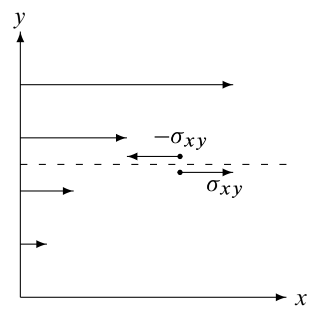
14 Viscosity

All fluids are viscous, except for a component of liquid helium close to absolute zero in temperature. Air, water and oil all put up resistance to flow, and a part of the money we spend on transport by plane, ship or car goes to overcome fluid friction, and all the energy in the fuel eventually contributes a small amount to heating the atmosphere and the sea.

It is primarily the interplay between the mechanical inertia of a moving fluid and its viscosity which gives rise to all the interesting and beautiful phenomena, the whirling and the swirling that we are so familiar with. If a volume of fluid is set into motion, inertia would dictate that it continue in its original motion, were it not checked by the action of internal shear stresses. Viscosity acts as a brake on the free flow of a fluid and will eventually make it come to rest in mechanical equilibrium, unless external driving forces continually supply energy to keep it moving. In an Aristotelian sense the "natural" state of a fluid is thus at rest with pressure being the only stress component. Disturbing a fluid at rest slightly, setting it into motion with spatially varying velocity field, will to first order of approximation generate stresses that depend linearly on the spatial derivatives of the velocity field. Fluids with a linear relationship between stress and velocity gradients are said to be *Newtonian*, and the coefficients in this relationship are material constants that characterize the strength of viscosity.

In this chapter the formalism for Newtonian viscosity will be set up, culminating in the formulation of the Navier-Stokes equation for incompressible fluids (compressible fluids will be treated in chapter 17). Superficially simple, the Navier-Stokes equation is a nonlinear differential equation for the velocity field which nevertheless continues to be a formidable challenge to engineers, physicists and mathematicians. It is a central theme for the remainder of this book.



Shear viscosity in laminar (layered) flow. The fluid above the dashed line moves slightly faster than below and exerts a positive shear stress σ_{xy} on the fluid below. By Newton's third law the fluid below will exert an opposite stress $-\sigma_{xy}$ on the fluid above.

14.1 Shear viscosity

Consider a fluid flowing steadily along the x-direction with a velocity field $v_x(y)$ which is independent of x but may vary with y. Such a field could, for example, be created by enclosing a fluid between moving plates, and is an elementary example of *laminar* or layered flow. If the velocity field has no y-dependence there should not be any internal stresses, because the fluid is then in uniform motion along the x-axis. If, on the other hand, the velocity grows with y, so that its gradient is positive $dv_x(y)/dy > 0$, we expect that the fluid immediately *above* a plane $y =$ const will drag along the fluid immediately *below* because of fluid friction and thus exert a *positive* shear stress, $\sigma_{xy}(y) > 0$, on this plane.

Table 14.1. Table of density and dynamic and kinematic viscosity for common substances (at the indicated temperature and at atmospheric pressure). Some of the values are only estimates. Note that air has greater kinematic viscosity than water, and hydrogen greater than olive oil. Glass is usually viewed as a solid, but there are claims (not very well substantiated [Zan98]) that it flows very slowly like a liquid over long periods of time even at normal temperatures.

	T [°C]	ρ [kg m^{-3}]	η [Pa s]	ν [m^{2}s^{-1}]
Hydrogen	20	0.084	8.80×10^{-6}	1.05×10^{-4}
Air	20	1.18	1.82×10^{-5}	1.54×10^{-5}
Water	20	1.00×10^{3}	1.00×10^{-3}	1.00×10^{-6}
Ethanol	25	0.79×10^{3}	1.08×10^{-3}	1.37×10^{-6}
Mercury	25	13.5×10^{3}	1.53×10^{-3}	1.13×10^{-7}
Whole blood	37	1.06×10^{3}	2.7×10^{-3}	2.5×10^{-6}
Olive oil	25	0.9×10^{3}	6.7×10^{-2}	7.4×10^{-5}
Castor oil	25	0.95×10^{3}	0.7	7.4×10^{-4}
Glycerol	20	1.26×10^{3}	1.41	1.12×10^{-3}
Honey(est)	25	1.4×10^{3}	14	1×10^{-2}
Pitch	20	1.1×10^{3}	2.3×10^{8}	2×10^{8}
Glass (est)	20	2.5×10^{3}	$10^{18} - 10^{21}$	$10^{15} - 10^{18}$

