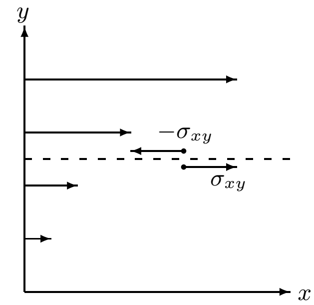
# 17 Viscosity

All fluids are viscous, except for a component of liquid helium close to absolute zero in temperature. Air, water, and oil all put up resistance to flow, and a part of the money we spend on transport by plane, ship or car goes to overcome fluid friction, and eventually goes to heating the atmosphere, the sea, or the bearings of the car. It may be true that money makes the world go around, but viscosity requires you to have a continuous supply!

It is primarily the interplay between the mechanical inertia of a moving fluid and its viscosity which gives rise to all the interesting and beautiful phenomena, the whirling and the swirling that we are so familiar with. If a volume of fluid inside a larger volume is set into motion, inertia would dictate that it continue in its original motion, were it not checked by the action of internal shear stresses. Viscosity acts as a brake on the free flow of a fluid and will eventually make it come to rest in mechanical equilibrium, unless external driving forces continually supply energy to keep it moving.

In an Aristotelian sense the "natural" state of a fluid is thus at rest with pressure being the only stress component. Disturbing a fluid at rest slightly, setting it into motion with spatially varying velocity field, will to first order of approximation generate stresses that depend linearly on the spatial derivatives of the velocity field. Fluids with a linear relationship between stress and velocity gradients are called Newtonian, and the coefficients in this linear relationship are material constants characterizing viscosity.

In this chapter the formalism for Newtonian viscosity will be set up and we shall finally arrive at the famous Navier-Stokes equations for fluids. Although simple on the surface, these non-linear differential equations remain a formidable challenge to both physicists and mathematicians.




*If fluid moves faster above a boundary surface, it will exert a positive shear stress on the boundary surface, and conversely.*

## 17.1 Shear viscosity

Consider a fluid flowing steadily along the $x$-direction with a velocity field $v_x(y)$ which is independent of $x$, but changes with $y$. The fluid streams in layers parallel with the $xz$-plane, and is an example of *laminar* flow. Due to the translational symmetry in the $xz$-plane there ought to be a constant shear stress $\sigma_{xy}(y)$ acting on any planar surface at a given value of $y$. A velocity field without $y$-dependence should not give rise to friction, because the fluid is then in simple uniform motion along the $x$-axis. If on the other hand the velocity grows with $y$, *i.e.* for $dv_x(y)/dy > 0$, we expect that the layers above the boundary surface at $y$ will drag it along because of friction and thus exert a positive shear stress, $\sigma_{xy}(y) > 0$, on the boundary surface, and conversely if the velocity decreases with $y$. It also seems reasonable to expect that a stronger velocity gradient $dv_x(y)/dy$ will evoke a stronger stress.

Such arguments justify that the shear stress in this example should be proportional to the gradient of the velocity field,

$$\boxed{\sigma_{xy}(y) = \eta \frac{dv_x(y)}{dy}} \, . \tag{17-1}$$

This is *Newton's law of viscosity (1687)*, and the constant of proportionality, $\eta$, is called the coefficient of *shear viscosity*, the *dynamic viscosity*, or simply *the* viscosity. It is a measure of how strongly the fluid layers are coupled by friction and is a material constant of the same nature as the shear modulus for elastic materials. We shall see below that there is also a bulk coefficient of viscosity corresponding to the elastic bulk modulus, but that is rather unimportant in ordinary applications.

The viscosities of naturally occurring fluids range over many orders of magnitude (see table 17.1). Since $dv_x/dy$ has dimension of inverse time, the unit for viscosity $\eta$ is Pa s (pascal seconds). Although it is sometimes called Poiseuille, there is no special SI-name for it. In the now defunct cgs-system it used to be called poise.

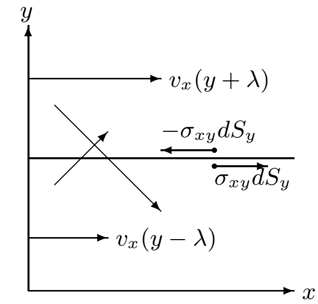
### Molecular origin of viscosity

In gases where molecules are far apart, internal stresses are caused by the incessant molecular bombardment of a boundary surface, transferring momentum in both directions across it. In liquids where molecules are in closer contact, internal stress is caused partly by molecular motion as in gases, and partly by intermolecular forces. The resultant stress in a liquid is a quite complicated combination of the two effects, and we shall for this reason limit the following discussion to the molecular origin of stress in gases.

