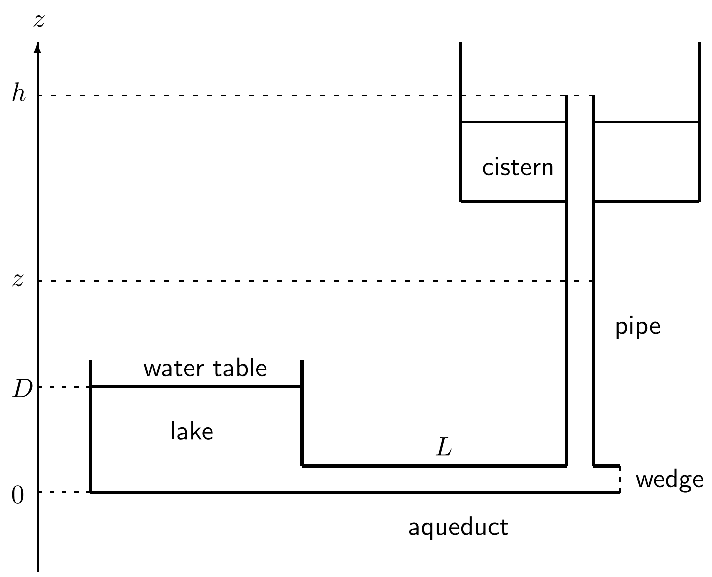


Figure 16.1: *The Roman water-hammer pump consists of a long aqueduct draining water from a large reservoir or lake. At regular intervals a slave will quickly insert a wedge to block the flow, which makes the water rise vertically through a pipe and flow into a cistern, thereby supplying the noble household with running water.*

There are clearly three phases in the cyclic operation of the water-hammer pump. 1) The aqueduct is opened and the water runs freely through it until it reaches the terminal steady velocity U. 2) After the wedge has been put in and the aqueduct closed, the water rises in the vertical pipe, until it overflows into the cistern. 3) Water then flows with decreasing speed into the cistern until it stops. The wedge is finally removed and the water starts to run freely again.

Phase 1: Accelerating to terminal speed.

Bernoulli's theorem tells us that the terminal speed is equal to the free-fall speed through the drop D,

$$U_1 = \sqrt{2g_0 D} \; . \tag{16-57}$$

For $D = 1$ m, this becomes $U_1 \approx 4.5$ m/s. The time it takes for the water to reach 76 % of terminal speed is given by (16-54)

$$\tau_1 \approx \frac{2L}{U_1} \; , \tag{16-58}$$

which for $L = 100$ m and $D = 1$ m comes to about 45 s.

Phase 2: Rise of the water

After closing the aqueduct, the water must change direction and rise in the vertical pipe. At the moment the wedge is put in, it is hit by a horizontal force which may be calculated from (16-23) to be $\mathcal{F} \approx \rho_0 U_1^2 A = 2\rho_0 g_0 D A$, which is equivalent to the hydrostatic pressure at a vertical depth $2D$. For $D = 1$ m this comes to about 0.2 bar, which does not sound of so much. But even for a fairly small aqueduct cross section, say $A = 10 \times 10$ cm^2, the force on the wedge nevertheless jumps to about 200 N, corresponding to a weight of 20 kg. The water slams into the wedge at considerable strength (and noise), as if the wedge were hit by a hammer (wherefrom the name).

Energy conservation

We choose a comoving control volume consisting of all the water the lake, the aqueduct, and the pipe. Ignoring the kinetic energy of the water in the lake, which is vanishingly small for a lake with a large surface area, we find

$$\mathcal{E} = \frac{1}{2}\rho_0 A(L+z)v^2 + \frac{1}{2}g_0\rho_0 A z^2 \; , \tag{16-59}$$

where z is the current height of the water column in the pipe.

Since the atmospheric pressure produces no power (see section 16.8), and since there is no flux of energy through the surface of the comoving control volume, the total energy must be conserved. Equating the current energy with the initial energy we obtain

$$(L+z)v^2 + g_0 z^2 = L U_1^2 \; , \tag{16-60}$$

after dividing out the common factor $\frac{1}{2}\rho_0 A$.

Solution

For $z \ll L$, we may replace $L + z$ by L, and we get a differential equation for z,

$$v = \frac{dz}{dt} = \sqrt{U_1^2 - \frac{g_0}{L} z^2} \ . \tag{16-61}$$

Using the initial conditions that $z = 0$ for $t = 0$, the solution is harmonic (see also section 16.1)

$$z = \sqrt{2LD} \sin\left(t\sqrt{\frac{g_0}{L}}\right) \ . \tag{16-62}$$

The maximal height that the water can be raised to is therefore $\sqrt{2LD}$, which comes to about 14 m for $L = 100$ m and $D = 1$ m.

