13 Vibrations in solids

Sound is the generic term for harmonic running waves of deformation in materials, be they solid, liquid or gaseous. Our daily existence as humans, communicating in and out of sight, is strongly dependent on sound transmission in air, and only rarely — for example in the dentists chair — do we notice primary effects of sound in solids. What we do experience in our daily lives is mostly secondary effects of vibrations in solids transferred to air as sound waves, for example a mouse scratching on the other side of a wooden wall, or more insidiously the neighbor's drilling into concrete.

In this chapter we shall see that there are essentially two kinds of vibrations in solids, longitudinal pressure waves and transverse shear waves. The two kinds of waves are generally transmitted with different phase velocities even in isotropic elastic solids, because such materials respond differently to pressure and shear stress. Sound waves in ideal elastic materials do not dissipate energy while propagating, but energy can be lost to spatial infinity through radiation of sound.

There are also wave motions in elastic solids, for example earthquakes that we would hardly call sound, except sometimes it is called infrasound. We don't hear these phenomena directly but experience earthquakes rather as a motion of the ground, though usually accompanied by audible sound. Infrasound may be felt through an increased level of anxiety and may one day be recognized as a stress factor.

In this chapter we shall study the basic properties of vibrations in isotropic elastic materials. The equations of motion for small displacements in isotropic elastic materials are derived from Newton's second law and applied to a few typical situations. This chapter is important because it is the first time we study continuous media in motion, the main theme for the remainder of this book.

13.1 Elastic waves

The instantaneous state of a deformable material is described by a time-dependent displacement field $\boldsymbol{u}(\boldsymbol{x}, t)$ which indicates how much a material particle at time t is displaced away from its reference position $\boldsymbol{x}$. Since the actual position of the particle is $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}, t)$, its actual velocity is evidently $\boldsymbol{v}(\boldsymbol{x}', t) = \partial \boldsymbol{u}(\boldsymbol{x}, t)/\partial t$ and its acceleration $\boldsymbol{w}(\boldsymbol{x}', t) = \partial^2 \boldsymbol{u}(\boldsymbol{x}, t)/\partial t^2$ without any non-linear terms.

The equation of motion for a displaced material particle becomes

$$\rho(\boldsymbol{x}, t) \frac{\partial^2 u_i(\boldsymbol{x}, t)}{\partial t^2} = \sum_j \nabla_j \sigma_{ij}(\boldsymbol{x}, t) + f_i(\boldsymbol{x}, t) \ , \tag{13-1}$$

where we on the right hand side have ignored the small differences in position created by the displacement itself and by the small displacement gradients (see however problem 11.11). The right hand side is now rewritten in the same way as in section 11.1. In a linear, homogeneous, and isotropic elastic material the contact forces are given by the stress-strain relation (10-7), and inserting Cauchy's strain tensor (9-17), we get

$$\boxed{\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \boldsymbol{\nabla}^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla} \mathsf{div}\, \boldsymbol{u} + \boldsymbol{f}} \ . \tag{13-2}$$

This equation is called *Navier's equation of motion* (1821) and reduces to Navier's equilibrium equation (11-3) for a time-independent displacement.

The field $\boldsymbol{u}(\boldsymbol{x}, t)$ should as before be understood as the displacement away from a reference state which may itself already be highly stressed and deformed. There are, for example, huge static stresses in balance with gravity in the pylons and girders of a bridge, but when the wind acts on the bridge, small vibrations obeying (13-2) may arise around the static state. Time-dependence is in fact often driven by such surface forces which like the wind on the bridge impose time-varying stresses on the surface a body. If you hit a nail with a hammer or stroke the strings of a violin, time varying displacement fields obeying the above equation are set up in the material.

The force density $\boldsymbol{f}$ should likewise be understood as the extra forces driving time dependence. The Moon's tidal deformation of the rotating Earth is caused by time-dependent gravitational body forces, acting on top of the gravitational force of Earth itself. Magnetostrictive, electrostrictive, and piezoelectric materials deform under the influence of electromagnetic fields, and are for example used in loudspeakers to set up vibrations that can be transmitted to air as sound.