Gas molecules normally move at much greater speeds than the gas itself, and the fluid velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ should as discussed before be understood as the center-of-mass velocity of a large collection of molecules. For the case of

| | density [kg m$^{-3}$] | viscosity [Pa s] | kinematic viscosity [m$^2$s$^{-1}$] |
|---|---|---|---|
| Hydrogen | | $8.80 \times 10^{-6}$ | $1.10 \times 10^{-4}$ |
| Air | | $1.82 \times 10^{-5}$ | $1.57 \times 10^{-5}$ |
| Water | $1 \times 10^{0}$ | $8.90 \times 10^{-4}$ | $8.64 \times 10^{-7}$ |
| Ethanol | | $1.08 \times 10^{-3}$ | $1.08 \times 10^{-3}$ |
| Mercury | | $1.53 \times 10^{-3}$ | $1.14 \times 10^{-7}$ |
| Blood | | $4 \times 10^{-3}$ | |
| Engine oil | | $1.75 \times 10^{-2}$ | $2.03 \times 10^{-5}$ |
| Olive oil | | $6.70 \times 10^{-2}$ | $6.70 \times 10^{-2}$ |
| Castor oil | | $7.00 \times 10^{-1}$ | $7.00 \times 10^{-1}$ |
| Glycerol | | $1.41 \times 10^{0}$ | $1.18 \times 10^{-3}$ |
| Ketchup | | $5 \times 10^{1}$ | |
| Tar | | $3 \times 10^{7}$ | |
| Glass | | $1 \times 10^{12}$ | |
| Lava | | | |
| Magma | | | |

Table 17.1: *Table of density, and dynamic and kinematic viscosity for common substances (at normal temperature and pressure). Some of the values are only estimates. Notice that air has greater kinematic viscosity than water and hydrogen greater than engine oil!*




*Layers of fluid moving with different velocities give rise to shear forces because they exchange molecules with different average velocities.*

steady laminar planar flow with velocity field $v_x(y)$, molecules crossing a surface element in the $xz$-plane from above will carry with them an excess of momentum and therefore a force in the $x$-direction to the material below. Similarly, the material below will exert an equal and opposite force on the material above.

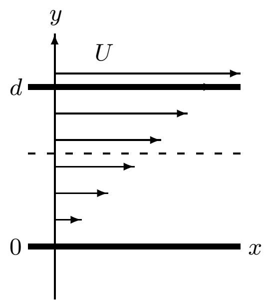
Let the typical distance between molecular collisions in the gas be $\lambda$ and the typical time between collisions $\tau$. The excess of momentum in the $x$-direction above an area element $dS_y$ in a layer of thickness $\lambda$ is of the order of

$$d\mathcal{P}_x \approx (v_x(y+\lambda) - v_x(y))\rho\lambda dS_y \approx \rho\lambda^2 \frac{dv_x(y)}{dy} dS_y \ .$$

This excess of momentum will be carried along by the fast molecular motion in all directions and about half of it will cross the surface in the time $\tau$. The shear stress may be estimated as the momentum transfer per unit of time and area, $\sigma_{xy} \approx d\mathcal{P}_x/\tau dS_y$, and takes indeed the form of Newton's law of viscosity (17-1) with a rough estimate of the shear viscosity,

$$\eta \approx \rho\frac{\lambda^2}{\tau} \ . \tag{17-2}$$

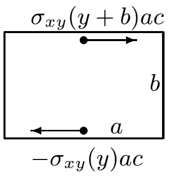
For gases this becomes of the right order of magnitude (see problem 17.1), but in general this estimate does not yield a useful value for the viscosity.




*A Newtonian fluid with spatially uniform properties between moving parallel plates. The velocity field varies linearly between the plates and satisfies the boundary conditions that the fluid is at rest relative to both plates (no-slip).*

## Velocity driven flow between parallel plates

Planar laminar flow described by a steady velocity field $v_x(y)$ may be realized by means of two —- in principle infinitely extended — plane parallel plates moving at constant relative speed $U$ a distance $d$ apart. The coordinate system is as before chosen with the $x$-axis in the direction of the plate velocity and the two plates situated at $y = 0$ and $y = d$ with the lower plate being at rest. Provided the upper plate does not move so fast that it creates turbulence, intuition tells us that the flow quickly will settle down into a steady pattern with all fluid moving along the $x$-direction with a velocity $v_x(y)$, that for symmetry reasons is independent of $x$ and $z$. We expect the fluid to assume the same speed as the plates, in other words $v_x(0) = 0$ and $v_x(d) = U$. This *no-slip* boundary condition will be discussed in more detail later.




*Shear forces on a volume $a \times b \times c$ of fluid in steady laminar flow. The shear forces must balance on the upper and lower sides, $\sigma_{xy}(y+b) = \sigma_{xy}(y)$. Since $b$ is arbitrary, the stress $\sigma_{xy}$ cannot depend on $y$.*

As each layer of fluid moves within itself with constant velocity, there can be no acceleration, and each layer must act on the neighboring layers with the same force (per unit of area). The shear stress $\sigma_{xy}$ must therefore be constant and take the same value everywhere in the fluid. From (17-1) we conclude that the velocity gradient is constant and the velocity itself linear in $y$. Using the boundary conditions $v_x(0) = 0$ and $v_x(d) = U$, this leads to

$$v_x(y) = \frac{y}{d}U \ , \tag{17-3}$$

and then by (17-1)

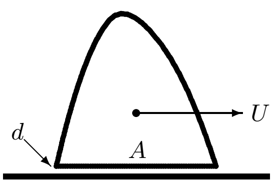
$$\sigma_{xy} = \eta\frac{U}{d} \ . \tag{17-4}$$

If the plate area is $A$, it is necessary to act with opposite external forces of magnitude $\mathcal{F} = A\sigma_{xy}$ on the plates. The rate of work of these forces is $P = \mathcal{F}U = \eta AU^2/d$, and equals the rate of heat production by friction in the fluid. If the external forces cease to act, the plates and fluid will eventually come to rest.

