

**Part I**

**Fluids at rest**

## Fluids at rest

In this part we only consider fluids at permanent rest without any macroscopic local or global motion. Microscopically, fluids like all kinds of matter are of course never completely at rest because of random molecular motion (heat), and internal atomic motion.

For a minimal curriculum only chapter 2 is essential.

### List of chapters

2. **Pressure:** The intuitive concept of pressure in static fluids is brought on a formal footing by means of the pressure field. The basic equations of hydrostatics are developed and applied to the sea, the atmosphere, and the outermost layer of the Sun.
3. **Buoyancy and stability:** Archimedes' principle is derived from the balance of forces and expanded using moment balance. The stability of floating objects is analyzed and applied to ships.
4. **Hydrostatic shapes:** The flattening of the Earth as well as the tides of Earth and other astronomical bodies is analyzed.
5. **Surface tension:** The physical origin and formal definition of surface tension are presented. Soap bubbles are described and analyzed. The Young-Laplace law is derived and the Rayleigh-Plateau instability analyzed. Capillary effects, menisci, and drop shapes are analyzed.

# 2

## Pressure

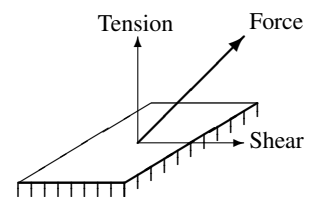
If the Sun did not shine, if no heat were generated inside the Earth and no energy radiated into space, all the winds in the air and the currents in the sea would die away, and the air and water on the planet would in the end come to rest in equilibrium with gravity. In the absence of external driving forces or time-dependent boundary conditions, and in the presence of dissipative contact forces, any fluid must eventually reach a state of *hydrostatic equilibrium*, where nothing moves anymore and all fields become constant in time. This state must be the first approximation to the sea, the atmosphere, the interior of a planet or a star.

In mechanical equilibrium of a continuous system there is everywhere a balance between short-range *contact forces* and long-range *body forces*, such as gravity. Contact interactions between material bodies or even between parts of the same body take place across *contact surfaces*. A contact force acting on a tiny patch of a surface can in principle take any direction relative to the surface, and may be resolved into a normal and a tangential component. If the normal component has the same orientation as the surface normal, it is called a *tension force*, and if opposite a *pressure force*. The component acting tangentially to the surface is called a *shear force* or a *traction force*. Fluids in motion, and solids at rest or in motion, are able to sustain shear forces, whereas fluids at rest can not. Should shear forces arise in a fluid at rest, it will begin to *flow* until it again reaches mechanical equilibrium without shear forces.

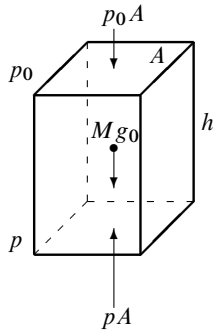
In this chapter we shall first investigate the basic properties of pressure, and afterwards develop the mathematical formalism which permits us to analyze hydrostatic equilibrium in the sea and the atmosphere. Along the way we shall recapitulate some basic rules of thermodynamics. In the following chapters we shall continue to study the implications of hydrostatic equilibrium for balloons and ships, and the shapes of large fluid bodies subject only to gravity and small fluid bodies subject mainly to surface tension.

### 2.1 What is pressure?

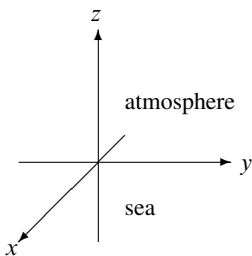
*Pressure* is defined as *normal force per unit of area*. The SI-unit for pressure is accordingly newton per square meter, but was in 1971 given the name *pascal* and the special symbol  $\text{Pa} = \text{N m}^{-2}$ . Earlier units for pressure were the *bar* ( $1 \text{ bar} = 10^5 \text{ Pa}$ ) and the *standard atmosphere* ( $1 \text{ atm} = 101325 \text{ Pa}$ ) which is close to the average air pressure at sea level. Modern television weather forecasters are now abandoning the older units and tend to quote air pressure in hectopascals rather than in millibars, even if they are exactly the same ( $1 \text{ hPa} = 100 \text{ Pa} = 10^{-3} \text{ bar} = 1 \text{ millibar}$ ).



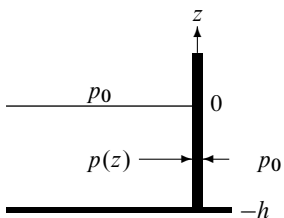
The force acting on the material underneath a small patch of a surface can always be resolved into a perpendicular pressure force and a tangential shear force. The pressure is positive, if the force is directed towards the patch, and negative if it (as here) is directed away from it.



A column of sea water. The pressure difference between bottom and top must carry the weight of the water in the box. Notice that the sum of the forces vanishes.



The flat-earth coordinate system.



Water stemmed up behind a sluice gate. The pressure varies linearly with depth  $-z$ .

**Evangelista Torricelli (1608–1647).** Italian physicist. Constructed the first mercury barometer in 1643 and noted that the barometric pressure varied from day to day. Served as companion and secretary for Galileo in the last months of Galileo's life. Died from typhoid fever at the age of 39.

### Case: The incompressible sea

Before presenting a formal definition of the pressure field, we use simple arguments to calculate it in the sea. In the first approximation water is incompressible and has everywhere the same mass density  $\rho_0$ . A vertical box of water in the sea with cross-sectional area  $A$  and height  $h$  has volume  $Ah$ , so that the total mass in the box is  $M = \rho_0 Ah$  and its weight  $Mg_0$  where  $g_0$  is standard gravity. The pressure  $p_0$  at the top acts on the box with a downwards force  $p_0 A$  whereas the pressure  $p$  at the bottom acts with an upwards force  $pA$ . In mechanical equilibrium, the upwards force  $pA$  must equal the sum of the downwards forces,

$$pA = Mg_0 + p_0 A. \quad (2.1)$$

If this equation is not fulfilled, the non-vanishing total vertical force on the column of water will begin to move it upwards or downwards.

Dividing by the area  $A$ , the pressure at the bottom of the box becomes,

$$p = p_0 + \rho_0 g_0 h. \quad (2.2)$$

To get an expression for the pressure field we introduce a *flat-earth coordinate system* with vertical  $z$ -axis and the surface of the sea at  $z = 0$ . Taking a box with the top at  $z = 0$  and the bottom at  $z = -h$ , we may write the pressure as a function of the coordinates, i.e. as a field,

$$p = p_0 - \rho_0 g_0 z, \quad (2.3)$$

where  $p_0$  is the surface pressure (equal to the atmospheric pressure). The pressure field is independent of the horizontal coordinates,  $x$  and  $y$ , because the flat-Earth sea has the same properties everywhere at the same depth.

With  $\rho_0 \approx 1000 \text{ kg m}^{-3}$  and  $g_0 \approx 10 \text{ m s}^{-2}$  the scale of the pressure increase per unit of depth in the sea becomes  $\rho_0 g_0 \approx 10^4 \text{ Pa m}^{-1} = 1 \text{ bar}/10 \text{ m}$ . Thus, at the deepest point in the sea—the Challenger Deep of the Mariana Trench in the Western Pacific Ocean—with  $z \approx -11 \text{ km}$  the pressure is a about 1100 bar. The assumption of constant density is reasonably well justified even to this depth, because the density of water is only about 5% higher there than at the surface (see page 33).

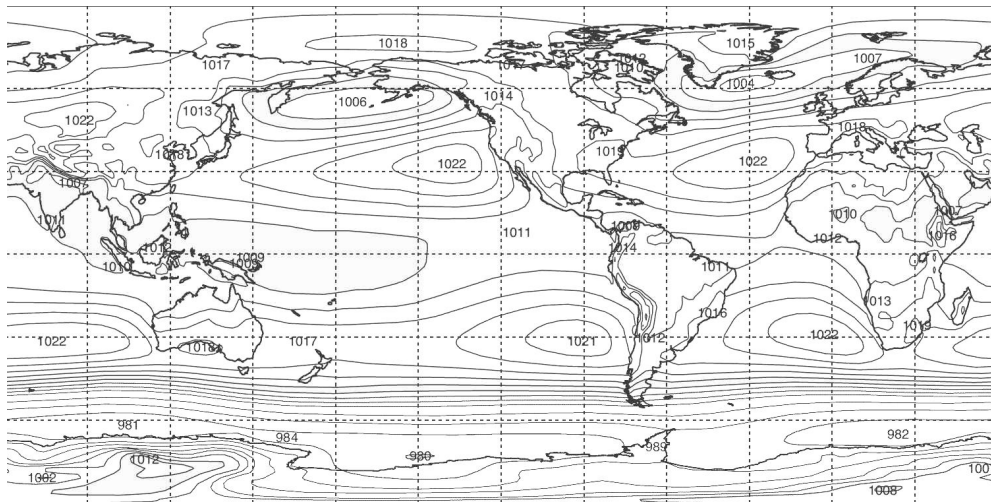
**Example 2.1 [Sluice gate force]:** Water is stemmed up to height  $h$  behind a sluice gate of width  $L$ . On the water surface and on the outer side of the gate there is atmospheric pressure  $p_0$ . What is the total horizontal force acting on the gate when  $h = 20 \text{ m}$  and  $L = 30 \text{ m}$ ?

Using that the pressure is the same in all directions in a fluid at rest (see the following section), we find by integrating the pressure excess,  $p(z) - p_0$ , on the inside of the gate,

$$\mathcal{F} = \int_{-h}^0 (p(z) - p_0) L dz = -\rho_0 g_0 L \int_{-h}^0 z dz = \frac{1}{2} \rho_0 g_0 L h^2. \quad (2.4)$$

This result should have been anticipated, because the pressure rises linearly with depth, such that the total force is simply the product of the area of the sluice gate  $Lh$  with the average pressure excess  $\langle p - p_0 \rangle = \frac{1}{2} \rho_0 g_0 h$  acting on the gate. With the given numbers the force becomes  $\mathcal{F} \approx 6 \times 10^7 \text{ N}$ , corresponding to the weight of 6,000 tons of water.