In a homogenous solids, the Lamé coefficients λ and μ are constants, and similarly the density may normally also be taken to be a constant, $\rho(\boldsymbol{x}, t) = \rho_0$. At the boundaries between homogeneous solids, the density and the elastic constants may change. In general the displacement field has to be continuous across such boundaries, unless the continuity of the material breaks down. Newton's third law requires as before the stress vector to be continuous across any boundary.

Adiabatic versus isothermal

As long as changes are *quasistatic*, a system will run through states of mechanical as well as thermodynamic equilibrium. No temperature differences will arise anywhere during an infinitely slow displacement, and elastic constants, for example the bulk modulus K_T, are in that case defined for constant temperature.

During rapid changes in deformation, temperature may not have time enough to adjust itself to the changes, and sufficiently fast processes will effectively proceed without heat transport, *i.e.* adiabatically or isentropically. This leads to a small local temperature changes and replaces the isothermal compressibility by the isentropic one,

$$\frac{1}{K_S} = \frac{1}{K_T} - \frac{\alpha^2 T}{C_p} \,, \tag{13-3}$$

where T is the absolute temperature of the body, α the thermal expansion coefficient, and C_p the specific heat capacity at constant pressure. The sign of the second term shows that the adiabatic bulk modulus is always larger than the isothermal one, $K_S > K_T$. One may understand this as a consequence of the universal tendency for matter to expand when heated and for expansion to cause an increase in pressure and thus an increase in the resistance against compression, *i.e.* the incompressibility K. In the same approximation, the isothermal and isentropic shear moduli are identical, $\mu_S = \mu_T$.

Whether vibrations in solids are isothermal or isentropic depend on their frequency and on the thermal properties of the material. In the following discussion we shall for simplicity disregard this normally tiny difference and just denote the bulk modulus by K.

13.2 Free elastic waves

Elastic waves are *free*, if there are no time-dependent body or surface forces. This is very much analogous to the equation for free electromagnetic waves, that also may be studied by themselves, even if we know that they always owe their existence to time-dependent interactions with matter in the environment. Taking $\boldsymbol{f} = \mathbf{0}$ and $\rho = \rho_0$ in (13-2), the equation for free displacement waves becomes

$$\boxed{\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \boldsymbol{\nabla}^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla} \mathsf{div}\, \boldsymbol{u}} \,. \tag{13-4}$$

As for electromagnetic waves, this is an equation for a vector field, but the difference is that it contains two material constants, whereas the equation for free electromagnetic waves in vacuum or in isotropic media only depends on one, namely the velocity of light in the material. The effect is that there will be two kinds of displacement waves moving with different phase velocities in isotropic solids.

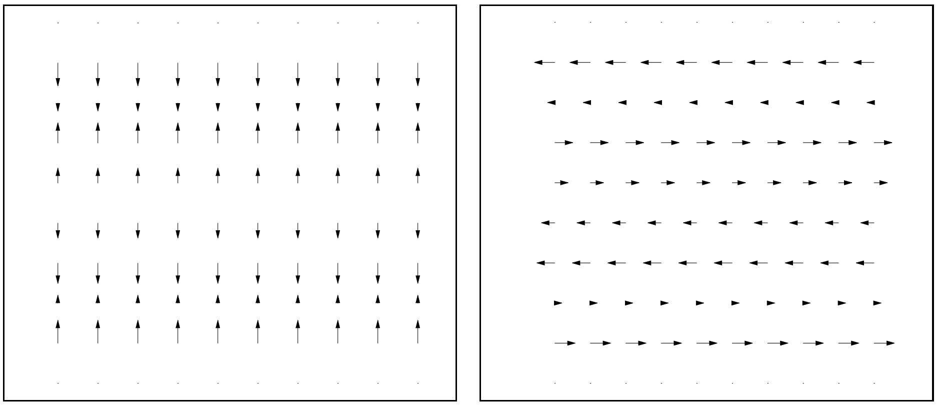


Figure 13.1: *Displacement fields for longitudinal (left) and transverse (right) plane waves. The waves are moving upwards on the page and have the same wave length. The small arrows show the instantaneous direction of the displacement field in an array of points. The compressional and shear nature of the two types of waves is quite visible.*