The University of Queensland pitch drop experiment, started in 1927. Until now eight drops have fallen, the last on november 28, 2000, although none have actually been seen to fall. The viscosity is determined to be about 2×10^8 Pa s [EDP84]. Image courtesy Wikimedia Commons.

It also seems reasonable to expect that a larger velocity gradient will evoke stronger stress. In *Newton's law of viscosity* the shear stress is simply made proportional to the gradient[1],

$$\sigma_{xy}(y) = \eta \frac{dv_x(y)}{dy}. \tag{14.1}$$

The constant of proportionality, η, is called the coefficient of *shear viscosity*, the *dynamic viscosity*, or simply *the* viscosity. It is a measure of how strongly the moving layers of fluid are coupled by friction, and a material constant of the same nature as the shear modulus for elastic materials. We shall see later (page 300) that in compressible fluids there is also a bulk coefficient of viscosity corresponding to the elastic bulk modulus, but that turns out to be rather unimportant in ordinary applications.

The viscosities of naturally occurring fluids range over many orders of magnitude (see table 14.1 and the margin figure). Since dv_x/dy has dimension of inverse time, the unit for viscosity η is Pa s (pascal seconds). Although this unit is sometimes called Poiseuille, there is in fact no special name for it in the standard (SI) system of units[2].

Molecular origin of viscosity in gases

In gases where molecules are far apart, internal stresses are caused by the incessant molecular bombardment of a boundary surface, transferring momentum in both directions across it. In liquids where molecules are in closer contact, internal stress is caused partly by molecular motion as in gases, and partly by intermolecular forces. The resultant stress in a liquid is a quite complicated combination of the two effects, and we shall for this reason limit the following discussion to the molecular origin of shear stress in gases.

Gas molecules move nearly randomly in all directions at speeds much higher than the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, representing the average non-random component of the molecular motion. In steady laminar planar flow with velocity $v_x(y)$ and positive velocity gradient $dv_x(y)/dy$,

[1]In this book we use the letter η rather than μ for viscosity to avoid a conflict with the shear elastic modulus.
[2]In the older cgs-system it used to be called poise $= 0.1$ Pa s.

a molecule of mass m crossing a surface element dS_y from above will carry an average momentum in the x-direction which is a little larger than $mv_x(y)$. Similarly, a molecule crossing dS_y from below will carry a little less than $mv_x(y)$. Since the same number of molecules pass from above and below, the result will be a net transfer of momentum from the fluid above to the fluid below.

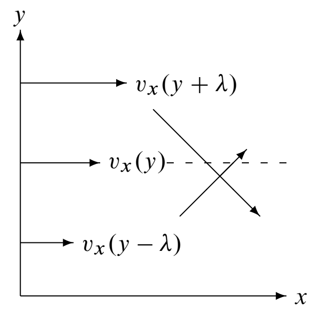
To make a quantitative estimate, let the typical distance between molecular collisions in the gas be λ and the typical time between collisions τ. Disregarding all factors of order unity, a layer of thickness λ above an area element dS_y carries an excess of momentum in the x-direction,

$$d\mathcal{P}_x \approx (v_x(y+\lambda) - v_x(y))\rho\lambda dS_y \approx \rho\lambda^2 \frac{dv_x(y)}{dy} dS_y.$$

The shear stress may be estimated from the transfer of this momentum per unit of time and area, $\sigma_{xy} \approx d\mathcal{P}_x/\tau dS_y$, and indeed takes the form of Newton's law of viscosity (14.1) with a rough estimate of the shear viscosity,

$$\eta \approx \rho\frac{\lambda^2}{\tau} \approx \rho\lambda v_{\rm mol}. \tag{14.2}$$

In the last expression we used that $\lambda/\tau \approx v_{\rm mol}$ where $v_{\rm mol} = \sqrt{3p/\rho} = \sqrt{3RT}$ is the root-mean-square molecular velocity. Over the years the estimate has been refined by means of the kinetic theory of gases, leading to about half the above value [Loeb 1961]. In practice one uses the resulting expression to determine the rather ill-defined molecular diameter from the measured viscosity (see the margin table).