**Example 2.2 [Torricelli's mercury barometer]:** The mercury barometer is a glass tube, sealed at one end, filled with mercury and placed vertically with the open end in a little dish containing mercury. If the tube is long enough (about one meter), the mercury will sink down until the height of the mercury column stabilizes at about 76 centimeter, depending on the ambient air pressure  $p$ . Above the mercury column there is nearly vacuum with a bit of mercury vapor exerting only a small pressure  $p_0 \approx 0$ . With  $\rho_0 = 13.534 \text{ g cm}^{-3}$  and  $p = 1 \text{ atm}$  we get indeed  $h = (p - p_0)/\rho_0 g_0 = 76.34 \text{ cm}$  from eq. (2.2). Conversely, from the variations in the actual height the actual atmospheric pressure can be determined. Until fairly recently a mercury-based sphygmomanometer with a U-shaped glass tube of about 30 cm height open in one side was always used to measure resting blood pressure (see problem 2.1).



**Figure 2.1.** The air pressure at sea level is not constant, but varies by a few percent around its mean. This picture shows the world-wide sea level air pressure averaged over a 22-year period (1979–2001). The contours are isobars and the numbers are millibars. Adapted from OceanWorld (permission to be obtained).

### Case: Incompressible atmosphere?

Since air is highly compressible, it makes little sense to use the linear pressure field (2.3) for the atmosphere, except just above the sea. If anyway continued beyond this region the pressure would reach zero at a height  $z = h_0$  where,

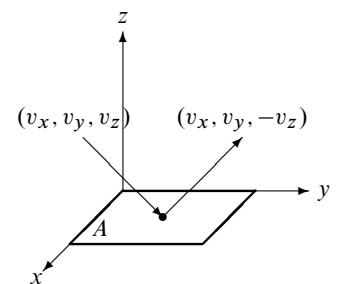
$$h_0 = \frac{p_0}{\rho_0 g_0}. \quad (2.5)$$

With surface pressure  $p_0 = 1 \text{ atm}$  and air density  $\rho_0 = 1.2 \text{ kg m}^{-3}$  at  $25^\circ\text{C}$  we get  $h_0 = 8.73 \text{ km}$  which is a tiny bit lower than the height of Mount Everest (8.848 km). This is, of course, meaningless, since climbers have reached the summit of that mountain without oxygen masks. Nevertheless, this height sets the correct scale for major changes in the atmospheric properties. As seen in figure 2.2 on page 35, the linear decrease of air pressure with height is quite a decent approximation for the first two kilometers.

### Microscopic origin of pressure in gases

In a liquid the molecules directly touch each other and the container walls, and liquid pressure may be seen as a result of these contacts. A gas consists mostly of vacuum with the molecules moving freely along straight lines between collisions with each other, and the gas pressure on a solid wall arises from the incessant random molecular bombardment of the wall.

We shall under very general and reasonable assumptions obtain the pressure from the average rate of molecular momentum transfer to a small flat area  $A$  of the wall orthogonal to the  $z$ -axis. Let us begin with the (evidently unreasonable) assumption that all molecules have the same velocity vector  $\mathbf{v} = (v_x, v_y, v_z)$  with  $v_z < 0$ , such that they all eventually will strike the wall at  $z = 0$ . The molecules that collectively hit the area  $A$  in a small time interval  $dt$  must all lie within a “striking distance”  $dz = -v_z dt$  from the wall, so that the total mass of these molecules becomes  $dM = \rho A dz = \rho A (-v_z) dt$  where  $\rho$  is the mass density of the gas. Next we assume that all molecules are reflected perfectly from the wall, thereby changing the  $z$ -component of the velocity from  $v_z$  to  $-v_z$  while leaving the other components unchanged. The momentum transferred to the area  $A$  in the time interval  $dt$



A molecule of mass  $m$  that is reflected from the wall changes the sign of its  $z$ -component but leaves the other components unchanged, such that the momentum transferred to the wall is  $2mv_z$ .

becomes  $d\mathcal{P}_z = 2v_z dM = -2\rho v_z^2 A dt$ , and the corresponding force

$$\mathcal{F}_z = \frac{d\mathcal{P}_z}{dt} = -2\rho v_z^2 A \quad (2.6)$$

Notice the sign, which shows that the force is always directed into the wall, as it should.

Finally, we take into account that the many molecular collisions creates a mixture of all possible velocities. We do not need to know this mixture, except that it ought to have the same probability for all velocity directions. Denoting the average over the mixture of velocities by a bracket  $\langle \cdot \rangle$ , we get by symmetry  $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{1}{3} \langle v_x^2 + v_y^2 + v_z^2 \rangle = \frac{1}{3} \langle v^2 \rangle$ . As only half the molecules, those with  $v_z < 0$ , can hit the wall, the pressure defined as the average force acting on a unit of area becomes,

$$p \equiv \frac{1}{2} \frac{\langle -\mathcal{F}_z \rangle}{A} = \rho \langle v_z^2 \rangle = \frac{1}{3} \rho \langle v^2 \rangle, \quad (2.7)$$

and solving for the root-mean-square molecular velocity, we obtain

$$v_{\text{mol}} \equiv \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3p}{\rho}}. \quad (2.8)$$

We shall see later that this velocity—as one might suspect—is of the same order of magnitude as the velocity of sound. At a pressure of  $p = 1 \text{ atm}$  and temperature  $T = 18^\circ \text{ C} = 291 \text{ K}$ , the density of air is  $\rho \approx 1.2 \text{ kg m}^{-3}$ , and the molecular velocity calculated from the above expression becomes  $v \approx 500 \text{ m s}^{-1}$ .

## 2.2 The pressure field

Pressure is a contact force acting on surfaces. A surface needs not be a real interface where material properties change dramatically but may just be an imaginary surface separating two parts of the same body from each other. The surface can be divided into a huge number of tiny flat vector surface elements, each being the product of its area  $dS$  and the unit vector  $\hat{n}$  in the direction of the normal to the surface,

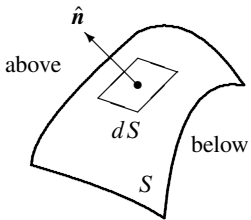
$$d\mathcal{S} = \hat{n} dS. \quad (2.9)$$

There is nothing intrinsic in a surface which defines the orientation of the normal, i.e. whether the normal is really  $\hat{n}$  and not  $-\hat{n}$ . A choice must, however, be made, and having done that, one may call the side of the surface element into which the normal points *positive* and the other of course *negative* (or alternatively *above* and *below*). Neighboring surface elements are required to be oriented towards the same side. By universal convention the normal of a closed surface is chosen to be directed out of the enclosed volume, so that the enclosed volume always lies at the negative side (more details can be found in appendix C).

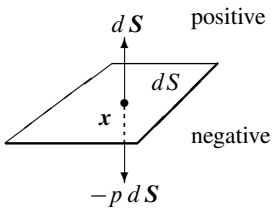
A fluid at rest cannot sustain shear forces, so all contact forces on a surface must act along the normal at every point of the surface. The force exerted by the material at the *positive* side of the surface element  $d\mathcal{S}$  (near  $\mathbf{x}$ ) on the material at the *negative* side is written,

$$d\mathcal{F} = -p(\mathbf{x}) d\mathcal{S}, \quad (2.10)$$

where  $p(\mathbf{x})$  is the pressure field. Convention dictates that a positive pressure exerts a force directed *towards* the material on the negative side of the surface element, and this explains the minus sign. A negative pressure that pulls at a surface is sometimes called a *tension*.



All normals to an oriented open surface must have the same orientation (excluding the Möbius band and similar constructs).

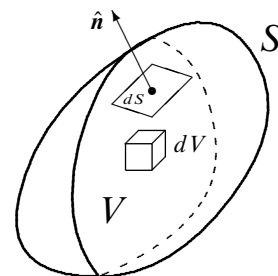


The force on a vector surface element under positive pressure is directed against the normal towards the material on the negative side.

The total pressure force acting on any oriented surface  $S$  is obtained by adding (i.e. integrating) all the little vector contributions from each surface element,

$$\mathcal{F} = - \int_S p d\mathbf{S}. \quad (2.11)$$

Here we have again suppressed the explicit dependence on  $\mathbf{x}$ . This is the force which acts on the cork in the champagne bottle, moves the pistons in the cylinders of your car engine, breaks a dam, and fires a bullet from a gun. It is also this force that lifts fish, ships, and balloons.



A volume  $V$  defined by the closed surface  $S$  has by convention all normals oriented towards the outside.

### Total pressure force on a material particle

A material particle in the fluid is like any other body subject to pressure forces from all sides, but being infinitesimal it is possible to derive a general expression for the resultant force. Let us choose a material particle in the shape of a small rectangular box with sides  $dx$ ,  $dy$ , and  $dz$ , and thus a volume  $dV = dxdydz$ . Since the pressure is slightly different on opposite sides of the box the  $z$ -component of total pressure force (2.11) becomes,

$$d\mathcal{F}_z = (p(x, y, z) - p(x, y, z + dz))dxdy \approx -\frac{\partial p(x, y, z)}{\partial z} dxdydz.$$

Here we have used that the normals to the closed surface of the box all point out of the box, so that  $dS_z = \pm dxdy$  on the faces of the box orthogonal to the  $z$ -axis and  $dS_z = 0$  on the other faces. Including similar results for the other coordinate directions we obtain the total pressure force,

$$d\mathcal{F} = -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) dV = -\nabla p dV. \quad (2.12)$$

Here we have introduced the *gradient operator* or *nabla*,  $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ . If you need more details, please consult appendix C where all these things are explained.