Longitudinal and transverse waves

An arbitrary vector field may always be resolved into *longitudinal* and *transverse* components (see problem 13.1),

$$\boldsymbol{u} = \boldsymbol{u}_L + \boldsymbol{u}_T \ , \tag{13-5}$$

where the longitudinal component has no curl, $\mathsf{curl}\, \boldsymbol{u}_L = \mathbf{0}$, and the transverse component has no divergence, $\mathsf{div}\, \boldsymbol{u}_T = 0$. By the "double-cross" rule it follows that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}_L) = \boldsymbol{\nabla} \mathsf{div}\, \boldsymbol{u}_L - \boldsymbol{\nabla}^2 \boldsymbol{u}_L = \mathbf{0}$, or $\boldsymbol{\nabla} \mathsf{div}\, \boldsymbol{u}_L = \boldsymbol{\nabla}^2 \boldsymbol{u}_L$, so that the free wave equation (13-4) for purely longitudinal and transverse waves becomes

$$\rho_0 \frac{\partial^2 \boldsymbol{u}_L}{\partial t^2} = (2\mu + \lambda) \boldsymbol{\nabla}^2 \boldsymbol{u}_L \ , \tag{13-6}$$

$$\rho_0 \frac{\partial^2 \boldsymbol{u}_T}{\partial t^2} = \mu \boldsymbol{\nabla}^2 \boldsymbol{u}_T \ . \tag{13-7}$$

Both of these equations are in the form of the standard equation for waves with phase velocity c,

$$\frac{1}{c^2} \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{\nabla}^2 \boldsymbol{u} \ . \tag{13-8}$$

The phase velocity of longitudinal waves may now be read off to be,

$$\boxed{c_L = \sqrt{\frac{\lambda + 2\mu}{\rho_0}} = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho_0}}} \ , \tag{13-9}$$

and for transverse waves

$$\boxed{c_T = \sqrt{\frac{\mu}{\rho_0}}} \ . \tag{13-10}$$

We have mentioned before that elastic materials with vanishing shear modulus, $\mu = 0$, behave like ideal fluids (without viscosity). In such materials, the transverse velocity vanishes, $c_T = 0$, so that transverse waves cannot propagate. We shall later see that although transverse waves may be created in viscous fluids, they die quickly out. The lack of transverse wave propagation is in fact a strong argument for the liquid nature of the Earth's core.

The ratio between the transversal and longitudinal velocities is a useful parameter

$$q = \frac{c_T}{c_L} = \sqrt{\frac{\mu}{\lambda + 2\mu}} = \sqrt{\frac{1-2\nu}{2(1-\nu)}} . \tag{13-11}$$

It depends only on Poisson's ratio, ν, and is a monotonically decreasing function. Its maximal value $\sqrt{3}/2$ is obtained for $\nu = -1$ so the transverse velocity is always smaller than the longitudinal one. In practice there are no materials with $\nu < 0$, so the practical upper limit to the ratio is instead $1/\sqrt{2}$. For the typical value $\nu = \frac{1}{3}$ we get $q = 1/2$, and longitudinal waves run with the double of the speed of transverse waves.

The pressure generated by the deformation is

$$p = -K \operatorname{div} \boldsymbol{u} = -K \operatorname{div} \boldsymbol{u}_L, \tag{13-12}$$

according to (10-14), and depends only on the longitudinal part of a wave. Longitudinal waves are for this reason also called *pressure* waves or *compressional* waves and sometimes denoted by P (for *primary*). Transverse waves, on the other hand, generate no pressure changes in the material, only shear, and are therefore also called *shear* waves and sometimes denoted by S (for *secondary*). In fig. 13.1 the displacement fields are shown for planar longitudinal and transverse waves.

Even if a displacement field may always be resolved into longitudinal and transverse parts, this does not mean that there will exist purely longitudinal and transverse solutions to any particular problem. In general, this will only be the case far from any boundaries, because boundary conditions tend not to respect the separation of longitudinal and transverse waves and will mix them with each other (see section 13.3).