Layers of gas moving with different velocities give rise to shear forces because they exchange molecules with different average velocities.

	$10^9\lambda$	$10^{12}d$
H_2	122	271
He	193	216
Ne	137	256
N_2	65	371
O_2	71	356
Ar	69	360
CO_2	44	454
Air	67	368
	m	m

Mean free path and effective molecular diameter, determined from measured viscosities [2] at 20 Celsius and 1 atmosphere (in SI-units).

Temperature dependence of viscosity

The viscosity of any material depends on temperature. Common experience from kitchen and industry tells us that most liquids become "thinner" when heated, indicating that the viscosity falls with temperature. Gases on the other hand become more viscous at higher temperatures, simply because the molecules move faster and thus transport more momentum across a surface per unit of time.

For an ideal gas, it follows from eq. (1.11) on page 8 that the expression $\lambda\rho$ is a combinations of constants, so that the viscosity becomes, $\eta \sim v_{\rm mol} \sim \sqrt{T}$. Thus, if the viscosity is η_0 at temperature T_0, it may be estimated as

$$\eta = \eta_0\sqrt{\frac{T}{T_0}}, \tag{14.3}$$

at temperature T. Notice that the viscosity is independent of the pressure. Empirically, the viscosity grows slightly faster with temperature because of molecular attraction.

Kinematic viscosity

The viscosity estimate (14.2) seems to point to another measure of viscosity, called the *kinematic viscosity*[3],

$$\nu = \frac{\eta}{\rho}. \tag{14.4}$$

Since the estimate, $\nu \approx \lambda^2/2\tau$, does not depend on the unit of mass, this parameter is measured in purely kinematic units[4] of $m^2\ s^{-1}$ (see table 14.1). In fluids with constant density, it varies with temperature in the same way as the dynamic viscosity η. In an ideal gas we have $\rho \propto p/T$, so that the kinematic viscosity will depend on both temperature and pressure, $\nu \propto T^{3/2}/p$. In isentropic gases it always decreases with temperature (problem 14.1).

[3]The conflicting use of ν for both the kinematic viscosity and Poisson's ratio is pervasive in the literature.
[4]In the older cgs-system, the corresponding unit was called stokes = $cm^2\ s^{-1} = 10^{-4}\ m^2\ s^{-1}$.

It is as we shall see below the kinematic viscosity ν which appears in the Navier-Stokes equation for the velocity field, rather than the dynamic viscosity η. Normally, we would think of air as less viscous than water and hydrogen as less viscous than olive oil, but under suitable conditions it is really the other way around. If a flow is driven by inflow of fluid with a certain velocity, air behaves in fact as if it were 10–20 times more viscous than water. If instead driven by the same external forces, air is much easier to set into motion than water because its density is a thousand times smaller, and that is what fools our intuition.

14.2 Velocity-driven planar flow

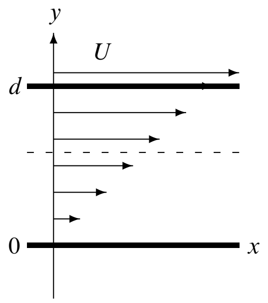
Before turning to the derivation of the Navier–Stokes equations for viscous flow, we shall explore the concept of shear viscosity a bit further for the simple case of planar flow. Let us, as before, assume that the flow is laminar and planar with the only non-vanishing velocity component being $v_x = v_x(y, t)$, now also allowing for time dependence. It is rather clear that there can be no advective acceleration in such a field, and formally we also find $(\boldsymbol{v} \cdot \boldsymbol{\nabla})v_x = v_x \nabla_x v_x = 0$. In the absence of volume and pressure forces, the Newtonian shear stress (14.1) will be the only non-vanishing component of the stress tensor, and Cauchy's dynamical equation (12.26) on page 198 reduces to

$$\rho \frac{\partial v_x}{\partial t} = f_x^* = \nabla_y \sigma_{xy} = \eta \frac{\partial^2 v_x}{\partial y^2}.$$

Dividing by the density (which is assumed to be constant) we get

$$\boxed{\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2},} \tag{14.5}$$

where ν is the kinematic viscosity (14.4) . This is a simplified version of the Navier–Stokes equation, particularly well suited for the discussion of the basic physics of shear viscosity.