For the pump to work, we must require the pipe exit height to be less than the maximum, *i.e.* $h < \sqrt{2LD}$. The time it takes for the water to rise to this level is

$$\tau_2 = \sqrt{\frac{L}{g_0}} \arcsin \frac{h}{\sqrt{2LD}} \ , \tag{16-63}$$

which comes to about 2.5 s for $h = 10$ m. The speed of the water at the moment it flows over into the cistern is

$$U_2 = U_1 \sqrt{1 - \frac{h^2}{2LD}} \ , \tag{16-64}$$

which with the given parameter values becomes $U_2 \approx 3.2$ m/s.

Phase 3: Filling the cistern

Now the lake is being drained through a negative "drop", and the dynamics is described by (16-52) with the drop z replaced by $-h + D$, or

$$\frac{dv}{dt} = \frac{-U_3^2 - v^2}{2L} \ , \tag{16-65}$$

where

$$U_3 = \sqrt{2(h - D)g_0} \tag{16-66}$$

is the free-fall velocity from the pipe exit in the cistern to the water level in the lake, which comes to about $U_3 \approx 13.4$ m/s with the given parameters.

The above differential equation may be solved exactly (problem 16.7), but since in our example $v^2/U_3^2 < U_2^2/U_3^2 \approx 1/16$, it is permitted to drop v^2. The solution to the resulting differential equation which has $v = U_2$ for $t = 0$ is

$$v = U_2 - \frac{U_3^2}{2L} t \ . \tag{16-67}$$

The velocity vanishes for $t = \tau_3$ where

$$\tau_3 = \frac{2LU_2}{U_3^2} \ . \tag{16-68}$$

In the present case, this becomes about 3.6 s.

The total volume of water that arrives in the cistern during a single beat of the pump becomes

$$Q = \int_0^{\tau_3} Av\,dt = AL\left(\frac{U_2}{U_3}\right)^2 \ , \tag{16-69}$$

which is a fraction,

$$\frac{Q}{AL} = \left(\frac{U_2}{U_3}\right)^2 \approx \frac{D}{h} - \frac{h}{2L} \ , \tag{16-70}$$

of the amount of water in the aqueduct. For $L = 100$ m, $D = 1$ m, and $h = 10$ m, about 5% of the water in the aqueduct is delivered to the cistern with each beat of the pump.

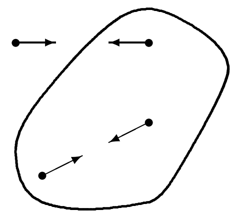
Completing the cycle

The cycle is now complete. The slave can remove the wedge and let the water flow again. We have seen that after picking up speed, it takes only a few seconds for the water to rise to the cistern, deliver the load and come to a full stop. Altogether the water-hammer pump cycle takes about one minute, with at least 90 % of the time spent in waiting for the water to reach terminal speed. The slave probably didn't mind.

∗ 16.10 Global sources

So far we have loosely discussed the integrated sources that were needed in the global equations of balance. Now we shall more systematically analyze the relations between the sources and the internal and external forces acting on a body.

The effective force (8-20) consists of a true volume force $\boldsymbol{f}$ plus a contribution from the contact forces, expressed as the divergence of the stress tensor. The material inside the control volume may be subject to *external* volume forces, $\boldsymbol{f}_{\rm ext}$, for example gravity, arising from material outside the volume, as well as *internal* volume forces, $\boldsymbol{f}_{\rm int}$, created by the material in the volume itself, for example the self-gravity of a planet or star. The true density of volume forces is the sum of the external and internal part



External and internal forces in a control volume.

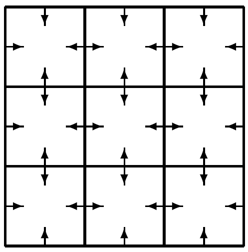
$$\boldsymbol{f} = \boldsymbol{f}_{\rm ext} + \boldsymbol{f}_{\rm int} \tag{16-71}$$

It is worth to note that the split between internal and external forces depends on the control volume.

Total force

The total *external* force acting on a control volume is the sum of the integrated external volume forces and the integrated stress on its surface,

$$\boxed{\mathcal{F}_i^{\text{ext}} = \int_V f_i^{\text{ext}}\, dV + \oint_S \sum_j \sigma_{ij}\, dS_j} \ . \qquad \text{(16-72)}$$



Internal contact forces (here due to pressure) cancel on internal boundaries between material particles, but not at the external surface, where they must be balanced by external pressure.