Harmonic waves

A harmonic displacement field obeys the equation

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = -\omega^2 \boldsymbol{u} \tag{13-13}$$

where ω is the (circular) frequency. Solutions to the harmonic equation are superpositions of two *standing* waves,

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_1(\boldsymbol{x}) \cos \omega t + \boldsymbol{u}_2(\boldsymbol{x}) \sin \omega t . \tag{13-14}$$

It is often most convenient instead to write the harmonic wave as the real part of a complex field,

$$\boldsymbol{u}(\boldsymbol{x},t) = \mathcal{R}e\left[\boldsymbol{u}(\boldsymbol{x})e^{-i\omega t}\right] , \tag{13-15}$$

where the complex standing wave

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_1(\boldsymbol{x}) + i\boldsymbol{u}_2(\boldsymbol{x}) \tag{13-16}$$

is also called the *amplitude field.* For the standard wave equation (13-8), standing waves, real or complex, must satisfy the equation

$$\boxed{\boldsymbol{\nabla}^2\boldsymbol{u} = -\frac{\omega^2}{c^2}\boldsymbol{u}} , \tag{13-17}$$

which may be viewed as an *eigenvalue-equation* for the displacement field. For a finite body, only a discrete spectrum of frequencies will be possible.

Plane waves

Infinitely extended material bodies do not exist, but deeply inside a finite body, far from the boundaries, conditions are almost as if the body were infinite. Intuitively it seems clear that this requires the typical wave lengths involved in the wave to be much smaller than the dimensions of the body or, more precisely, much smaller than the distance to the boundaries. For waves with significant amplitude in wave lengths comparable to the distance to the surface of the body, or to boundaries between different materials, a special approach is necessary. We shall return to this problem in section 13.3.

In an infinitely extended medium, Fourier's theorem tells us that an arbitrary wave is a superposition of plane waves with complex amplitude field of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}} , \tag{13-18}$$

where $\boldsymbol{k}$ is the wave vector and $\boldsymbol{u}_0$ is the (constant) *polarization vector.* Inserting this into the standard wave equation (13-8), we obtain

$$\omega^2 = c^2\boldsymbol{k}^2 , \tag{13-19}$$

where c is the phase velocity.

For longitudinal plane waves with phase velocity c_L, we must have $\operatorname{curl}\boldsymbol{u}_L \sim \boldsymbol{k}\times\boldsymbol{u}_0 = \boldsymbol{0}$, so that the polarization vector is proportional to the wave vector

$$\boldsymbol{u}_0 \sim \boldsymbol{k} . \tag{13-20}$$

For transverse waves with phase velocity c_T, the polarization vector must be orthogonal to the wave vector

$$\boldsymbol{k}\cdot\boldsymbol{u}_0 = 0 , \tag{13-21}$$

and there will in general be two linearly independent transverse polarizations.

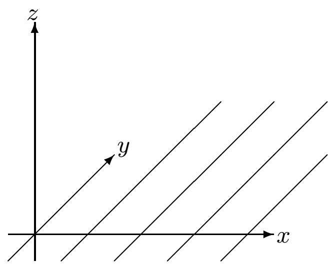
∗ 13.3 Rayleigh waves

At the surface of a body, boundary conditions will put limits on the free variation of the displacement field. If for example the surface is free from external stresses, the stress vector acting on the surface must vanish, $\sum_j \sigma_{ij} n_j = 0$, and that places restrictions on the spatial derivatives of the displacement field. The clean separation between free longitudinal and transverse plane waves is for this reason not possible near the surface, and only certain superpositions of longitudinal and transverse waves will be allowed. Lord Rayleigh discovered that there are special types of waves near a free surface which cannot penetrate into the depth of the material, but decay exponentially with the distance to the surface.

John William Strutt, 3rd Baron Rayleigh (1842–1919). Discovered and isolated the rare gas Argon for which he got the Nobel Prize (1904). Published the influential book "The Theory of Sound" on vibrations in solids and fluids in 1877-78.

Planar surface

The simplest case which differs from from an infinitely extended medium is a semi-infinite medium bounded by a plane surface. We take the surface to be the xy-plane of the coordinate system with the material corresponding to negative z values, whereas there is vacuum for $z > 0$. In a sense we are looking at the surface from "above". In the x- and y-directions there are no restrictions, and we expect to be able to choose the wave vectors, k_x and k_y, freely. In discussing a particular wave we may always choose the x-axis so that $k_y = 0$ and $k_x = k$. In other words, the wave propagates along the x-axis with wave vector k and has constant phase in the y-direction.