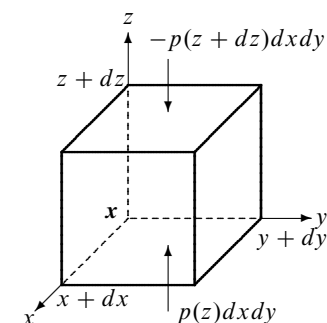
The resultant of all pressure forces acting on a tiny material particle is apparently equivalent to a volume force with a density of force equal to the negative gradient of the pressure,  $d\mathcal{F}/dV = -\nabla p$ . It is, however, not a true long-range volume force like gravity or electromagnetism but an effective expression for the total contact force on the material particle. We shall in chapter 6 see that a similar result holds when shear contact forces are at play.

### Gauss' theorem

The total pressure force on a body of volume  $V$  can thus be obtained in two ways: either by adding (integrating) all the pressure forces that act on its surface  $S$ , or by adding (integrating) all the forces on its constituent material particles. Changing the sign, we must in other words have,

$$\oint_S p d\mathbf{S} = \int_V \nabla p dV. \quad (2.13)$$

The circle through the surface integral is only there to remind us that the surface is closed. This is one version of *Gauss' theorem*, a purely mathematical relation between the integral of an arbitrary function  $p(\mathbf{x})$  over a closed surface  $S$  and the integral of its gradient  $\nabla p(\mathbf{x})$  over the enclosed volume  $V$ .



Pressure forces in the  $z$ -direction on a material particle in the shape of a rectangular box with sides  $dx$ ,  $dy$ , and  $dz$ . The  $x$  and  $y$  coordinates are suppressed in the pressure.



**Johann Carl Friedrich Gauss (1777–1855).** German mathematician of great genius. Contributed to number theory, algebra, non-Euclidean geometry, and complex analysis. In physics he developed the magnetometer. The older (cgs) unit of magnetic strength is named after him.

Usually *Gauss' theorem* is formulated as a relation between the surface integral over a vector field  $\mathbf{U}(\mathbf{x})$  and the volume integral over its *divergence*  $\nabla \cdot \mathbf{U} = \nabla_x U_x + \nabla_y U_y + \nabla_z U_z = \partial U_x / \partial x + \partial U_y / \partial y + \partial U_z / \partial z$ ,

$$\oint_S \mathbf{U} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{U} dV. \quad (2.14)$$

The following argument shows that the two formulations are equivalent. Starting with the  $x$ -component of (2.13) with  $p = U_x$  we obtain  $\oint_S U_x dS_x = \int_V \nabla_x U_x dV$ . Adding the similar contributions from the two other directions we obtain (2.14). Conversely, we may choose  $\mathbf{U} = (p, 0, 0)$  in (2.14) to obtain the  $x$ -component of (2.13), and similarly for the other directions. A rigorous mathematical proof of the theorem is found in appendix C.

### Same pressure in all directions?

One might think that the pressure could depend on the orientation of the surface element  $d\mathbf{S}$ , but that is actually not the case. Newton's third law guarantees that the pressure force exerted by the material below on the material above is  $-d\mathcal{F}$ . Since the surface element also changes sign we can write the definition (2.10) as  $-d\mathcal{F} = -p(\mathbf{x})(-d\mathbf{S})$ , which shows that the pressure  $p(\mathbf{x})$  is the same, whether you calculate the force from above or below. This conclusion is part of broader theorem, called *Pascal's law*, which says that *the pressure in a fluid at rest is the same in all directions*. Pressure is, in other words, a true field that only depends on space (and possibly also time).



**Blaise Pascal (1623–1662).** French mathematician and physicist. Founded probability theory. Wrote the first book on the systematic theory of hydrostatics. Constructed what may be viewed as the first digital calculator. He spent his later years with religious thinking in the Cistercian abbey of Port-Royal. More than one property of pressure goes under the name of Pascal's law.

The simple reason for pressure being the same in all directions in hydrostatic equilibrium is that the pressure acts on the surface of a body whereas a body force like gravity acts on the volume. If we let the body shrink, the contribution from the body force will vanish faster than the contribution from the surface force because the volume vanishes faster than the surface area. In the limit of vanishing body size only the surface force is left, but it must then itself vanish in hydrostatic equilibrium where the total force on all parts of a body has to vanish.

**\* Proof of Pascal's law:** Consider a material particle in the shape of a right tetrahedron oriented along the coordinate axes, as shown in the margin figure. Each of the three triangles making up the sides of the tetrahedron in the coordinate planes is in fact the projection of the front face  $d\mathbf{S} = (dS_x, dS_y, dS_z)$  onto that face. By elementary geometry the areas of the three projected triangles are  $dS_x$ ,  $dS_y$  and  $dS_z$ . Assume now that the pressure is actually different on the four faces of the tetrahedron:  $p$  on the front face and  $p_x$ ,  $p_y$ , and  $p_z$  on the faces in the coordinate planes. The total pressure force acting on the body is then,

$$\begin{aligned} d\mathcal{F}_S &= -pd\mathbf{S} + p_x dS_x \hat{\mathbf{e}}_x + p_y dS_y \hat{\mathbf{e}}_y + p_z dS_z \hat{\mathbf{e}}_z \\ &= ((p_x - p)dS_x, (p_y - p)dS_y, (p_z - p)dS_z). \end{aligned}$$

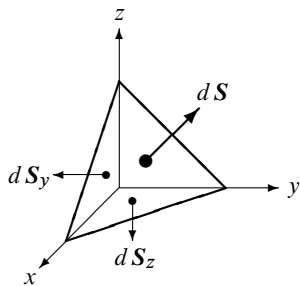
For a fluid at rest the sum of the total surface force  $d\mathcal{F}_S$  and a volume force  $d\mathcal{F}_V = \mathbf{f} dV$  (for example gravity) must vanish,

$$d\mathcal{F}_S + d\mathcal{F}_V = \mathbf{0}. \quad (2.15)$$

The idea is now to show that for sufficiently small tetrahedrons the volume force can be neglected. To convince yourself of this, consider a geometrically congruent tetrahedron with all lengths scaled down by a factor  $\lambda < 1$ . Since the volume scales as the third power of  $\lambda$  whereas the surface areas only scale as the second power, the above equation for the scaled tetrahedron becomes  $\lambda^2 d\mathcal{F}_S + \lambda^3 d\mathcal{F}_V = \mathbf{0}$  or equivalently

$$d\mathcal{F}_S + \lambda d\mathcal{F}_V = \mathbf{0}. \quad (2.16)$$

In the limit of  $\lambda \rightarrow 0$ , it follows that the total contact force must vanish, i.e.  $d\mathcal{F}_S = \mathbf{0}$ , and that is only possible for  $p_x = p_y = p_z = p$ . The pressure must indeed be the same in all directions.



A right tetrahedron with three sides in the coordinate planes. The vector normals to the sides are all pointing out of the volume ( $dS_x$  is hidden from view).



## 2.3 Hydrostatics

Two classes of forces are at play in standard continuum physics. The first class consists of *body forces* that act throughout the volume of a body. The only known body forces are gravity and electromagnetism, but in standard hydrostatics we shall only discuss gravity. The second class consists of *contact forces* acting only on the surface of a body. Pressure is the only contact force that is possible in fluids at rest, but there are others in fluids in motion and in solids. The full set of possible contact forces will be introduced in chapter 6.

### Global hydrostatic equilibrium

The total force acting on a fluid volume  $V$  enclosed by the surface  $S$  receives two contributions: one from gravity and one from pressure. The gravitational acceleration field  $\mathbf{g} = \mathbf{g}(\mathbf{x})$  is a body force which acts on every material particle with a force  $d\mathcal{F} = \mathbf{g} dM$  where  $dM = \rho dV$  is the mass of the particle. Pressure acts as described in the preceding section with a force  $d\mathcal{F} = -p d\mathbf{S}$  on every surface element of the body. Explicitly, these two contributions to the total force on the fluid body are,

$$\mathcal{F}_G = \int_V \rho \mathbf{g} dV, \quad \mathcal{F}_B = -\oint_S p d\mathbf{S}. \quad (2.17)$$

The force of gravity is called the *weight* of the fluid body, and the force due to pressure is called the *buoyancy* force.

In mechanical equilibrium the total force must vanish,

$$\mathcal{F} = \mathcal{F}_G + \mathcal{F}_B = \int_V \rho \mathbf{g} dV - \oint_S p d\mathbf{S} = \mathbf{0} \quad (2.18)$$

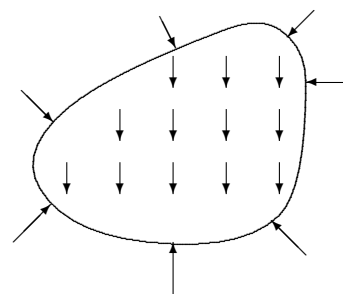
for *each and every* volume  $V$  of fluid. This is the *equation of global hydrostatic equilibrium*. It states that *the weight of the fluid in the volume must exactly balance its buoyancy*. In section 2.1 we used intuitively that the weight of a vertical box of sea water must be in balance with the pressure forces on the bottom and the top of the box, and could in this way derive the pressure in the sea. If the balance is not perfect, as for example if a small volume of fluid is heated or cooled relative to its surroundings, the fluid *must* start to move, either upwards if the buoyancy force wins over the weight or downwards if it loses.