## Viscous friction

A thin layer of viscous fluid may be used to lubricate the interface between solid objects, and by means of the above solution we can estimate the friction force, or *drag*, exerted on the body by the viscous lubricant. In chapter 23 we shall at length investigate lubrication and calculate both the *lift* and the drag from a layer of lubricant, and how it influences the motion of the body.

If the would-be contact area between the body and the surface on which it slides is $A$, the fluid must exert a force $\mathcal{F} \approx -\sigma_{xy}A = -\eta AU/d$ on the body, where $U$ is the velocity of the object and $d$ the thickness of the layer of lubricant. For an object of mass $M$, not subject to other forces, the (quasi-steady) equation of motion becomes




*A solid object sliding on a plane lubricated surface.*

$$M\frac{dU}{dt} = -\eta\frac{A}{d}U \ . \tag{17-5}$$

Provided the lubricant thickness $d$ stays constant, the solution $U(t)$ to this differential equation decreases exponentially with time, a behavior which is quite different from the linear decrease in the velocity due to solid friction (section 8.1). Although the assumption of constant lubricant thickness is probably not very well fulfilled, the velocity dependent deceleration is presumably what makes the ice sport of curling interesting — and braking a car on ice scary. In these cases, as well as in ice-skating, a thin layer of liquid water acts as the lubricant.

## Kinematic viscosity

The viscosity estimate (17-2) seems to point to another measure of viscosity, called the *kinematic viscosity*[1],

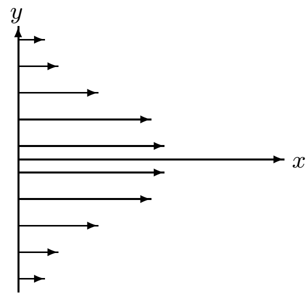
$$\boxed{\nu = \frac{\eta}{\rho}} \,. \tag{17-6}$$

In fluids with constant density, it is a material constant at equal footing with the dynamic viscosity $\eta$ (see table 17.1). Since the estimate of this parameter is $\nu \approx \lambda^2/\tau$, it does not depend on the unit of mass, but is measured in purely kinematic units of $\mathsf{m}^2/\mathsf{s}$ (in the cgs-system, the corresponding unit $\mathsf{cm}^2/\mathsf{s}$ was called stokes). We shall see below that this parameter rather than the shear viscosity is the natural controlling parameter in most of fluid dynamics.

Kinematic viscosity may turn our intuitive understanding of different fluids upside down. Normally, we would think of air as less viscous than water and hydrogen as less viscous than engine oil, but under suitable conditions it is really the other way around. The lightness of air makes it much easier to set in motion with a given force than water, say by waving the hands, and that is what fools our intuition. Given the same velocity pattern, air is in fact 10–20 times more viscous than water, depending on the temperature.

## Momentum diffusion

Until now we have assumed that the velocity field $v_x(y)$ is steady, *i.e.* independent of time, and we found that the balance of forces requires this field to vary linearly with $y$. In principle nothing prevents us from creating an arbitrary planar velocity distribution $v_x(y)$ at $t = 0$, a 'river' in the 'sea' of fluid, but then it cannot be steady. Instead it must evolve with time and we shall for simplicity assume that it retains its laminar form and becomes $v_x(y,t)$ at a later time $t$. This is by no means evident. If the original flow pattern $v_x(y)$ imposed on the fluid varies too strongly with $y$, non-laminar turbulent flow may ensue.




*Velocity distribution for a planar 'river' in an 'ocean' of fluid.*

With these reservations, we may write down Cauchy's equation of motion (14-32). Since the assumed velocity field $v_x(y,t)$ is independent of $x$, there can be no inertial acceleration, $(\boldsymbol{v} \cdot \boldsymbol{\nabla})v_x = v_x \nabla_x v_x = 0$, and in the absence of pressure forces and gravity, the equation of motion becomes

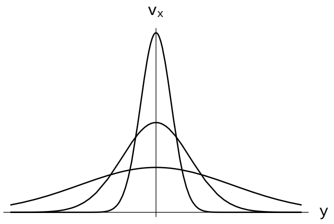
$$\rho \frac{\partial v_x}{\partial t} = \frac{\partial \sigma_{xy}}{\partial y} = \eta \frac{\partial^2 v_x}{\partial y^2}$$

[1]There clash between the uses of $\nu$ for kinematic viscosity and for Poisson's ratio will in general not be a problem.