A plane surface wave moving along the x-direction with no amplitude in the y-direction.

We are interested in elementary surface waves that are exponentially damped in the material for $z < 0$,

$$\boldsymbol{u} = \boldsymbol{u}_0 e^{\kappa z} e^{i(kx-\omega t)}, \tag{13-22}$$

where $\kappa > 0$ corresponds to an imaginary wave number $k_z = -i\kappa$. Apart from that, the formalism for longitudinal and transverse waves is unchanged. Longitudinal waves have amplitude proportional to the (now complex) wave vector $\boldsymbol{u}_0 \sim \boldsymbol{k}$, and frequency $\omega^2 = c_L^2(k^2 + k_z^2)$. Transverse waves are orthogonal to the wave vector with frequency $\omega^2 = c_T^2(k^2 + k_z^2)$. Solving for $k_z = -i\kappa$, we obtain the decay constants for longitudinal and transverse waves

$$\kappa_L = \sqrt{k^2 - \frac{\omega^2}{c_L^2}}\,, \qquad \kappa_T = \sqrt{k^2 - \frac{\omega^2}{c_T^2}}\,. \tag{13-23}$$

The most general decaying superposition of the one longitudinal and the two transverse waves becomes

$$\boldsymbol{u}(\boldsymbol{x}, t) = \left[A(k, 0, -i\kappa_L)e^{\kappa_L z} + \big(B(\kappa_T, 0, -ik) + C(0, 1, 0)\big)e^{\kappa_T z}\right] e^{i(kx-\omega t)}\,, \tag{13-24}$$

where A, B, and C are complex constants. One may verify that the longitudinal wave is indeed proportional to its (complex) wave vector, and that the transverse waves are orthogonal to their wave vector and to each other.

Boundary conditions

The free boundary conditions imply that the stress vector must vanish, $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, at the surface $z = 0$. From Hooke's law (10-7) we get for $z = 0$ (since $u_{yy} = \nabla_y u_y = 0$ everywhere)

$$(2\mu + \lambda)u_{zz} + \lambda u_{xx} = 0 \;, \tag{13-25a}$$
$$u_{xz} = 0 \;, \tag{13-25b}$$
$$u_{yz} = 0 \;. \tag{13-25c}$$

Expressing the strains in terms of the displacement field via Cauchy's strain tensor (9-17), we get for $z = 0$ (and dropping the common oscillating factor)

$$\begin{aligned} u_{xx} &= \nabla_x u_x = ik(Ak + B\kappa_T) \;, \\ u_{zz} &= \nabla_z u_z = -i(\kappa_L^2 A + k\kappa_T B) \;, \\ 2u_{xz} &= \nabla_x u_z + \nabla_z u_x = 2k\kappa_L A + (k^2 + \kappa_T^2)B \;, \\ 2u_{yz} &= \nabla_y u_z + \nabla_z u_y = \kappa_T C \;. \end{aligned}$$

The boundary conditions thus lead to $C = 0$ and

$$(k^2 + \kappa_T^2)A + 2k\kappa_T B = 0 \;, \tag{13-26a}$$
$$2k\kappa_L A + (k^2 + \kappa_T^2)B = 0 \;. \tag{13-26b}$$

In the first equation we have used that $\lambda/(\lambda + 2\mu) = 1 - 2c_T^2/c_L^2$, and that $c_L^2\kappa_L^2 = c_T^2\kappa_T^2 + (c_L^2 - c_T^2)k^2$, which follows from the definitions of κ_T and κ_L. These equations can only have a non-vanishing solution for A and B, if the determinant vanishes, or

$$(k^2 + \kappa_T^2)^2 = 4k^2\kappa_T\kappa_L \;. \tag{13-27}$$

Eliminating κ_L and κ_T, we obtain

$$\left(2k^2 - \frac{\omega^2}{c_T^2}\right)^2 = 4k^2\sqrt{\left(k^2 - \frac{\omega^2}{c_T^2}\right)\left(k^2 - \frac{\omega^2}{c_L^2}\right)} \tag{13-28}$$