A Newtonian fluid with spatially uniform properties between moving parallel plates. The velocity field varies linearly between the plates and satisfies the no-slip boundary condition that the fluid is at rest relative to both plates. The stress must be the same on any plane in the fluid parallel with the plates (dashed).

Case: Steady planar flow

In steady flow the left-hand side of (14.5) vanishes, and from the vanishing of the right-hand side it follows that the general solution must be linear, $v_x = A + By$, with arbitrary integration constants A and B. We shall imagine that the flow is maintained between (in principle infinitely extended) solid plates, one at rest at $y = 0$ and the other moving with constant velocity U at $y = d$. Where the fluid makes contact with the plates, we require it to assume the same speed as the plates, in other words $v_x(0) = 0$ and $v_x(d) = U$ (this *no-slip* boundary condition will be discussed in more detail later). Solving these conditions we find $A = 0$ and $B = U/d$ such that the field between the plates becomes

$$v_x(y) = \frac{y}{d} U, \tag{14.6}$$

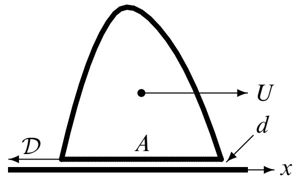
independent of the viscosity. From this expression we obtain the shear stress,

$$\sigma_{xy} = \eta \frac{dv_x}{dy} = \eta \frac{U}{d}. \tag{14.7}$$

It is independent of y, as one might have expected, because in stationary flow the balance of forces (and planar symmetry) requires the stress on any plane parallel with the plates to be the same.

Case: Viscous friction

A thin layer of viscous fluid is often used to lubricate the interface between solid objects. From the above solution to steady planar flow we may calculate the friction force, or *drag*, exerted on the body by the layer of viscous lubricant (see also section 16.4). Let the would-be contact area between the body and the surface on which it slides be A, and let the thickness of the fluid layer be d everywhere. If the layer is thin, $d \ll \sqrt{A}$, we may disregard edge effects and simply multiply the planar stress (14.7) by the contact area to get the drag force,



A solid object sliding on a plane lubricated surface with velocity U is subject to a viscous drag $\mathcal{D}$ opposite to the velocity.

$$\mathcal{D} \approx \eta \frac{A}{d} U. \tag{14.8}$$

The velocity-dependent viscous drag is quite different from the constant drag experienced in solid friction (see section 6.1 on page 97). The decrease in drag with falling velocity makes the object seem to want to slide "forever", and this is what makes ice sports such as skiing, skating, sledging and curling interesting. It is scary to brake a car on ice or to aquaplane, because the decreasing deceleration as the speed drops makes the car appear to run away from you. In these cases, a thin layer of liquid water acts as the lubricant.

The quasi-steady horizontal equation of motion for an object of mass M, not subject to forces other than viscous drag opposite the direction of the velocity, becomes

$$M \frac{dU}{dt} = -\eta \frac{A}{d} U. \tag{14.9}$$

Assuming that the thickness of the lubricant layer stays constant (and that is by no means evident) the solution to (14.9) is

$$U = U_0 e^{-t/\tau}, \qquad \tau = \frac{Md}{\eta A}, \tag{14.10}$$

where U_0 is the initial velocity and τ is the characteristic exponential decay time for the velocity. Integrating this expression we obtain the total stopping distance

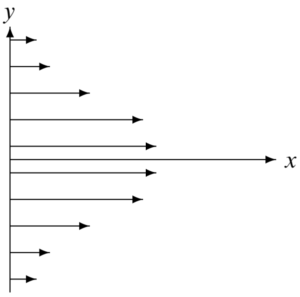
$$L = \int_0^\infty U \, dt = U_0 \tau = \frac{U_0 M d}{\eta A}. \tag{14.11}$$

Although it formally takes infinite time for the sliding object to come to a full stop, it does so in a finite distance! The stopping length grows with the mass of the object which is quite unlike solid friction, where the stopping length is independent of the mass.