The problem with the global equilibrium equation is that we have to know the force density  $\rho \mathbf{g}$  and the pressure  $p$  in order to calculate the integrals. Sometimes symmetry considerations can get us a long way (see the example below), but in general we need a local form of the equation of hydrostatic equilibrium valid at each point  $\mathbf{x}$  in any geometry, to be able to determine the hydrostatic pressure anywhere in a fluid.

**Example 2.3 [The incompressible sea]:** In constant gravity and using flat-earth coordinates, we have  $\mathbf{g}(\mathbf{x}) = \mathbf{g}_0 = (0, 0, -g_0)$ . For symmetry reasons the sea on the flat Earth ought to have the same properties for all  $x$  and  $y$ , such that the pressure can only depend on  $z$ . Taking a rectangular vertical box with height  $h$  and cross sectional area  $A$ , we find

$$\mathcal{F}_G = -\rho_0 g_0 A h \hat{\mathbf{e}}_z, \quad \mathcal{F}_B = (-p(0)A + p(-h)A) \hat{\mathbf{e}}_z, \quad (2.19)$$

because the pressure contributions from the sides of the box cancel. Global hydrostatic balance immediately leads back to eq. (2.2).



A body field like gravity acts on a body over its entire volume, while a contact field like pressure only acts on its surface.

### Local hydrostatic equilibrium

The total force on a material particle is of course also the sum of its weight  $d\mathcal{F}_G = \mathbf{g} dM$  and its effective buoyancy  $d\mathcal{F}_B = -\nabla p dV$  from eq. (2.12),

$$d\mathcal{F} = d\mathcal{F}_G + d\mathcal{F}_B = (\rho \mathbf{g} - \nabla p) dV. \quad (2.20)$$

The expression in the parenthesis is called the *effective force density*,

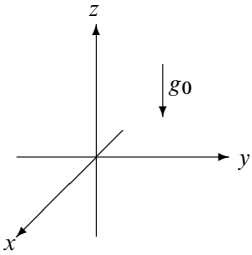
$$\mathbf{f}^* = \rho \mathbf{g} - \nabla p. \quad (2.21)$$

It must again be emphasized that the effective force density is the sum of the true long-range force (gravity) and the resultant of all the short-range pressure forces acting on its surface.

In hydrostatic equilibrium the total force  $d\mathcal{F} = \mathbf{f}^* dV$  must vanish for every material particle, implying that the effective force density must vanish everywhere,  $\mathbf{f}^* = \mathbf{0}$ , or

$$\nabla p = \rho \mathbf{g}. \quad (2.22)$$

This is the *local equation of hydrostatic equilibrium*. It is a differential equation for the pressure, valid everywhere in a fluid at rest, and it encapsulates in an elegant way all the physics of hydrostatics. Gauss' theorem (2.13) allows us to convert the local equation back into the global one (2.18), showing that there is complete mathematical equivalence between the local and global formulations of hydrostatic equilibrium.



The flat-earth coordinate system with constant gravity pointing against the direction of the  $z$ -axis.

### Hydrostatics in constant gravity

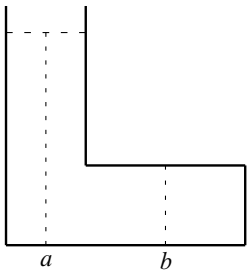
In a flat-earth coordinate system, the constant field of gravity is  $\mathbf{g}(\mathbf{x}) = (0, 0, -g_0)$  for all  $\mathbf{x}$ . Written out explicitly in coordinates, the local equilibrium equation (2.22) becomes three individual equations,

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g_0. \quad (2.23)$$

The two first equations express that the pressure does not depend on  $x$  and  $y$  but only on  $z$ , which confirms the previous argument based on symmetry. It also shows that, independently of the shape of a fluid container, the pressure will always be the same at a given depth (in constant gravity). For the special case of constant density,  $\rho(z) = \rho_0$ , the last equation may immediately be integrated to yield the previous result (2.3) for the pressure field in the incompressible sea.

### Paradoxes of hydrostatics

It is easily understood why the pressure can only depend on  $z$  in an open liquid container, because it must everywhere carry the weight of the liquid column above. But what if the liquid column does not reach all the way to the surface, as in a “boot” filled with water (see the margin figure)? Since there is only a short column of water in the “toe”, it raises the question of where the constant pressure comes from. What is it “up against”? The only possibility is that the material of the boot must supply the necessary downwards forces to compensate for the missing weight of the water column. These forces arise as a response to the elastic extension of the material of the boot caused by the pressure. If the material in the toe cannot deliver a sufficient response, it will break and a fountain of water will erupt from the toe.



A schematic “boot”. The pressure at a horizontal bottom under the open shaft of the boot ( $a$ ) has to carry the full weight of the liquid above, but what about the pressure in the toe of the boot ( $b$ ) where the water column is much shorter?

Another “paradox” goes back to Pascal. Wine is often stored for aging in large wooden casks containing cubic meters of the precious liquid. Such casks may only be able to withstand perhaps 30% pressure difference between liquid and atmosphere. Even if it would seem natural to mount a pipe line to transport the wine to and from the casks, such a system comes with its own problems. If, for example, the pipe entrance lies five meters above the casks (that often are placed in cellars), and if the unwary employee fills the cask and the whole pipe with wine, half an atmosphere extra pressure will be added to the cask which may as a consequence develop a leak or even burst. Paradoxically, this does not depend on the diameter of the pipe.

Finally, what happens if someone presents you with a fluid density  $\rho(x, y, z)$  that actually depends on the horizontal coordinates? Such a situation could, for example, arise when you remove a separating wall in a container with petrol on one side and water on the other. The hydrostatic equations (2.23) cannot be fulfilled because they imply that  $-g_0 \partial \rho / \partial x = \partial^2 p / \partial x \partial z = 0$ , contradicting the assumption that  $\rho$  varies with  $x$ . The conclusion must be that the liquid cannot be static but must flow until the dependence on the horizontal coordinates has gone. All the petrol will in the end lie in a horizontal layer on top of the water. Non-horizontal separating surfaces between immiscible fluids at rest can simply not exist.

## 2.4 Equation of state

The local equation of hydrostatic equilibrium (2.22) with an externally given gravitational field is not enough in itself, but needs some relation between pressure and density. In the preceding sections we mostly thought of fluids with constant density and could then integrate the hydrostatic equation and determine the pressure, but in general that is not possible.

Ordinary thermodynamics provides us with a relationship between density  $\rho$ , pressure  $p$ , and absolute temperature  $T$ , called the *equation of state*, valid for any macroscopic amount of homogeneous isotropic fluid in thermodynamic equilibrium. Its most general form is,

$$F(\rho, p, T) = 0, \quad (2.24)$$

The actual form of the equation of state for a particular substance is derived from the properties of the molecular interactions, a subject that falls outside the scope of this book.

In continuum physics where conditions can change from point to point, we shall always assume that each material particle is in thermodynamic equilibrium with its environment, such that the equation of state holds for the local values of density, pressure and temperature fields in every point of space,

$$F(\rho(\mathbf{x}), p(\mathbf{x}), T(\mathbf{x})) = 0. \quad (2.25)$$

In full generality the function  $F$  could also depend explicitly on  $\mathbf{x}$ , for example if the molecular composition depends on the position, but we shall not consider that possibility.

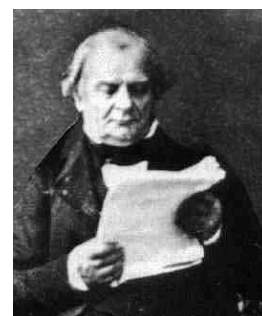
### The ideal gas law

The *ideal gas law* is the oldest and by far the most famous equation of state, normally credited to Clapeyron (1834) although there were other contributors, as nearly always in the history of physics. It is usually presented in terms of the volume  $V$  of a mass  $M$  of the gas and the number  $n = M/M_{\text{mol}}$  of moles in this volume,

$$pV = nR_{\text{mol}}T, \quad (2.26)$$

where  $R_{\text{mol}} = 8.31447 \text{ J K}^{-1} \text{ mol}^{-1}$  is the *universal molar gas constant*<sup>1</sup>.

<sup>1</sup>In physics and chemistry the universal molar gas constant is nearly always denoted  $R$ . We shall however use the more systematic notation  $R_{\text{mol}}$  and leave the single-letter symbol  $R$  free for other uses, for example the specific gas constant  $R = R_{\text{mol}}/M_{\text{mol}}$ . It occurs naturally in the local form of the ideal gas law (2.27), and simplifies the formalism. Since this notation is slightly atypical, we shall always make appropriate remarks when it is used.



**Benoit Paul Émile Clapeyron (1799–1864).** French engineer and physicist. Formulated the ideal gas law from previous work by Boyle, Mariotte, Charles, Gay-Lussac, and others. Established what is now called Clapeyron’s formula for the latent heat in the change of state of a pure substance.

	$10^3 M_{\text{mol}}$	$10^{-3} R$
H <sub>2</sub>	2.0	4.157
He	4.0	2.079
Ne	20.2	0.412
N <sub>2</sub>	28.0	0.297
O <sub>2</sub>	32.0	0.260
Ar	39.9	0.208
CO <sub>2</sub>	44.0	0.189
Air	29.0	0.287
	$\frac{\text{kg}}{\text{mol}}$	$\frac{\text{J}}{\text{K kg}}$

The molar mass,  $M_{\text{mol}}$ , and the specific gas constant,  $R = R_{\text{mol}}/M_{\text{mol}}$ , for a few gases (in SI-units).