Dividing by the density (which is assumed to be constant) we get

$$\boxed{\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}}, \tag{17-7}$$



*A Gaussian 'river' widens and slows down in the course of time because of viscosity.*

where $\nu$ is the kinematic viscosity (17-6). If the velocity field is linear in $y$, the right hand side vanishes, and we recover the result that such a field is independent of time, *i.e.* steady.

This equation of motion is a typical *diffusion equation* with diffusion constant, or *diffusivity*, equal to the kinematic viscosity, $\nu$. In general, such an equation leads to a spreading of the distribution of the diffused quantity, which in this case is the velocity field, or perhaps better, the momentum density $\rho v_x$. The generic example of time-dependent planar flow is the Gaussian 'river',

$$v_x(y,t) = v_0 \frac{a}{\sqrt{a^2 + 4\nu t}} \exp\left(-\frac{y^2}{a^2 + 4\nu t}\right), \tag{17-8}$$

which may be verified to be a solution by insertion into (17-7). This 'river' starts out at $t = 0$ with width $a$ and spreads out with time so that it at time $t$ has width $\sqrt{a^2 + 4\nu t}$. Although momentum diffuses away from the center of the river, the total momentum must remain constant because there are no external forces acting on the fluid. Kinetic energy is on the other hand dissipated into heat (see problem 17.3).

For sufficiently large times, $4\nu t \gg a^2$, the shape of the Gaussian has become independent of the original width $a$. The shape of *any* bounded 'river' will eventually become proportional to $\exp(-y^2/4\nu t)$ (see problem 17.4), a factor that drops to about 10% of its central value for $y = 3\sqrt{\nu t}$. The characteristic momentum spread of 90%-range $\delta \approx 3\sqrt{\nu t}$ orthogonally to the flow is a *universal* feature of viscous fluid motion which we shall return to in the discussion of boundary layers (chapter 24).

## 17.2 Isotropic, Newtonian fluids

Fluids obeying a linear law of viscosity are called *Newtonian*. Although there are numerous everyday examples of non-Newtonian fluids (tomato ketchup, paint, jelly, putty, blood, *etc*) we shall in this book mostly be concerned with Newtonian fluids which cover a large number of other everyday fluids (water, air, oil, antifreeze, *etc*).

### Isotropy

Newton's law of viscosity (17-1) is a linear relation between the stress and the velocity gradient, only valid in a particular geometry. In general we want a definition of viscous stress which takes the same form for any geometry and in any coordinate system, so that we are free to choose our own geometry and coordinates.

Most ordinary fluids are not only Newtonian, but also *isotropic*. Liquid crystals are anisotropic, but so special that we shall not consider them here. In an isotropic fluid at rest there are no directions defined at all and the stress tensor must be diagonal $\sigma_{ij} = -p\,\delta_{ij}$, where $p$ is the (hydrostatic) pressure. When such a fluid is set in motion, the velocity field $v_i(\boldsymbol{x}, t)$ defines a direction in every point of space, but as we have argued before the velocity in a point cannot itself provoke stress in the fluid. It is the variations in velocity from point to point that cause stress, in other words the velocity gradients $\nabla_i v_j$.

These arguments show that the most general form of the stress tensor in an isotropic fluid in motion must be

$$\sigma_{ij} = a\delta_{ij} + b\nabla_i v_j + c\nabla_j v_i \ , \tag{17-9}$$

where the coefficients $a$, $b$, and $c$, may depend on the density $\rho$ and temperature $T$ as well as on the velocity gradients.

If we impose symmetry on the stress tensor, $\sigma_{ij} = \sigma_{ji}$, then it follows that $b = c$. The common value may immediately be identified with the shear viscosity defined in (17-1). For if we insert $\boldsymbol{v} = (v_x(y), 0, 0)$ in the above expression, we obtain the shear stress $\sigma_{xy} = \sigma_{yx} = b\nabla_y v_x(y)$, showing that $b = c = \eta$.

The coefficient $a$ is related to the average pressure (8-12). Calculating the trace of the above stress tensor, we obtain

$$p = -\frac{1}{3}\sum_i \sigma_{ii} = -a - \frac{2}{3}\eta \,\mathsf{div}\,\boldsymbol{v} \ .$$

Eliminating $a$, we have finally arrived at an expression for the complete stress tensor in a Newtonian fluid

$$\boxed{\sigma_{ij} = -p\,\delta_{ij} + \eta\left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\,\mathsf{div}\,\boldsymbol{v}\,\delta_{ij}\right)} \ , \tag{17-10}$$

an expression due to Stokes (1845). Comparing with (8-15), we recognize the second term as being the deviatoric stress tensor.

## Bulk viscosity

In the linear approximation we may disregard any velocity dependence in the shear viscosity $\eta$, but not in the average pressure $p$. Since the pressure is a scalar, it can only depend on $\mathsf{div}\,\boldsymbol{v}$ which is the only scalar linear combination of velocity gradients. Consequently, in the linear approximation the most general form of the pressure must be

$$\boxed{p = p_0 - \zeta\,\mathsf{div}\,\boldsymbol{v}} \ , \tag{17-11}$$

with coefficients $p_0$ and $\zeta$ that may depend on the density $\rho$ and temperature $T$. Taking $\boldsymbol{v} = \mathbf{0}$ we conclude that $p_0(\rho, T)$ must be understood as the hydrostatic pressure determined from the equation of state.