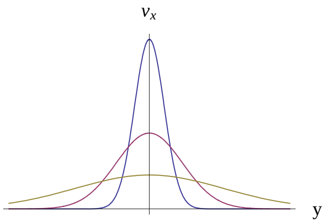
Example 14.1 [Curling]: In the ice sport of *curling*, a "stone" with mass $M \approx 20$ kg is set into motion with the aim of bringing it to a full stop at the far end of an ice rink of length $L \approx 40$ m. The area of the highly polished contact surface towards the ice is $A \approx 700\ \text{cm}^2$ and the initial velocity about $U_0 \approx 3\ \text{m s}^{-1}$. From (14.11) we obtain the thickness of the fluid layer $d \approx 43\ \mu\text{m}$ which does not seem unreasonable, and neither does the decay time $\tau \approx 13$ s. The players' intense sweeping of the ice in front of the moving stone presumably serves to smooth out tiny irregularities in the surface, which could otherwise slow down the stone.

Case: Momentum diffusion

The dynamic equation (14.5) is a typical *diffusion equation* with diffusion constant equal to the kinematic viscosity, ν, also called the *momentum diffusivity*. In general, such an equation leads to a spreading of the distribution of the diffused quantity, which in this case is the velocity v_x, or perhaps better, the momentum density ρv_x.



Velocity distribution for a planar Gaussian "river" in an "ocean" of fluid.



A Gaussian "river" widens and slows down in the course of time because of momentum diffusion, but retains its Gaussian shape.

The simplest example of a flow with momentum diffusion is a Gaussian "river", which starts out at $t = 0$ with shape $v_x = U \exp\left(-y^2/a^2\right)$ where a is a parameter that sets the scale for the width of the river. By direct insertion into the planar equation of motion (14.5) , it may be verified that the solution at time t is,

$$v_x(y,t) = U \frac{a}{\sqrt{a^2 + 4\nu t}} \exp\left(-\frac{y^2}{a^2 + 4\nu t}\right). \tag{14.12}$$

This river spreads with time but stays Gaussian, so that at time t it has width parameter $\sqrt{a^2 + 4\nu t}$. Although momentum diffuses away from the center of the river, the total momentum must remain constant because there are no external forces acting on the fluid. Kinetic energy is on the other hand dissipated and ends up as heat. The apparent paradox that the kinetic energy can vanish while momentum stays constant is resolved in problem 14.3.

At sufficiently large times, $t \gg a^2/4\nu$, the shape of the Gaussian becomes independent of the original width a. This is, in fact, a general feature of *any* bounded "river" flow: for large times it becomes proportional to $\exp(-y^2/4\nu t)$, as shown in problem 14.5. The Gaussian factor drops sharply to zero beyond $y \simeq 2\sqrt{\nu t}$, giving momentum diffusion a fairly well-defined front at a distance $L \simeq 2\sqrt{\nu t}$ from the origin of the velocity disturbance. A velocity disturbance may similarly be characterized by the time, $t \simeq L^2/4\nu$, it takes for it to spread through a transverse distance L by diffusion.

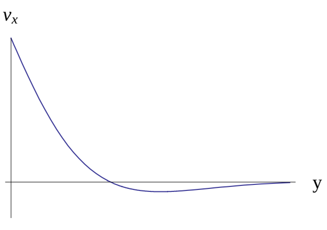
In the simple case discussed here, momentum diffusion takes place orthogonally to the general direction of motion of the fluid. Even if momentum diffuses away from the center in the y-direction, there is no mass flow in the y-direction because $v_y = 0$. In less restricted flows there may be more direct competition between mass flow and diffusion. If the velocity scale of a flow is $|\boldsymbol{v}| \sim U$, it would take the time $t_{\text{flow}} \sim L/U$ for a mass of fluid to move through the distance L. The ratio of the the diffusion time scale $t_{\text{diff}} \sim L^2/\nu$ to the mass flow time scale t_{flow} becomes a dimensionless number, first introduced by Reynolds,

$$\mathsf{Re} \approx \frac{t_{\text{diff}}}{t_{\text{flow}}} \sim \frac{UL}{\nu} \tag{14.13}$$

When this number is large compared to unity, momentum diffusion through a given distance takes much longer time than mass flow and plays only a small role, whereas when it is small, momentum diffusion is much faster than mass flow and dominates the flow pattern.