Since  $\rho = M/V = nM_{\text{mol}}/V$  we may write the ideal gas law in the local form,

$$p = R\rho T, \quad R = \frac{R_{\text{mol}}}{M_{\text{mol}}}. \quad (2.27)$$

In the margin the specific gas constant  $R$  is tabled for a few gases. Various extensions of the ideal gas law take into account the excluded molecular volume as well as the strong molecular repulsion (see problem 2.7). The ideal gas law is not only valid for pure gases but also for a mixture of pure gases provided one uses the average molar mass (see page 4). Real air with  $M_{\text{mol}} = 28.97 \text{ g mol}^{-1}$  is quite well described by the ideal gas law at normal temperatures, although in precise calculations it may be necessary to include various corrections (see for example [Kaye and Laby 1995]).

### \* Microscopic origin of the ideal gas law

Even if the derivation of the equation of state belongs to statistical mechanics, it is nevertheless of interest to see how the ideal gas law arises in the simplest possible molecular picture of a gas at rest, already discussed on page 23. The gas consists of particles of mass  $m$  that are free except for random collisions with the walls and with each other, such that the velocity vector,  $\mathbf{v} = (v_x, v_y, v_z)$ , of a typical particle incessantly undergo thermal fluctuations around  $\mathbf{v} = \mathbf{0}$ .

In statistical mechanics one derives (under certain conditions that we shall not discuss here) the *equipartition theorem* which states that each independent kinematic variable on average carries the thermal energy  $\frac{1}{2}k_B T$  where  $k_B = 1.38065 \times 10^{-23} \text{ J K}^{-1}$  is Boltzmann's constant. Accordingly, the average kinetic energy of a molecule of mass  $m$  becomes

$$\frac{1}{2}m \langle v^2 \rangle = \frac{1}{2}m \langle v_x^2 \rangle + \frac{1}{2}m \langle v_y^2 \rangle + \frac{1}{2}m \langle v_z^2 \rangle = \frac{3}{2}k_B T, \quad (2.28)$$

and inserting this into (2.7) we obtain

$$p = \frac{1}{3}\rho \langle v^2 \rangle = \rho \frac{k_B T}{m}. \quad (2.29)$$

Multiplying numerator and denominator of the fraction on the right by Avogadro's number  $N_A$  and using that the molar mass is  $M_{\text{mol}} = N_A m$ , we obtain the ideal gas law with the molar gas constant  $R_{\text{mol}} = N_A k_B$ , as it should be.

### Case: Isothermal atmosphere

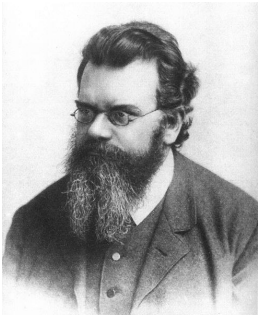
Everybody knows that the atmospheric temperature varies with height, but if we nevertheless assume that it has constant temperature,  $T(\mathbf{x}) = T_0$ , and combine the equation of hydrostatic equilibrium (2.23) with the ideal gas law (2.27), we obtain

$$\frac{dp}{dz} = -\rho g_0 = -\frac{g_0 p}{RT_0}. \quad (2.30)$$

This is an ordinary differential equation for the pressure, and using the initial condition  $p = p_0$  for  $z = 0$ , we find the solution

$$p = p_0 e^{-z/h_0}, \quad h_0 = \frac{RT_0}{g_0} = \frac{p_0}{\rho_0 g_0}. \quad (2.31)$$

In the last step we have again used the ideal gas law at  $z = 0$  to show that the expression for  $h_0$  is identical to the incompressible atmospheric scale height (2.5).



**Ludwig Boltzmann (1844–1906).** Austrian theoretical physicist. Made fundamental contributions to statistical mechanics and the understanding of the relations between the macroscopic and microscopic descriptions of nature. Believed firmly in the reality of atoms of molecules, even if most of his contemporaries did not. His recurrent depressions eventually led him to suicide.

In the isothermal atmosphere the pressure thus decreases exponentially with height on a characteristic length scale again set by  $h_0$ . The pressure at the top of Mount Everest ( $z = 8,848$  m) is now finite and predicted to be 368 hPa. As seen in figure 2.2, the isothermal approximation is better than the incompressible one and fits the data to a height of about 4 kilometers. In section 2.6 we shall obtain a third estimate, the homentropic atmosphere, lying between the two.

### Barotropic equation of state

The equation of state is in general not a simple relation between density and pressure which may be plugged into the equation of local hydrostatic equilibrium, but also involves temperature. To solve the general problem we need a further dynamic equation for the temperature, called the heat equation, expressing energy balance (see chapter 22). Only if the temperature is somehow given in advance, can we use the equation of state as it stands and solve the equilibrium equation.

Sometimes there exists a so-called *barotropic* relationship between density and pressure,

$$F(\rho(\mathbf{x}), p(\mathbf{x})) = 0, \quad (2.32)$$

which does not depend on the temperature  $T(\mathbf{x})$ . This is in fact not as far-fetched as it might seem at first. The condition of constant density  $\rho(\mathbf{x}) = \rho_0$  which we used in the preceding section to calculate the pressure in the sea is a trivial example of such a relationship in which the density is independent of both pressure and temperature. A less trivial example is obtained if the walls containing a fluid at rest are held at a fixed temperature  $T_0$ . The omnipresent heat conduction will eventually cause all of the fluid to attain this temperature,  $T(\mathbf{x}) = T_0$ , and in this state of *isothermal equilibrium* the equation of state (2.25) simplifies to,  $F(\rho(\mathbf{x}), p(\mathbf{x}), T_0) = 0$ , which is indeed a barotropic relationship. In section 2.6 we shall obtain another barotropic relation, corresponding to the opposite extreme where heat conduction can be completely disregarded, so that no heat transfer can take place.

### Effective potential

In elementary physics it is shown that the gravitational field is conservative and can be obtained from the gradient of the *gravitational potential*  $\Phi(\mathbf{x})$ ,

$$\mathbf{g}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}). \quad (2.33)$$

The function  $\Phi(\mathbf{x})$  represents the *potential energy per unit of mass*, so that a body of mass  $m$  at the point  $\mathbf{x}$  has a gravitational potential energy  $m\Phi(\mathbf{x})$ . For constant gravity in flat-earth coordinates we have  $\mathbf{g}(\mathbf{x}) = (0, 0, -g_0)$ , and the potential is simply  $\Phi = g_0z$ .

In terms of the potential, the equilibrium equation (2.22) may now be written as,

$$\nabla\Phi + \frac{\nabla p}{\rho} = 0. \quad (2.34)$$

If the density is constant,  $\rho(\mathbf{x}) = \rho_0$ , it follows that

$$\boxed{\Phi^* = \Phi + \frac{p}{\rho_0}} \quad (2.35)$$

is also a constant, independent of  $\mathbf{x}$ . In flat-earth gravity, where  $\Phi = g_0z$ , the constancy of  $\Phi^* = g_0z + p/\rho_0$  leads directly to the linear pressure field (2.3) in the incompressible sea.

More generally, it is always possible to integrate the hydrostatic equation for any barotropic fluid with  $\rho = \rho(p)$ . To do this, we define the so-called *pressure potential*,

$$w(p) = \int \frac{dp}{\rho(p)}, \quad (2.36)$$

which becomes  $w = p/\rho_0$  for constant density. It then follows from the chain rule for differentiation that

$$\nabla w(p) = \frac{dw}{dp} \nabla p = \frac{1}{\rho} \nabla p,$$

so that hydrostatic equilibrium (2.34) may be written as,  $\nabla \Phi^* = \mathbf{0}$ , where

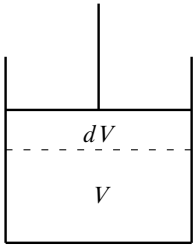
$$\Phi^* = \Phi + w(p). \quad (2.37)$$

The requirement  $\Phi^*(\mathbf{x}) = \text{const}$  is completely equivalent to the equation of hydrostatic equilibrium for barotropic fluids. There is no agreement in the literature about a name for  $\Phi^*$ , but we have chosen to call it the *effective potential*, because it combines the true gravitational potential with the pressure potential, and because the effective force density (2.21) in general is given by  $\mathbf{f}^* = \rho \mathbf{g}^*$  with  $\mathbf{g}^* = -\nabla \Phi^*$ .

**Example 2.4 [Isothermal atmosphere]:** Under isothermal conditions, the pressure potential of an ideal gas is calculated by means of the ideal gas law (2.27),

$$w = \int RT_0 \frac{dp}{p} = RT_0 \log p. \quad (2.38)$$

In flat-earth gravity, the constancy of  $\Phi^* = g_0 z + RT_0 \log p$  immediately leads to the exponentially decreasing pressure (2.31) in the isothermal atmosphere.



The archetypal thought-experiment in thermodynamics: A cylindrical chamber with a movable piston.

## 2.5 Bulk modulus

The archetypal thermodynamics experiment is carried out on a fixed amount  $M = \rho V$  of a fluid placed in a cylindrical container with a moveable piston. When you slightly increase the force on the piston, the volume of the chamber decreases, so that  $dV < 0$ . The pressure in the fluid must necessarily always increase,  $dp > 0$ , for if the pressure instead always diminished, an arbitrarily small extra force would send the piston right down to the bottom of the chamber leaving zero volume for the gas. Since a larger volume diminishes proportionally more for a given pressure increase, we define the *bulk modulus* as the pressure increase  $dp$  per *fractional decrease* in volume,  $-dV/V$ , or

$$K = \frac{dp}{-dV/V} = \frac{dp}{d\rho/\rho} = \rho \frac{dp}{d\rho}. \quad (2.39)$$

In the second step we have used the constancy of the mass  $M = \rho V$  of the fluid in the volume to derive that  $dM = \rho dV + V d\rho = 0$ , from which we get  $-dV/V = d\rho/\rho$ .