Surface wave velocity

The above equation only depends on the ratio $c = \omega/k$. Defining the parameter $\xi = (c/c_T)^2$ and introducing the ratio q given by (13-11), we arrive at the equation $(2 - \xi)^2 = 4\sqrt{1 - \xi}\sqrt{1 - q^2\xi}$ and upon squaring this we finally obtain the third degree polynomial equation

$$\xi^3 - 8\xi^2 + 8(3 - 2q^2)\xi - 16(1 - q^2) = 0 \;. \tag{13-29}$$

A third degree equation always has at least one real root. For the typical value of $q = 1/2$ there is only one real root at $\xi = 0.870$ corresponding to $c = 0.933\,c_T$.

The surface wave velocity is normally very close to the velocity of free transverse waves (see problem 13.3).

The amplitude of the surface waves decays exponentially with the depth below the surface as shown by the exponentials in (13-24). The transverse part of the wave decays with the rate $\kappa_T = k\sqrt{1-\xi}$ whereas the rate for the longitudinal part is $\kappa_L = k\sqrt{1-q^2\xi}$ which is larger than the transverse rate, κ_T, because $q < 1$.

* 13.4 Radial waves

Due to the problem of surface waves it turns out that radial oscillations in a spherical geometry is one of the few exactly solvable problems for finite bodies. In this subsection we shall solve the spherical case in general for the case of purely radial oscillations.

We assume as in section 11.6 that the form of the instantaneous displacement is radial $\boldsymbol{u}(\boldsymbol{r},t) = u(r,t)\boldsymbol{e}_r$. Then the equation of motion for a harmonically oscillating field $u(r,t) = u(r)\exp(-i\omega t)$ becomes

$$-\frac{\omega^2\rho_0}{\lambda+2\mu}u = \frac{d}{dr}\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}\right) = \left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2}\right]u \ . \tag{13-30}$$

Since the radial oscillations by their form are purely longitudinal with no transversal components this expression only depends on the longitudinal velocity, and it is convenient to introduce the dimensionless variable $s = kr$ with $k = \omega/c_L$. In this variable the equation becomes

$$\frac{d}{ds}\left(\frac{1}{s^2}\frac{d(s^2u)}{ds}\right) = \left[\frac{d^2}{ds^2} + \frac{2}{s}\frac{d}{ds} - \frac{2}{s^2} + 1\right]u = 0 \ . \tag{13-31}$$

The general solution to this equation is a linear combination of the spherical Bessel function $j_1(s) = \sin s/s^2 - \cos s/s$ and the spherical Neumann function $n_1(s) = -\cos s/s^2 - \sin s/s$, as one may easily verify.

Free massive sphere

In a finite body with free boundaries, the waves with definite frequency are always standing waves, because running waves can never fulfill the boundary conditions. Since the Neumann function $n_1(s)$ is singular for $s = 0$ the displacement field for a massive elastic sphere must be proportional to the Bessel function, $u \propto j_1(s)$. The radial strain is simply $u_{rr} = du/dr$, and the radial stress becomes

$$\sigma_{rr} = \lambda \mathsf{div}\, \boldsymbol{u} + 2\mu u_{rr} = \lambda(\frac{du}{dr} + 2\frac{u}{r}) + 2\mu\frac{du}{dr} \ ,$$

$$\propto (\lambda+2\mu)\frac{\sin s}{s} + 4\mu\left(\frac{\cos s}{s^2} - \frac{\sin s}{s^3}\right) \ .$$