The parameter $\zeta$ is variously called the *bulk viscosity*, the *second viscosity*, or the *expansion viscosity*. Its presence implies that there is an extra *dynamic pressure* of size $\Delta p = -\zeta \operatorname{div} \boldsymbol{v}$ exerted by a viscous fluid in motion. The dynamic pressure is negative during expansion of a fluid ($\operatorname{div} \boldsymbol{v} > 0$), positive during contraction ($\operatorname{div} \boldsymbol{v} < 0$), and vanishes for incompressible fluids. It is as the derivation shows unrelated to the changes in (hydrostatic) pressure caused by density variations.

Bulk viscosity is normally of little consequence, because most fluids, even gases, behave as if they are incompressible under normal circumstances where the flow velocity is much smaller than the velocity of sound in the fluid (see page 241). In an effectively incompressible fluid we have $\operatorname{div} \boldsymbol{v} = 0$ so that the dynamic pressure vanishes. This is also why bulk viscosity is hard to measure, because one must set up physical conditions such that expansion and contraction become important, for example by means of high frequency sound waves. Fluids transmit sound like solids, but because of viscosity only pressure waves are able propagate through any distance. Even then, pressure waves are always damped or *attenuated* by viscosity. From the attenuation of ultrasound in fluids it is possible to extract values for the bulk modulus which turn out to be quite frequency dependent, although generally of the same magnitude as the shear viscosity (see chapter 20).

## Rate of deformation

Like Hooke's law (10-7) for the stress tensor in linear elastic materials, the stress tensor depends here on two material parameters (or "constants"), the shear viscosity $\eta$ and the bulk viscosity $\zeta$. Since a fluid particle is displaced by $\delta \boldsymbol{u} = \boldsymbol{v}\, \delta t$ in a small time interval $\delta t$, fluid motion may be seen as a continuous sequence of infinitesimal deformations with strain tensor, $\delta u_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)\, \delta t$. The symmetrized velocity gradients $\delta u_{ij}/\delta t = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$ may thus be understood as the *rate of deformation* or *rate of strain* of the fluid material. Comparison with (14-7)) shows that the divergence $\operatorname{div} \boldsymbol{v} = \operatorname{div} \delta \boldsymbol{u}/\delta t$ may be understood as the relative *rate of expansion*.

## The Navier-Stokes equation

Cauchy's equation of motion (14-32) depends only on the effective density of force $f_i^* = f_i + \sum_j \nabla_j \sigma_{ij}$. Inserting the stress tensor (17-10), we find

$$\sum_j \nabla_j \sigma_{ij} = -\nabla_i p + \eta \left( \boldsymbol{\nabla}^2 v_i + \frac{1}{3} \nabla_i \operatorname{div} \boldsymbol{v} \right) ,$$

where we have assumed that the shear viscosity does not depend on $\boldsymbol{x}$. If the viscosity varies, for example due to differences in temperature or composition, this will change the right hand side. Inserting this expression into Cauchy's equation of motion and converting to ordinary vector notation we obtain the field equation,

$$\boxed{\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = -\boldsymbol{\nabla} p + \eta \boldsymbol{\nabla}^2 \boldsymbol{v} + \frac{1}{3}\eta\boldsymbol{\nabla}\operatorname{div}\boldsymbol{v} + \boldsymbol{f}} \; . \tag{17-12}$$

This is the most general form of the *Navier-Stokes equation* for fluids (Navier (1822), Stokes (1845)). Together with the equation of continuity (14-25) and the relation of pressure to the equation of state (17-11), we have arrived at a set of five equations for the five mechanical fields $v_x, v_y, v_z$, $\rho$, and $p$. The system is complete.

Although relatively simple to look at, the Navier-Stokes equation contains all the complexity of real fluid flow, including that of Niagara Falls! It is therefore clear that one cannot in general expect to find simple solutions. Exact solutions can only be found in fairly simple geometries and under strong simplifying assumptions concerning the nature of the flow, as we did for the planar laminar flow between two plates in section 17.1.

## 17.3 Major classes of flow

The Navier-Stokes equation is a partial differential equation (or rather three) in time and space, implementing the local balance of momentum in a fluid around any point of space at any time, or equivalently Newton's second law.

Alternatively one may look at the Navier-Stokes equation as a dynamic law expressing the instantaneous rate of change of velocity

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} = -\rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\nabla} p + \eta \boldsymbol{\nabla}^2 \boldsymbol{v} + \frac{1}{3}\eta\boldsymbol{\nabla}\operatorname{div}\boldsymbol{v} + \boldsymbol{f} \; . \tag{17-13}$$

The right hand side is a sum of terms, each with its own interpretation, and often it will turn out that one term in the sum dominates the others. This must in fact be the usual case, unless there is some kind of "conspiracy" which coerces two terms to be of comparable magnitude in the same region of space and time. Such conspiracies do occur under special circumstances, for example near solid boundaries, but by and large we are justified in expecting only one term to be really important at any given point of space and time.