Case: Shear wave

Consider an infinitely extended plate in the xz-plane immersed in an infinite sea of fluid. Let the plate oscillate in its own plane with circular frequency $\omega = 2\pi/\tau$, so that its instantaneous velocity in the x-direction is $U(t) = U_0 \cos \omega t$. The motion of the plate is transferred to the neighboring fluid because of the no-slip condition and then spreads into the fluid at large.



Shape of the velocity amplitude of a shear wave at $t = 0$.

By direct insertion it may be verified that the following field satisfies eq. (14.5) as well as the no-slip boundary condition $v_x = U(t)$ for $y = 0$,

$$v_x(y,t) = U_0 e^{-ky} \cos(\omega t - ky), \qquad k = \sqrt{\frac{\omega}{2\nu}}. \tag{14.14}$$

Evidently, this is a damped wave spreading from the oscillating plate into the fluid. Since the velocity oscillations take place in the x-direction whereas the wave propagates in the y-direction it is a *transverse* or *shear* wave. The wavenumber k determines the wavelength $\lambda = 2\pi/k = 2\sqrt{\pi\nu\tau}$, as well as the decay length of the exponential, also called the *penetration depth* $d = 1/k = \lambda/2\pi$. The wave is strongly damped and penetrates only a fraction of a wavelength into the fluid. It is really not much of a wave.

A shear wave of frequency 1000 Hz penetrates only 71 μm in air and 18 μm in water at normal temperature and pressure.

Case: The Stokes layer

Consider again an infinite flat plate in the xz-plane immersed in a fluid. The fluid and plate are initially at rest for $t < 0$, but at time $t = 0$, the plate is suddenly set into motion with velocity U along the x-direction. The no-slip boundary condition will drag the fluid along at the plate, and viscosity will spread its momentum into the fluid at large. The aim is to calculate the spreading of this *boundary layer* with time, also called Stokes' First Problem.

Assuming that the flow is planar with $v_x = v_x(y,t)$ and $v_y = v_z = 0$, we may use (14.5) to find the solution. The linearity of this equation guarantees that the velocity everywhere must be proportional to U, and since there is no intrinsic length or time scale in the definition of the problem, the velocity field can only be of the form,

$$v_x(y,t) = Uf(s), \qquad s = \frac{y}{2\sqrt{\nu t}}. \tag{14.15}$$

The definition of the problem requires the —so far unknown—shape function $f(s)$ to obey the boundary conditions $f(0) = 1$ and $f(\infty) = 0$. The factor two in the denominator is just a convenient choice, suggested by the characteristic scale for momentum diffusion.

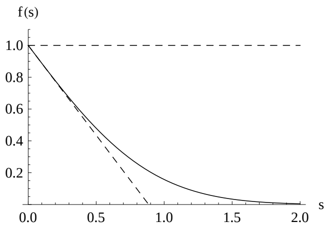
Inserting the above flow into the planar flow equation (14.5) we are led to an ordinary second-order differential equation for $f(s)$,

$$f''(s) + 2sf'(s) = 0. \tag{14.16}$$

It is a first-order differential equation for $f'(s)$, with the solution $f'(s) \sim \exp(-s^2)$. Integrating once more and applying the boundary conditions, the final result becomes (see the margin figure)

$$f(s) = \frac{2}{\sqrt{\pi}} \int_s^\infty e^{-u^2}\, du = \text{erfc}(s), \tag{14.17}$$

where erfc(·) is known in mathematics as the *complementary error function*.