The above definition makes immediate sense for a barotropic fluid, where  $p = p(\rho)$  is a function of density. For general fluid states it is necessary to specify the conditions under which the bulk modulus is defined, for example whether the temperature is held constant (isothermal) or whether there is no heat transfer (adiabatic). Thus, the equation of state (2.27) for an ideal gas implies that the isothermal bulk modulus is

$$K_T = \left( \rho \frac{\partial p}{\partial \rho} \right)_T = p, \quad (2.40)$$

where the index — as commonly done in thermodynamics — indicates that  $T$  is held constant.

Liquid	$T$ [°C]	$K$ [GPa]
Mercury	20	24.93
Glycerol	0	3.94
Water	20	2.18
Benzene	25	1.04
Ethanol	20	0.89
Methanol	20	0.82
Hexane	20	0.60

Isothermal bulk modulus for common liquids [Lide 1996].

The definition of the bulk modulus (and the above equation) shows that it is measured in the same units as pressure, for example pascals, bars or atmospheres. The bulk modulus is actually a measure of *incompressibility*, because the larger it is, the greater is the pressure increase that is needed to obtain a given fractional increase in density. The inverse bulk modulus  $\beta = 1/K$  may be taken as a measure of *compressibility*.

**Example 2.5 [Water compression]:** For sea water the bulk modulus is  $K \approx 23.2$  kbar = 2.32 GPa [Kaye and Laby 1995]. As long as the pressure change is much smaller than the bulk modulus,  $dp \ll K$ , we may calculate the relative change in density from (2.39) to be  $d\rho/\rho \approx dp/K$ . In the deepest abyss of the sea the pressure is about 1100 bar, implying that the relative density change is  $d\rho/\rho \approx 1/20 \approx 5\%$ .

### Case: Polytropic water

The bulk modulus of seawater is nearly constant, although it does increase slowly with pressure, nearly doubling between 1 and 3000 bar [Kaye and Laby 1995]. Disregarding any temperature dependence, we model the data by a linear function,

$$K = K_0 + \gamma(p - p_0), \quad (2.41)$$

with  $K_0 = 23.2$  kbar,  $p_0 = 1$  bar and  $\gamma = 6$ .

Inserting this expression into the definition (2.39) it becomes,

$$\frac{dp}{K_0 + \gamma(p - p_0)} = \frac{d\rho}{\rho}.$$

Integrating both sides of this equation, and fixing the integration constant by setting the density to  $\rho = \rho_0$  at  $p = p_0$ , we finally arrive at the following barotropic equation of state for water, valid well beyond the depth of the deepest abyss,

$$p = p_0 + \frac{K_0}{\gamma} \left[ \left( \frac{\rho}{\rho_0} \right)^\gamma - 1 \right]. \quad (2.42)$$

In the next section we shall see that an ideal gas under certain conditions obeys a similar so-called *polytropic* equation of state.

Differentiating with respect to  $z$  we get,

$$\frac{dp}{dz} = \frac{K_0}{\rho_0^\gamma} \rho^{\gamma-1} \frac{d\rho}{dz}.$$

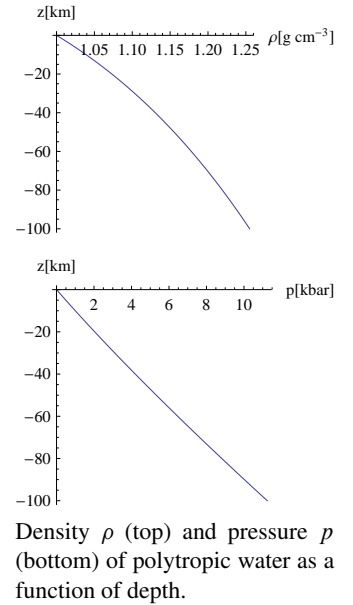
In flat-earth equilibrium we have  $dp/dz = -\rho g_0$ , and a dividing both sides with  $\rho$ , the differential equation may immediately be integrated to yield,

$$-g_0 z + \frac{K_0}{(\gamma-1)\rho_0} = \frac{K_0}{(\gamma-1)\rho_0^\gamma} \rho^{\gamma-1},$$

where the integration constant on the left hand side has been determined by demanding  $\rho = \rho_0$  at  $z = 0$ . Solving for the density we get

$$\rho(z) = \rho_0 \left( 1 - \frac{z}{h_1} \right)^{1/(\gamma-1)} \quad h_1 = \frac{K_0}{(\gamma-1)\rho_0 g_0}, \quad (2.43)$$

With the given parameters the length scale becomes  $h_1 = 47.3$  km. The pressure as a function of depth  $z$  is obtained by inserting this into the equation of state (2.42). Since the resulting exponent,  $\gamma/(\gamma-1) = 1.2$ , is close to unity the pressure increases nearly linearly with depth, as seen in the margin figures. At the deepest point of the sea,  $z = -10.924$  km, the pressure is  $p = 1.094$  kbar and the density is  $\rho = 1.042$  g cm<sup>-3</sup> which is 4.2% larger than the surface density. The pressure at the deepest point is increased by about 23 bar above the pressure that would have been obtained if the density had been constant.

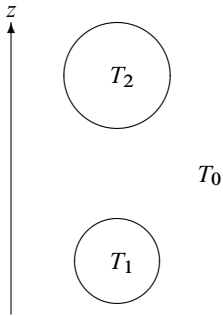


## 2.6 Homentropic atmosphere

Anyone who has flown in a modern passenger jet and listened to the pilot can testify that the atmospheric temperature falls with height. So the atmosphere is *not* in isothermal equilibrium, and this is perhaps not so surprising, since the “container walls” of the atmosphere, the ground and outer space, have different temperatures. There must be a heat flow through the atmosphere between the ground and outer space, maintained by the inflow of solar radiation and the outflow of geothermal energy. But air is a bad conductor of heat, so although heat conduction does play a role, it is not directly the cause of the temperature drop in the atmosphere.

Of much greater importance are the indirect effects of solar heating, the *convection* which creates air currents, winds, and local turbulence, continually mixing different layers of the atmosphere. The lower part of the atmosphere, the *troposphere*, is quite unruly and vertical mixing happens at time scales that are much shorter than those required to reach thermal equilibrium. There is in fact no true hydrostatic equilibrium state for the real atmosphere. Even if we disregard large-scale winds and weather systems, horizontal and vertical mixing always takes place at small scales, and a realistic model of the atmosphere must take this into account.

### Vertical mixing



Swapping air masses from different heights. If the air has temperature  $T_0$  before the swap, the compressed air would be warmer  $T_1 > T_0$  and the expanded colder  $T_2 < T_0$ .

Let us imagine that we take a small amount of air and exchange it with another amount with the same mass, taken from a different height with a different volume and pressure. In order to fill out the correct volume, one air mass would have to be compressed and the other expanded. If this is done quickly, there will be no time for heat exchange with the surrounding air, and one air mass will consequently be heated up by compression and the other cooled down by expansion. If the atmosphere initially were in isothermal equilibrium, the temperature of the swapped air masses would not be the same as the temperature of the surrounding air in the new positions, and the atmosphere would be brought out of equilibrium.

If, however, the surrounding air initially had a temperature distribution, such that the swapped air masses after the expansion and compression would arrive at precisely the correct temperatures of their new surroundings, a kind of “equilibrium” could be established, in which the omnipresent vertical mixing had essentially no effect. Intuitively, it is reasonable to expect that the end result of fast vertical mixing and slow heat conduction might be precisely such a state. It should, however, not be forgotten that this state is not a true equilibrium state but rather a dynamically balanced state depending on the incessant small-scale motion in the atmosphere.

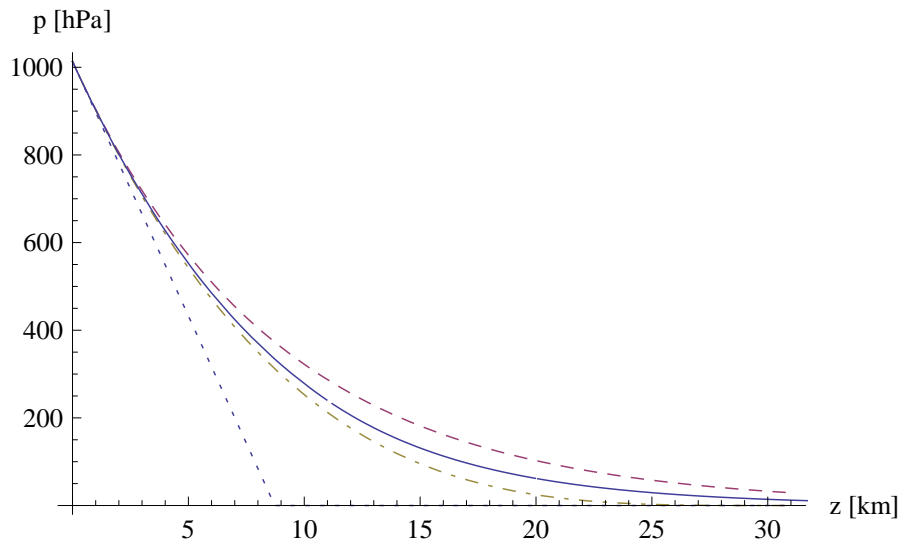
### Isentropic processes in ideal gases

A process that takes place without exchange of heat between a system and its environment is said to be *adiabatic*. If furthermore the process is *reversible*, it will conserve the entropy and is called *isentropic*. In that case it follows from the thermodynamics of ideal gases (see appendix E) that an isentropic compression or expansion of a fixed amount  $M$  of an ideal gas which changes the volume from  $V_0$  to  $V$  and the pressure from  $p_0$  to  $p$ , will obey the rule,

$$pV^\gamma = p_0V_0^\gamma, \quad (2.44)$$

Here  $\gamma$  is the so-called *adiabatic index* which for a gas like air with diatomic molecules is approximately  $\gamma \approx 7/5 = 1.4$ . Monatomic gases like Helium have  $\gamma = 5/3$  and multiatomic gases like water vapor have  $\gamma = 4/3$ .