The most interesting phenomena in fluid dynamics arise from the competition between the inertia of the fluid represented by the advective term $\rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ and the viscosity represented by $\eta\boldsymbol{\nabla}^2\boldsymbol{v}$. Inertia attempts to continue the motion of a fluid once it is started whereas viscosity acts as a brake. If inertia is dominant we may leave out the terms due to viscosity, arriving again at Euler's equation (15-1) describing lively, *nearly ideal* fluid flow (see chapter 15). If on the other hand viscosity is dominant, we may drop the inertia term, and thereby obtain the basic equations for sluggish *creeping* flow (see chapter 22).

## Reynolds number

As a measure of how much an actual flow is the one or the other, one may make a rough estimate, called the *Reynolds number*, of the ratio between the advective term $\rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ and the viscous term $\eta\boldsymbol{\nabla}^2\boldsymbol{v}$. If the velocity typically changes by $U$ over a region of typical size $L$, then the order of magnitude of the first order spatial derivatives of the velocity will typically be $U/L$, and similarly the second order derivatives will be of the order of $U/L^2$. Consequently, the Reynolds number becomes

$$\mathsf{Re} \approx \frac{|\rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}|}{|\eta\boldsymbol{\nabla}^2\boldsymbol{v}|} \approx \frac{\rho U^2/L}{\eta U/L^2} = \frac{\rho U L}{\eta} = \frac{UL}{\nu} \ . \tag{17-14}$$

For small values of the Reynolds number, $\mathsf{Re} \ll 1$, the flow creeps along, whereas for large values, $\mathsf{Re} \gg 1$, it tends to be lively. The streamline pattern of creeping flow is orderly and layered, also called *laminar*, as is easily observed when mixing cocoa into dough to make a chocolate cake. The laminar pattern continues quite far beyond $\mathsf{Re} \simeq 1$, but depending on the flow geometry and other circumstances, there will be a Reynolds number, typically in the region of thousands, where *turbulence* sets in with its characteristic tumbling and chaotic behavior.

It is often quite easy to estimate the Reynolds number from the geometry and boundary conditions of a flow pattern, as is done in the following examples and in table 17.2.

**Example 17.3.1:** Getting out of a bathtub you create flows with speeds of say $U \approx 1$ m/s over a distance of $L \approx 1$ m. The Reynolds number becomes $\mathsf{Re} \approx 10^6$ and you are definitely creating visible turbulence in the water. Similarly, when jogging you create air flows with $U \approx 3$ m/s and $L \approx 1$ m, leading to a Reynolds number around $2 \times 10^5$, and you know that you must leave all kinds of little invisible turbulent eddies in the air behind you. The fact that the Reynolds number is smaller in air than in water despite the higher velocity is a consequence of the kinematic viscosity being larger for air than for water.

**Example 17.3.2:** For planar flow between two plates (section 17.1), the velocity scale is set by the velocity difference $U$ between the plates whereas the length scale is set by the distance $d$ between the plates. Using this model for a bearing lubricated with a layer of engine oil of thickness $d \approx 0.1$ mm and a relative speed of the rotating shaft of $U \approx 1$ m/s, the Reynolds number becomes $\mathsf{Re} \approx Ud/\nu \approx 5$ which indicates that the system is far from turbulence and the laminar assumption is thus justified.

**Example 17.3.3:** A typical 1/2" water pipe has diameter $d \approx 1.25$ cm and that sets the length scale. If the volume flux of water is $Q = 0.1$ liter/s, the average water speed becomes $U = Q/\pi a^2 \approx 1.3$ m/s and we get a Reynolds number $\mathsf{Re} = \rho U d/\eta \approx 15,000$ which brings the character of the flow just into the turbulent regime. For olive oil under otherwise identical conditions we get $\mathsf{Re} \approx 0.19$, indicating creeping flow.

| | fluid | size $L$ [m] | velocity $U$ [$\mathsf{ms}^{-1}$] | Reynolds number |
|---|---|---|---|---|
| submarine | water | 100 | 15 | $1.7 \times 10^9$ |
| airplane | air | 50 | 200 | $6.3 \times 10^8$ |
| blue whale | water | 30 | 10 | $3.4 \times 10^8$ |
| car | air | 5 | 30 | $9.4 \times 10^6$ |
| swimming human | water | 2 | 1 | $2.3 \times 10^6$ |
| running human | air | 2 | 3 | $3.8 \times 10^5$ |
| herring | water | 0.3 | 1 | $3.8 \times 10^5$ |
| golf ball | air | 0.043 | 40 | $2.2 \times 10^5$ |
| ping-pong ball | air | 0.040 | 10 | $5 \times 10^4$ |
| fly | air | 0.01 | 1 | 600 |
| flea | air | 0.001 | 3 | 190 |
| gnat | air | 0.001 | 0.1 | 6 |
| bacterium | water | $10^{-6}$ | $10^{-5}$ | $10^{-5}$ |
| bacterium | blood | $10^{-6}$ | $4 \times 10^{-6}$ | $10^{-9}$ |