The surface integral is first converted into a volume integral using Gauss' theorem (6-3), and we get

$$\mathcal{F}_i^{\text{ext}} = \int_V \left(f_i^{\text{ext}} + \sum_j \nabla_j \sigma_{ij} \right) dV \ .$$

This is almost the same as the total force (16-17), except that the contribution from internal volume forces is missing. Thus the total internal force must be

$$\mathcal{F}_i^{\text{int}} = \mathcal{F}_i - \mathcal{F}_i^{\text{ext}} = \int_V f_i^{\text{int}}\, dV \ . \qquad \text{(16-73)}$$

The reason that there is no contribution from internal contact forces is that they cancel against each other due to Newton's third law. In most cases, and certainly for gravity, Newton's third law is also fulfilled for the volume forces, which means that the internal volume forces will add up to zero, leading to $\boldsymbol{\mathcal{F}}_{\text{int}} = \mathbf{0}$. If this were not the case, an isolated body could start to accelerate all by itself.

Consequently, the total force equals the external force,

$$\boxed{\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_{\text{ext}}} \ . \qquad \text{(16-74)}$$

As the derivation shows, this is true in general for all types of contact forces. It is a strong help in setting up global momentum balance to know that only external forces need to be taken into account, just as in Newtonian particle mechanics.

Total moment of force

The total external moment of force acting on a control volume is calculated from the moments of external volume and the contact forces acting on the surface of the volume,

$$\boxed{\boldsymbol{\mathcal{M}}_{\text{ext}} = \int_V \boldsymbol{x} \times \boldsymbol{f}_{\text{ext}}\, dV + \oint_S \boldsymbol{x} \times \boldsymbol{\sigma} \cdot d\boldsymbol{S}} \ . \qquad \text{(16-75)}$$

Using index notation this looks more complicated,

$$\mathcal{M}_i^{\text{ext}} = \int_V \sum_{jk} \epsilon_{ijk} x_j f_k^{\text{ext}} + \oint_S \sum_{jk} \epsilon_{ijk} x_j \sum_l \sigma_{kl}\, dS_l \ , \qquad \text{(16-76)}$$

but is more amenable to analysis. The surface integral is again converted to a volume integral by means of Gauss' theorem, and using that

$$\nabla_l(x_j\sigma_{kl}) = x_j\nabla_l\sigma_{kl} + \sigma_{kj}\delta_{jl} \ ,$$

the surface integral may be rewritten as

$$\mathcal{M}_i^{\rm ext} = \int_V \sum_{jk} \epsilon_{ijk} x_j \left(f_k^{\rm ext} + \sum_l \nabla_l \sigma_{kl} \right) dV + \int_V \sum_{jk} \epsilon_{ijk}\sigma_{kj}\, dV \ , \tag{16-77}$$

Writing the integrated source of angular momentum as $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_{\rm int} + \boldsymbol{\mathcal{M}}_{\rm ext}$, and using that $\boldsymbol{f}_{\rm ext} = \boldsymbol{f} - \boldsymbol{f}_{\rm int}$, we find

$$\boxed{\mathcal{M}_i^{\rm int} = \int_V \sum_{jk} \epsilon_{ijk} x_j f_k^{\rm int}\, dV + \int_V \sum_{jk} \epsilon_{ijk}\sigma_{jk}\, dV} \ , \tag{16-78}$$

where we have also replaced σ_{kj} by σ_{jk} in the last term (and changed the sign).

The moment of internal volume forces vanishes, provided they are central (see problem 16.9), which is normally the case. But there is evidently an extra internal moment with density $\sum_{jk} \epsilon_{ijk}\sigma_{jk}$. If the stress tensor is symmetric, $\sigma_{jk} = \sigma_{kj}$, this moment vanishes because of the antisymmetry of ϵ_{ijk}, but otherwise not. We saw in section 8.5 that it was in principle always possible to choose a symmetric stress tensor, although it might not be practical. We now learn that an asymmetric choice will always give rise to an internal moment of force. The total moment (16-29) is, however, insensitive to this choice, because it only depends on the effective force. And that is what counts.

Under the assumption of central forces and a symmetric stress tensor, we conclude that the total internal moment vanishes, $\boldsymbol{\mathcal{M}}_{\rm int} = \boldsymbol{0}$, from which we get

$$\boxed{\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_{\rm ext}} \ . \tag{16-79}$$

The total moment of forces may normally be calculated alone from the moment of the external forces.