**Figure 2.2.** The pressure as function of altitude (for  $T_0 = 25^\circ\text{C}$ ) in three different atmospheric models analyzed in this chapter: constant density (dotted), isothermal (dashed) and homentropic (dot-dashed). The solid curve is the Standard Atmosphere (1976) model (see page 37).

From the constancy of  $pV^\gamma$ , it follows that

$$0 = d \log(pV^\gamma) = d \log p + \gamma d \log V = \frac{dp}{p} + \gamma \frac{dV}{V},$$

and this allows us to calculate the *isentropic bulk modulus* (2.39) (indicated by  $S$ ),

$$K_S = \left( -V \frac{dp}{dV} \right)_S = \gamma p. \quad (2.45)$$

It is larger than the isothermal bulk modulus (2.40) by a factor of  $\gamma > 1$ , because adiabatic compression also increases the temperature of the gas which further increases the pressure.

### Homentropic equation of state

Expressed in terms of the density  $\rho = M/V$ , an isentropic process which locally changes density and pressure from  $\rho_0$  to  $\rho$  and  $p_0$  to  $p$  must obey the local rule

$$p\rho^{-\gamma} = p_0\rho_0^{-\gamma}. \quad (2.46)$$

In the dynamically balanced state of the atmosphere described above, this rule would not only be locally valid, but also apply to the swapping of two air masses from different heights, for example from the ground  $z = 0$  to any value of  $z$ . The lower part of the atmosphere, the troposphere, is at least approximatively in such a *homentropic state* in which this law is valid everywhere, with the right hand side being a constant expressed in terms of the ground values of pressure and density.

The pressure potential (2.36) is easily obtained using that  $p = C\rho^\gamma$  with  $C = p_0\rho_0^{-\gamma}$ ,

$$w = \int \frac{dp}{\rho} = \int C\gamma\rho^{\gamma-2} d\rho = C\frac{\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}. \quad (2.47)$$

Since the ideal gas law (2.27) tells us that  $p/\rho = RT$ , where  $R = R_{\text{mol}}/M_{\text{mol}}$  is the specific gas constant, this may also be written,

$$w = c_p T, \quad c_p = \frac{\gamma}{\gamma-1} R. \quad (2.48)$$

The constant  $c_p$  is in fact the *specific heat at constant pressure* of the gas (see appendix E for more details). For dry air with  $R \approx 0.287 \text{ J K}^{-1} \text{ g}^{-1}$  and  $\gamma = 7/5$ , its value is  $c_p \approx 1 \text{ J K}^{-1} \text{ g}^{-1}$  which is about 4 times smaller than for liquid water.

### Homentropic solution

It was shown before on page 32 that hydrostatic equilibrium implies that  $\Phi^* = \Phi + w(p) = g_0 z + c_p T$  is a constant, independent of  $x$ . Defining the value of this constant to be  $\Phi^* = c_p T_0$ , we find the temperature

$$T(z) = T_0 - \frac{g_0}{c_p} z. \quad (2.49)$$

where  $T_0$  is the sea-level temperature. It is convenient to rewrite this expression as

$$T = T_0 \left(1 - \frac{z}{h_2}\right), \quad h_2 = \frac{c_p T_0}{g_0} = \frac{\gamma}{\gamma-1} h_0, \quad (2.50)$$

where  $h_0$  is the scale height (2.31) for the isothermal atmosphere. For  $\gamma = 7/5$  and  $T_0 = 25^\circ\text{C}$ , we find the isentropic scale height  $h_2 \approx 31 \text{ km}$ .

The ideal gas law combined with the homentropic law (2.46) implies that  $T\rho^{1-\gamma}$  and  $p^{1-\gamma}T^\gamma$  are also constant, so that the density and pressure become,

$$\rho = \rho_0 \left(1 - \frac{z}{h_2}\right)^{1/(\gamma-1)}, \quad p = p_0 \left(1 - \frac{z}{h_2}\right)^{\gamma/(\gamma-1)}, \quad (2.51)$$

where  $\rho_0$  and  $p_0$  are the density and pressure at sea level. Temperature, density and pressure all vanish at  $z = h_2$ , which is of course unphysical.

In figure 2.2 the pressure in the various atmospheric models has been plotted together with the 1976 Standard Atmosphere model (see below). Even if the homentropic model gives the best fit, it fails at altitudes above 10 kilometers. The real atmosphere is more complicated than any of the models considered here.

### The atmospheric temperature lapse rate

The negative vertical temperature gradient  $-dT/dz = g_0/c_p$  is called the *atmospheric temperature lapse rate*. Its value is  $9.8 \text{ K km}^{-1}$  for dry air. At the top of Mount Everest ( $z = 8.848 \text{ km}$ ) the temperature is for  $T_0 = 25^\circ\text{C}$  predicted to be  $-61^\circ\text{C}$ , the density  $0.50 \text{ kg m}^{-3}$ , and the pressure  $306 \text{ hPa}$ .

Water vapor is always present in the atmosphere and will condense to clouds in rising currents of air. The latent heat released during condensation heats up the air, so that the temperature lapse rate becomes smaller than in dry air, perhaps more like  $6 - 7 \text{ K km}^{-1}$ , which corresponds to choosing an effective adiabatic index of  $\gamma \approx 1.24$ . Using this value

with  $T_0 = 25^\circ\text{C}$ , we now predict the pressure at the top of Mount Everest to be 329 hPa, and the temperature  $-33^\circ\text{C}$ , which is not so far from the measured average summit temperature of about  $-25^\circ\text{C}$ .

Clouds may eventually precipitate out as rain, and when the now drier air afterwards descends again, for example on the lee side of a mountain, the air will heat up at a higher rate than it cooled during its ascent on the windward side and become quite hot, a phenomenon called *föhn* in the Alps.

### Atmospheric stability

In the real atmosphere the temperature lapse rate will generally not be constant throughout the troposphere. Sunlight will often heat bubbles of air relative to the surrounding atmosphere because of local topographic variations. Since the pressure must be the same in the heated bubble and the atmosphere, the ideal gas law tells us that a higher temperature is always accompanied by lower mass density. The bubble will thus weigh less than the volume of air it displaces, and must according to Archimedes Principle begin to rise (see the following chapter). As it rises it will always be in balance with the pressure of the environment, but the temperature will in general not have time to equilibrate because of the low heat conductivity of air.

If the temperature lapse rate in the bubble is larger than outside, the bubble temperature will soon drop down to the temperature of its environment, and the bubble will stop rising. If, on the other hand, the bubble temperature lapse rate is smaller than outside, the bubble temperature will increase relative to the environment, and the bubble will continue to rise faster and faster. This will, for example, happen if a bubble of warm humid air rises in a relatively dry atmosphere. To begin with, it rises slowly because of the small initial temperature excess in the bubble, but when the humidity begins to condense, the released heat will make the bubble's lapse rate even smaller, making it rise even faster. Strong vertical currents may be generated in this way, for example in thunderstorms.

### \* The Standard Atmosphere (1976)

A number of standard models for the atmosphere have been created in the last half of the 20th century to aid pilots and flight engineers to know in advance the temperature, pressure and density of the air at the increasing altitudes reached by their aircraft. The US Standard Atmosphere (1976)—which is essentially the same as the International Standard Atmosphere (1976)—was created to provide a numerically trustworthy model valid up to about 100 kilometers. With minor corrections this has become the accepted atmospheric model for sub-orbital altitudes.

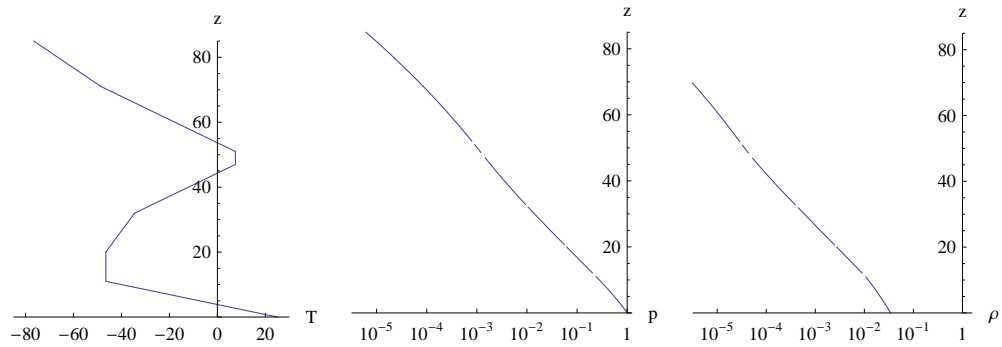
The input to the model is an empirical vertical temperature distribution  $T(z)$ . Using the ideal gas law (2.27), the mass density may be eliminated from the hydrostatic equation (2.23), to arrive at the differential equation

$$\frac{dp}{dz} = -\frac{g_0 p}{RT}, \quad (2.52)$$

Since  $T$  is assumed to be known, this equation can be immediately integrated to yield

$$p(z) = p_0 \exp\left(-\lambda_0 \int_0^z \frac{dz'}{T(z')}\right), \quad \lambda_0 = \frac{g_0}{R}, \quad (2.53)$$

where  $p_0$  is the pressure at  $z = 0$ . The exponential coefficient  $\lambda_0 = 34.16 \text{ K km}^{-1}$  is called the *atmospheric constant*.