Table 17.2: *Table of Reynolds numbers for some moving objects calculated on the basis of typical values of lengths and speeds. Viscosities are taken from 17.1. It is perhaps surprising that a submarine operates at a Reynolds number that is larger than that of a passenger jet at cruising speed.*

## Hydrodynamic similarity

The Reynolds number is *the* most important parameter that characterizes the flow. What matters is not the actual velocity, size, or kinematic viscosity, but the combination of velocity times size over kinematic viscosity that makes up the Reynolds number. Two flows in congruent geometries will, everything else being equal, follow the same streamline pattern if their Reynolds numbers are the same.

Air is about 15 times more kinematically viscous than water at normal temperature and pressure. The flow of water around an object, say a stone, has the same Reynolds number as the flow of air around the same object at 15 times higher speed. Water streaming at 6 $\mathsf{ms}^{-1}$ follows a pattern that is similar to hurricane winds at 90 $\mathsf{ms}^{-1}$, though the forces exerted by the two flows on solid objects are quite different (by roughly a factor 4 in favor of water). Air flowing at the same speed around an object 15 times larger will also follow this pattern.

The similarity of flows in congruent geometries can be exploited to study the flow around tiny objects by means of an enlarged models immersed in more viscous fluid. It is, for example, hard to study the air flow around wing of a fly in motion, which flaps perhaps 50 times per second. For a wing of size $L \approx 10$ mm and a flap-amplitude of 3 mm the tip velocity becomes $U \approx 300$ mm $\mathsf{s}^{-1}$ and the Reynolds number $\mathsf{Re} \approx 190$. The same Reynolds number can be obtained from a 10 cm metal wing of the same shape flapping slowly in water, once every 36 seconds, making the vortices that are are so important for flight much easier to study.

Flows in different geometries can only be compared in a coarse sense, even if they have the same Reynolds number. A running man has the same Reynolds number as a swimming herring, and a flying gnat the same Reynolds number as a man swimming in castor oil. In both cases the geometries presented to the flow are quite different, leading to different streamline patterns. Here the Reynolds number can only be used to indicate the character of the flow which tends to be turbulent around the running man, but laminar around the flying gnat.

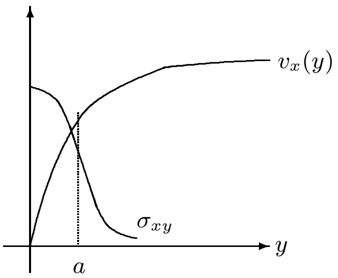
## 17.4 Boundary conditions

Differential equations always require boundary conditions. Field equations that are first order in time, like the Navier-Stokes equation (17-12), need initial values of the fields (and their spatial derivatives) in order to predict their values at later times. But what about physical boundaries, the containers of fluids, or even internal boundaries between different fluids? How do the fields behave there? Let us discuss the various fields that we have met one by one.

**Density:** The density is easy to dispose of, since it is allowed to be discontinuous and jump at a boundary between two materials, so this provides us with no condition at all. It is evident from the Navier-Stokes equation that a jump in density across a fluid boundary must somehow be accompanied by a jump in the derivatives of the other fields, but we shall not go into this question here.

**Stress:** Newton's third law requires the stress vector $\sum_j \sigma_{ij} n_j$ to be continuous across any boundary with normal $\boldsymbol{n}$. This simple picture is, however, complicated by surface tension, which can give rise to a discontinuous jump in stress across a boundary.

**Velocity:** The normal component $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ of the velocity field must be continuous across any boundary, for the simple reason that what goes in on one side must come out on the other. If this were not the case, material would collect at the boundary or holes would develop in the fluid. The latter kind of breakdown can actually happen in extreme situations (cavitation).



*Sketch of rapidly varying velocity and shear stress in a region of size $a$ near a boundary. For $a \to 0$ the velocity develops an abrupt jump and the stress becomes singular. The decrease of the the shear stress away from the discontinuity leads to spreading of a sharp discontinuity.*

The tangential component of velocity $\boldsymbol{v}_t = \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n})$ must also be continuous, but for different reasons. The linear relationship (17-1) between stress and velocity gradient implies that a tangential velocity field which changes rapidly along the direction normal to a surface, must create very large and rapidly varying shear stress. In the extreme case of a discontinuous jump in velocity, the shear stress would become infinite. Although such large stresses may be created, for example by hitting a fluid container with a hammer, they can however not be maintained for long, but are rapidly smoothed out by the action of viscosity.

Usually the whole velocity field, normal as well as tangential components, will therefore be assumed to be continuous across any boundary between Newtonian fluids. Since a solid wall may be viewed as an extreme Newtonian fluid with infinite viscosity, we recover the previously mentioned *no-slip condition*: a fluid has zero velocity relative to its containing walls. Viscous fluids never slip along the containing boundaries but adhere to them, and this is what makes viscous fluids *wet*.