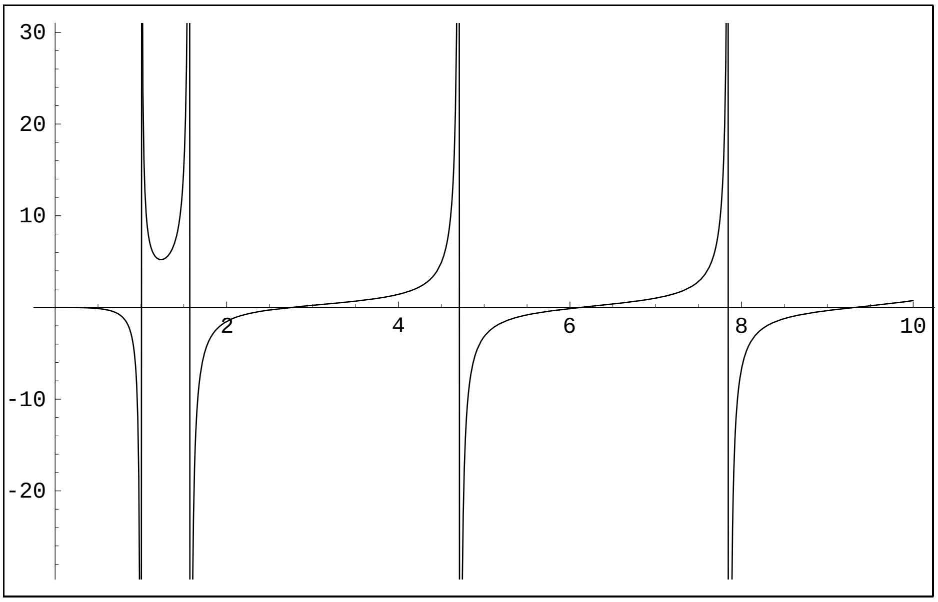


Figure 13.2: *Graph of eq.(13-32) for $q = 1/2$. The first solution is found for $s = 2.82$. The rest of the solutions come roughly spaced by π. There is no solution for $s = 1$.*

At the surface of the sphere the stress must vanish, $\sigma_{rr}(a) = 0$, and this leads to the following implicit equation for s

$$\tan s - \frac{s}{1 - \frac{s^2}{4q^2}} = 0 \ , \tag{13-32}$$

where q as before is the ratio between the transverse and longitudinal sound velocity in the material. There is an infinite sequence of solutions to this equation (shown graphically in Fig. 13.2) for $q = 1/2$ which may be denoted $s_n(q)$ for $n = 1, 2, \ldots$. In problem 13.4 an approximate expression is derived for s_n.

The corresponding frequencies are

$$\omega_n = \frac{c_L}{a} s_n(q) \ , \tag{13-33}$$

and these are the only possible frequencies for standing waves in a massive sphere. In problem 13.4 it is shown that

$$s_n = n\pi - \frac{4q^2}{n\pi} \tag{13-34}$$

is a reasonable approximation to the eigenvalues.

Problems

13.1 Show that an arbitrary vector field may be resolved into (not necessarily unique) longitudinal and transverse components, and that the longitudinal component may be chosen to be a gradient.

13.2 Show that the most general sound wave in an infinite isotropic and homogeneous medium is of the form

$$\boldsymbol{u}(\boldsymbol{x},t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \sum_{i=1}^{3} \boldsymbol{e}_i(\boldsymbol{k}) \left(a_i(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega_i t} + a_i(\boldsymbol{k})^{\times} e^{-i\boldsymbol{k}\cdot\boldsymbol{x} + i\omega_i} \right) , \tag{13-35}$$

where $\omega_1 = \omega_2 = c_T k$ and $\omega_3 = c_L k$. The set of vectors $\boldsymbol{e}_\lambda(\boldsymbol{k})$ ($\lambda = 1, 2, 3$) are called polarisation vectors and required to be orthonormal $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. The third $\boldsymbol{e}_3 = \boldsymbol{k}/k$ is defined to be longitudinal with respect to the wave number. The generally complex coefficients $a_i(\boldsymbol{k})$ are called the amplitude of the wave for wave number $\boldsymbol{k}$ and polarisation i.

13.3 In the equation for the surface wave velocity (13-29) the coefficients are numerically rather large compared to 1. Since for $\xi = 1$ the left hand side takes the value 1 one may expect that there is a root close to $\xi = 1$. Show that the solution near 1 is approximately given by

$$1 - \xi = \frac{1}{11 - 16q^2} , \tag{13-36}$$

and determine the magnitude of the error. Compare with the exact result $\xi = 0.869605\ldots$ for $q = 1/2$.

13.4 Show that for large values of s the solution to (13-32) is

$$s_n = n\pi - \frac{4q^2}{n\pi} . \tag{13-37}$$

Compare for $q = 1/2$ this approximation with the exact result.