Total power

The external power also consists of the power of external volume forces and the power of the contact forces acting on the surface of the volume,

$$\boxed{P_{\rm ext} = \int_{V(t)} \boldsymbol{v}\cdot\boldsymbol{f}_{\rm ext}\, dV + \oint_{S(t)} \boldsymbol{v}\cdot\boldsymbol{\sigma}\cdot d\boldsymbol{S}} \ . \tag{16-80}$$

Using Gauss' theorem on the surface integral, this becomes

$$
\begin{aligned}
P_{\rm ext} &= \int_V \left(\sum_i v_i f_i^{\rm ext} + \sum_{ij} \nabla_j (v_i \sigma_{ij}) \right) dV \\
&= \int_V \left(\sum_i v_i \left(f^{\rm ext} + \sum_j \nabla_j \sigma_{ij} \right) + \sum_{ij} \sigma_{ij} \nabla_j v_i \right) dV \ .
\end{aligned}
$$

Comparing with the total power (16-37), and using that $P = P_{\rm int} + P_{\rm ext}$, the internal power must be

$$
\boxed{P_{\rm int} = \int_{V(t)} \left(\boldsymbol{v} \cdot \boldsymbol{f}_{\rm int} - \sum_{ij} \sigma_{ij} \nabla_j v_i \right) dV} \ . \tag{16-81}
$$

Although the total internal volume force vanishes, its power does not have to vanish, because the velocity field in general varies over the volume. The second represents the work of internal contact forces, both compression and dissipation. The relation of this term to dissipation in viscous fluids is postponed until section 17.5.

Internal energy in Euler fluids

In Euler fluids, where $\sigma_{ij} = -p\delta_{ij}$, the internal power becomes

$$
P_{\rm int} = \int_{V(t)} (\boldsymbol{v} \cdot \boldsymbol{f}_{\rm int} + p \, \mathsf{div}\, \boldsymbol{v}) \, dV \ . \tag{16-82}
$$

The second term is negative during compression ($\mathsf{div}\, \boldsymbol{v} < 0$) and positive during expansion, and it must be interpreted as *rate of work of the internal pressure forces* acting between the material particles. Although the internal pressure forces cancel against each other in the total internal force, their total power does not vanish because neighboring material particles move with slightly different velocities in a compressible material. For an incompressible or effectively incompressible fluid, we have $\mathsf{div}\, \boldsymbol{v} = 0$, and the power of the internal pressure forces vanishes.

We have already seen (section 16.7) that time-independent external gravitational forces, $\boldsymbol{f}_{\rm ext} = \rho \boldsymbol{g}_{\rm ext}$ may be removed from the power and included in the energy density. We shall now see that the power of the internal pressure forces may likewise be included by defining the specific energy

$$
\widetilde{\epsilon} = \frac{1}{2}\boldsymbol{v}^2 + \Phi_{\rm ext} + w(p) - \frac{p}{\rho} \tag{16-83}
$$

where $w(p)$ is the pressure function (7-3) for barotropic matter. The local source of this energy is

$$\rho\frac{\delta\epsilon}{\delta t} = \boldsymbol{v}\cdot(\boldsymbol{f}^* - \boldsymbol{f}_{\mathrm{ext}}) + \frac{p}{\rho}\frac{\delta\rho}{\delta t} = \boldsymbol{v}\cdot\boldsymbol{f}_{\mathrm{int}} - \mathsf{div}\,(p\boldsymbol{v}) \tag{16-84}$$

where we in the last step have used the equation of continuity, $\delta\rho/\delta t = -\rho\,\mathsf{div}\,\boldsymbol{v}$. If there are no internal forces, $\boldsymbol{f}_{\mathrm{int}} = \mathbf{0}$, and if the total power of the pressure on the external surface vanishes, the total energy in a comoving control volume will be conserved.

Problems

16.1 Before the advent of modern high precision positioning systems, water levels could be transmitted across a strait by means of a long tube filled with water. Estimate the oscillation time of the water in the tube for the Danish Great Belt which is about 20 km wide.

16.2 A horizontal 1 inch water pipe bends horizontally 180°. Estimate the force that the water exerts on the pipe, when the water speed is 1 m/s.

16.3 A pipe bends from $\boldsymbol{n}_1$ to $\boldsymbol{n}_2$ in a constant gravitational field $\boldsymbol{g}_0$. Show that the reaction force from the pipe is

$$\boldsymbol{\mathcal{R}} = \rho_0 A\big((v^2 + \Delta p_2)\boldsymbol{n}_2 - (v^2 + \Delta p_1)\boldsymbol{n}_1 - L\boldsymbol{g}_0\big) \tag{16-85}$$

where L is the length of the pipe.