> In the above argument it was not essential that the fluid be strictly Newtonian, as long as the stresses grow with rate of deformation. If this is not the case, or if the continuum approximation breaks down, shear slippage may occur, for example in extremely rarified gases.

### Boundary layers

The dissipative viscous processes which tend to smooth out all velocity differences will eventually also bring an isolated fluid completely to rest. If the fluid is not isolated, but acted upon by external forces doing real work, the smoothing eventually stops and the flow may become steady. In that case boundary layers will form in which the transition from the velocity of the containing walls to the fluid at large takes place. For sufficiently strong external driving forces, large shear velocity gradients may in fact be maintained and make boundary layers very thin, approximately discontinuous. We shall return to a detailed discussion of boundary layers in chapter 24.

## * 17.5 Dissipation

From the stress tensor (17-10) we obtain

$$\begin{aligned}\sum_{ij} \sigma_{ij} \nabla_j v_i &= -p \operatorname{div} \boldsymbol{v} + \eta \sum_{ij} (\nabla_i v_j + \nabla_j v_i) \nabla_j v_i - \frac{2}{3} \eta (\operatorname{div} \boldsymbol{v})^2 \\ &= -p \operatorname{div} \boldsymbol{v} + \eta \sum_{ij} \left( \nabla_i v_j + \nabla_j v_i - \frac{2}{3} \operatorname{div} \boldsymbol{v}\, \delta_{ij} \right) \nabla_j v_i \ .\end{aligned}$$

Defining the traceless and symmetric velocity gradient tensor,

$$v_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i) - \frac{1}{3} \operatorname{div} \boldsymbol{v}\, \delta_{ij} \ , \tag{17-15}$$

we may write

$$\sum_{ij} v_{ij} \nabla_j v_i = \sum_{ij} v_{ij}^2 \ .$$

Finally, we obtain the following expression for the internal power (16-81) in a viscous fluid,

$$\boxed{P_{\text{int}} = \int_V \left( \boldsymbol{v} \cdot \boldsymbol{f}_{\text{int}} + p \operatorname{div} \boldsymbol{v} - 2\eta \sum_{ij} v_{ij}^2 \right) dV} \ , \tag{17-16}$$

where $p$ is the pressure (17-11) containing the bulk viscosity. The first term is the rate of work of the internal volume forces, and the second the rate of work of the pressure. The last term represents the rate of work done by shear viscosity. It is negative definite and thus corresponds to a loss of kinetic energy, which we identify with the dissipation of kinetic energy flow into heat.

## Problems

**17.1** a) Show that in an ideal gas the density of molecules (molecules per unit of volume) is

$$n = \frac{N_A p}{RT} \tag{17-17}$$

and find its value for normal temperature and pressure.

b) Show that in an ideal gas consisting of spherical molecules with diameter $d$ the mean free path between collisions is of the order of

$$\lambda \approx \frac{1}{n\pi d^2} \tag{17-18}$$

Estimate its magnitude in air where the molecular diameter may be taken to be about 0.3 nm (the kinetic theory of gases actually reduces the mean free path by a factor of $\sqrt{2}$).

c) Assume that the molecular velocity $v_0 = \lambda/\tau$ is of the same order of magnitude as the velocity of sound (the kinetic theory of gases actually makes it about a factor $\sqrt{2}$ larger), and use this to estimate the viscosity of air.

**17.2** In the sport of *curling*, a "stone" with mass $M \approx 20$ kg is set into motion on an ice rink approximately 40 m long. The contact area between stone and ice is $A \approx 700\ \mathrm{cm}^2$. One may assume that a thin layer of water of constant thickness $d$ acts as lubricant. Show that a stone given an initial velocity $U_0$ will only move a finite distance $x_0$, and estimate for $U_0 \approx 3$ m/s the thickness of the lubricating layer.

**17.3** Consider planar momentum diffusion (page 289) and assume that the flow of the "river" vanishes fast at infinity, as in the Gaussian case. a) Show that for any river flow the total flux of fluid in the $x$-direction is independent of time. b) Show that the total momentum per unit of length in the $x$-direction is likewise constant. c) Calculate the kinetic energy per unit of length in the $x$-direction as a function of time in the Gaussian case.

∗ **17.4** Show that the general solution to the momentum diffusion equation (17-7) is

$$v_x(y,t) = \frac{1}{2\sqrt{\pi\nu t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-y')^2}{4\nu t}\right) v_x(y',0)\, dy' \tag{17-19}$$

Use this to show that any bounded initial velocity distribution becomes Gaussian for $|y| \to \infty$.

**17.5** a) Show that the average of a unit vector $\boldsymbol{n}$ over all directions obeys

$$\langle n_i n_j \rangle = \frac{1}{3}\delta_{ij} \tag{17-20}$$

b) Use this to show that the average pressure 8-12 is also the average of the normal stress acting on an arbitrary surface element in a fluid.

**17.6** Estimate the Reynolds number for a) an ocean current, b) a water fall, c) a weather cyclone, d) a hurricane, e) a tornado, f) lava running down a mountainside, and g) plate tectonic motion.