**Figure 2.3.** Plots of temperature, pressure and density in the Standard Atmosphere (1976) for sea-level temperature  $T_0 = 25^\circ\text{C}$ . The vertical height is measured in kilometers, the temperature in Celsius, the pressure in bars, and the density in kilograms per cubic meter. Notice the logarithmic scale of pressure and density.

layer	$z$	$\frac{dT}{dz}$
mesopause	85	
high mesosphere	71	-2.0
low mesosphere	51	-2.8
stratopause	47	0.0
high stratosphere	32	+2.8
low stratosphere	20	+1.0
tropopause	11	0.0
troposphere	0	-6.5

The temperature gradient in the layered model of the US Standard Atmosphere (1976). The height  $z$  is measured in km and the gradient  $dT/dz$  in  $\text{K m}^{-1}$ .

In the margin table is shown the temperature gradient for the Standard Atmosphere (1976). It is given as a piecewise constant function, implying that the temperature itself is piecewise linear. This way of presenting the model is convenient for analytic or numeric calculations. The result for the pressure is shown as the fully drawn curve in figure 2.2. Plots of temperature, pressure and density are shown in figure 2.3 for  $T_0 = 25^\circ\text{C}$ .

## \* 2.7 The Sun's convective envelope

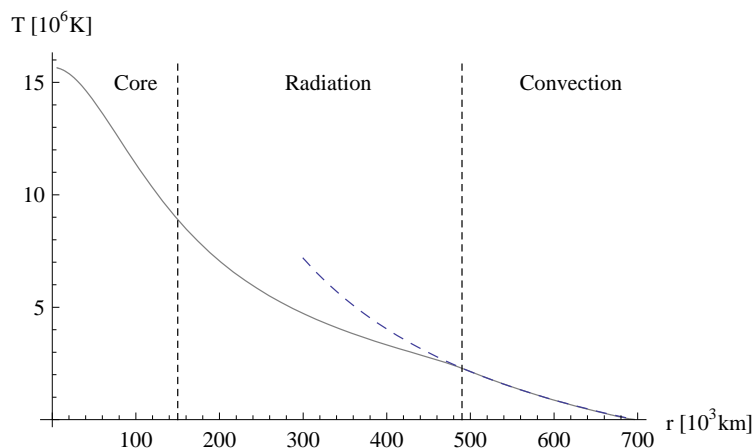
Stars like the Sun are self-gravitating, gaseous and almost perfectly spherical bodies that generate heat by thermonuclear processes in a fairly small region close to the center. The heat is transported to the surface by radiation, conduction and convection and eventually released into space as radiation. Like planets, stars also have a fairly complex structure with several layers differing in chemical composition and other physical properties.

Our Sun consists—like the rest of the universe—of roughly 75% hydrogen and 25% helium (by mass), plus small amounts of other elements (usually called “metals”). The radius of its photosphere is  $a = 700,000$  km, its mass  $M_0 = 2 \times 10^{30}$  kg, and its total luminosity (power)  $3.8 \times 10^{26}$  W. The gases making up the Sun are almost completely ionized, so that they form a *plasma* consisting of positively charged hydrogen ions ( $\text{H}^+$ ), doubly charged helium ions ( $\text{He}^{++}$ ), and negatively charged electrons ( $\text{e}^-$ ). The mean molar mass of the plasma is  $M_{\text{mol}} = 0.59$  g mol $^{-1}$ . It is smaller than unity because the nearly massless electrons make up about 52% of all the particles in the plasma (see problem 2.11).

In this section we shall ignore heat production and only model the convective outer layer as a homentropic ideal gas, in other words modeling a Sun that does not shine! A much more enlightening and comprehensive analysis of the structure of stellar interiors can, for example, be enjoyed in [Hansen and Kawaler 1994]. The generally accepted Standard Model for the Sun is found in [CDDA&96].

### The convective envelope

In the preceding section we argued for the case of Earth's troposphere that — provided the time scale for vertical convective mixing is fast compared to heat conduction — a *homentropic* dynamical equilibrium state will be established in the ideal gas in which  $p\rho^{-\gamma}$  is constant. We shall now apply this argument to the convective envelope of the Sun which has a thickness of about 200,000 kilometers. In the deeper layers heat transport is dominated by radiative or



**Figure 2.4.** The temperature distribution in the Sun as a function of the distance from the center. The vertical lines are boundaries between the three major layers of the Sun. The fully drawn curve is taken from the Standard Sun model [CDDA&96] and [3]. The dashed curve is the approximative homentropic solution (2.55) for the convective envelope.

material conduction, not by convection.

The density in the convective zone is so low that the zone contains only about 3% of the Sun's mass, so that we may approximate the gravitational potential in this region by that of a point particle with the Sun's mass,

$$\Phi = -G \frac{M_0}{r} \quad (2.54)$$

where where  $G = 6.67428(67) \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$  is the universal gravitational constant [1]. The pressure potential for a homentropic gas is as before given by eq. (2.48). Since the solar plasma consists of monatomic "molecules", the adiabatic index is  $\gamma = 5/3$ , leading to a specific heat capacity at constant pressure  $c_p = 35.1 \text{ J K}^{-1} \text{ g}^{-1}$ . This is 35 times that of the Earth's atmosphere, mainly because of the low average molar mass of the solar plasma.

Hydrostatic equilibrium (2.37) requires the effective potential  $\Phi^* = \Phi + c_p T$  to be constant. Fixing this constant by means of the surface temperature  $T_0$ , we obtain the expression

$$T(r) = T_0 + T_1 \left( \frac{a}{r} - 1 \right), \quad \text{where } T_1 = \frac{GM_0}{c_p a}. \quad (2.55)$$

With the Sun's parameters we find  $T_1 = 5.4 \times 10^6 \text{ K}$ , a value that sets the scale for the Sun's central temperature. The precise surface of the Sun is not too well defined, but normally chosen to be that spherical surface at radius  $r = a$  where the plasma turns into neutral gas and becomes transparent. The transition to complete transparency actually takes place over a few hundred kilometers. The surface temperature is about 6000 K, which is about 1000 times smaller than  $T_1$  and can mostly be ignored. At the surface of the Sun the temperature lapse rate becomes  $-dT/dr = T_1/a = 7.8 \text{ K km}^{-1}$  which accidentally is nearly the same as in the Earth's troposphere.

In figure 2.4 the solution is plotted together with the temperature distribution from the Standard Sun model. The agreement is nearly perfect in the convective zone. The density and pressure can as in the preceding section be determined from the adiabatic law. To do that one needs reliable values for the solar surface density and pressure, in which the mean molar mass variation due to incomplete ionization near the surface is taken into account. We shall not go further into this question here but refer the reader to the literature [CDDA&96, Hansen and Kawaler 1994].

## Problems

**2.1** The normal human systolic blood pressure is about  $16,000 \text{ Pa} = 0.16 \text{ Bar}$  above atmospheric pressure. Invent a manometer in the form of a mercury-filled U-tube and determine the tube length necessary to give it a measurement range of  $\pm 50\%$  around normal.

**2.2** Consider a canal with a dock gate which is  $12 \text{ m}$  wide and has water depth  $9 \text{ m}$  on one side and  $6 \text{ m}$  on the other side.

- (a) Calculate the pressures in the water on both sides of the gate at a height  $z$  over the bottom of the canal.
- (b) Calculate the total force on the gate.
- (c) Calculate the total moment of force around the bottom of the gate.
- (d) Calculate the height over the bottom at which the total force acts.

**2.3** An underwater lamp is covered by a hemispherical glass with a diameter of  $30 \text{ cm}$  and is placed with its center at a depth of  $3 \text{ m}$  on the side of the pool. Calculate the total horizontal force from the water on the lamp, when there is air at normal pressure inside.

**2.4** Using a manometer, the pressure in an open container filled with liquid is found to be  $1.6 \text{ atm}$  at a height of  $6 \text{ m}$  over the bottom, and  $2.8 \text{ atm}$  at a height of  $3 \text{ m}$ . Determine the density of the liquid and the height of the liquid surface.

**2.5** An open jar contains two non-mixable liquids with densities  $\rho_1 > \rho_2$ . The heavy layer has thickness  $h_1$  and the light layer on top of it has thickness  $h_2$ . (a) An open glass tube is lowered vertically into the liquids towards the bottom of the jar. Describe how high the liquids rise in the tube (disregarding capillary effects). (b) The open tube is already placed in the container with its opening close to the bottom when the heavy fluid is poured in, followed by the light. How high will the heavy fluid rise in the tube?

**2.6** Show that a mixture of ideal gases (see page 4) also obeys the equation of state (2.27).

**2.7** The equation of state due to van der Waals is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \quad (2.56)$$

where  $a$  and  $b$  are constants. It describes gases and their condensation into liquids. (a) Calculate the isothermal bulk modulus. (b) Under which conditions can it become negative, and what does that mean?

**2.8** The equation of state for water is (for pressures up to  $100\,000 \text{ bar}$ ) given by

$$\frac{p + B}{p_0 + B} = \left(\frac{\rho}{\rho_0}\right)^n \quad (2.57)$$

with  $B = 3000 \text{ atm}$ ,  $n = 7$ ,  $p_0 = 1 \text{ atm}$  and  $\rho_0 = 1 \text{ g cm}^{-3}$ . (a) Calculate the bulk modulus  $K$  for water. (b) Calculate the density and pressure distribution in the sea. (c) What is the pressure and the relative compression of the water at the deepest point in the sea ( $z = -10.924 \text{ km}$ )?

**2.9** Calculate the pressure and density in the flat-earth sea, assuming constant bulk modulus. Show that both quantities are singular at a certain depth and calculate this depth.

**2.10** Show that the constancy of the temperature in the homentropic atmosphere can easily be derived using the pressure potential.

**2.11** Calculate the mean molar mass of a plasma consisting of  $75\%$  hydrogen and  $25\%$  helium (by mass).