16.4 A firefighter bends a water hose through 90° close to the nozzle where the pressure is nearly atmospheric. The hose diameter is 5 cm and it discharges 40 liter per second. Calculate the magnitude of the force that the firefighter has to exert on the hose to bend it. Will he be able to hold the hose without using equipment?

16.5 Consider a circular drain (section 16.8) of radius a in the bottom of a large circular cistern of radius $b \gg a$. Assume that the average velocity of the water at a half sphere of radius r centered at the drain is $v(r) = (a/r)^2 v$ where v is the average velocity at the drain, so that the same amount of water passes through the half sphere for all $r > a$. Calculate the kinetic energy associated with this velocity distribution, and compare with the kinetic energies of

16.6 A large lake is drained through a thin pipe that slopes downwards at an angle α with the horizontal. The pipe is positioned near the surface of the lake. A valve is positioned a good distance L down the pipe, but the pipe continues downwards with the same slope far beyond the valve. a) Find the equation of motion for the water front when it has progressed a distance x past the valve. b) Show that the sum of the kinetic plus potential energy is conserved. c) Use this to calculate the acceleration of the water front as a function of x, and show that terminal speed is never reached.

16.7 Solve the phase-3 equation (16-65) exactly and calculate the total amount of water delivered in one beat of the pump. Compare with the estimate (16-70).

16.8 The engine of a rocket emits gases with density ρ_0 at constant average speed v_0 relative to the rocket through a nozzle with cross section area A. The rocket blasts vertically off from rest on the flat and airless Earth at $t = 0$ with initial mass M_0. a) Determine the equations of motion for the rocket from mass and momentum balance. b) Find the condition for the rocket to get off the ground. c) Solve the equations of motion with the given boundary conditions. d) Find the maximal height the rocket can reach.

16.9 A volume force is said to be central, if it acts along the line connecting two material particles and only depends on the distance between them. In that case, the internal density of force in a volume $V(t)$ takes the form

$$\boldsymbol{f}_{\text{int}}(\boldsymbol{x},t) = \int_{V(t)} (\boldsymbol{x}-\boldsymbol{y}) h(|\boldsymbol{x}-\boldsymbol{y}|, t)\, dV_y \tag{16-86}$$

a) Show that the total internal force vanishes. b) Show that the total internal moment of force also vanishes.

16.10 Show that

$$\frac{d}{dt}\int_{V(t)} \rho\, dV = \int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \oint_{S(t)} \rho\, \boldsymbol{v}_S \cdot d\boldsymbol{S} \tag{16-87}$$

16.11 A volume V contains N point-like particles of mass m_n with instantaneous positions $\boldsymbol{x}_n$ and velocities $\boldsymbol{v}_n$. The total mass is $M = \sum_n m_n$ and the center of mass is $\boldsymbol{x} = \sum_n m_n \boldsymbol{x}_n / M$. Define the relative positions $\boldsymbol{x}'_n = \boldsymbol{x}_n - \boldsymbol{x}$ and the relative velocities $\boldsymbol{v}'_n = \boldsymbol{v}_n - \boldsymbol{v}$ where $\boldsymbol{v} = \sum_n m_n \boldsymbol{v}_n / M$ is the center-of-mass velocity.

Assume that the relative positions and velocities are random and average out to zero,

$$\langle \boldsymbol{x}'_n \rangle = \boldsymbol{0} \tag{16-88}$$

$$\langle \boldsymbol{v}'_n \rangle = \boldsymbol{0} \tag{16-89}$$

Also assume that they are independent, uncorrelated and that the velocities are spherically distributed

$$\langle (\boldsymbol{x}'_n)_i (\boldsymbol{v}'_m)_j \rangle = 0 \tag{16-90}$$

$$\langle (\boldsymbol{v}'_n)_i (\boldsymbol{v}'_m)_j \rangle = U^2 \delta_{ij} \delta_{nm} \tag{16-91}$$

a) Show that the total angular momentum of the system is

$$\boldsymbol{\mathcal{L}} = M\boldsymbol{x} \times \boldsymbol{v} + \sum_n m_n \boldsymbol{x}'_n \times \boldsymbol{v}'_n \tag{16-92}$$

and that the second term averages to zero.

b) Show that the total kinetic energy is

$$\mathcal{T} = \frac{1}{2} M \boldsymbol{v}^2 + \frac{1}{2}\sum_n m_n (\boldsymbol{v}'_n)^2 \tag{16-93}$$

and calculate the average of the second term.