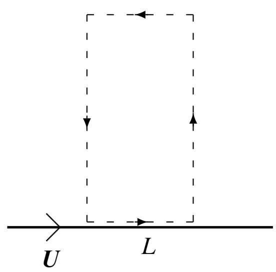
The Stokes layer shape function $f(s)$. The sloping dashed line is tangent at $s = 0$ with inclination $f'(0) = -2/\sqrt{\pi}$.

Infinite speed?: For large values of s the shape function $f(s)$ approaches zero with a Gaussian tail, $f(s) \sim \exp(-s^2) = \exp(-y^2/4\nu t)$, typical of momentum diffusion. The fluid is apparently in motion everywhere for $t > 0$, but how can that be when the plate was only started to move at time $t = 0$? The short answer is that we have assumed the fluid to be incompressible, and this—fundamentally untenable—assumption will in itself entail infinite signal speeds. At a deeper level, a diffusion equation like (14.5) is the continuum limit of the dynamics of random molecular motion in the fluid, and although high molecular speeds are strongly damped at speeds beyond $v_{\text{mol}} = \sqrt{3RT/M_{\text{mol}}}$, they may in principle occur.

The vorticity field has only one non-vanishing component,

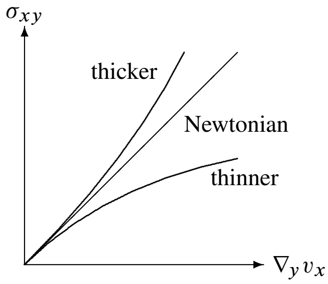
$$\omega_z(y,t) = -\frac{\partial v_x(y,t)}{\partial y} = -\frac{Uf'(s)}{2\sqrt{\nu t}} = \frac{U}{\sqrt{\pi\nu t}}\, e^{-y^2/4\nu t}. \tag{14.18}$$

Before the plate started to move the flow was everywhere irrotational. Afterwards there is vorticity everywhere in the fluid. Where did that come from? Consider a nearly infinite rectangle with support of length L on the plate. By Stokes' theorem (13.23) on page 217 the total flux of vorticity through the rectangle equals the circulation around its perimeter, $\Gamma = \int \boldsymbol{\omega} \cdot d\boldsymbol{S} = \oint \boldsymbol{v} \cdot d\boldsymbol{\ell}$. The fluid velocity is always $v_x = U$ on the plate, vanishes at infinity, and is orthogonal to the sides of the rectangle, so that we obtain $\Gamma = UL$. Since the circulation is constant in time, vorticity is not generated inside the Stokes layer itself during its growth. Instead, it must arise at the plate surface during the instantaneous acceleration to velocity U. For if the plate had followed a gentler road $U(t)$ from 0 to U, the circulation would have been $\Gamma(t) = U(t)L$, growing smoothly towards its final value. The conclusion is that vorticity is generated at the plate surface during acceleration, and afterwards diffuses away from the plate and into the fluid at large without changing the total flux.



The circulation around an infinitely tall rectangle with side L against the moving wall is $\Gamma = \oint \boldsymbol{v} \cdot d\boldsymbol{\ell} = UL$.

14.3 Dynamics of incompressible Newtonian fluids



In Newtonian fluids the shear stress σ_{xy} increases linearly with the strain-rate $\nabla_y v_x$, whereas non-Newtonian fluids mostly become thinner and only a few thicker.

Numerous everyday fluids obey Newton's law of viscosity (14.1) , for example water, air, oil, alcohol and antifreeze. A number of common fluids are only approximatively Newtonian, for example paint and blood, and others are strongly non-Newtonian, for example tomato ketchup, jelly and putty. There also exist *viscoelastic* materials that—depending on frequency—are both elastic and viscous. They are sometimes used in toys that can be slowly deformed like clay but also bounce like rubber balls when dropped on the floor.

In Newtonian fluids the shear stress σ_{xy} is directly proportional to the velocity gradient $\nabla_y v_x$—also called the *shear strain rate*—with proportionality constant equal to the constant shear viscosity η. Most non-Newtonian fluids become "thinner" as the shear strain rate increases, meaning that the shear stress grows slower than linearly. Even the most Newtonian of fluids, water, is thinner at shear strain rates above 10^{12} s^{-1}. Only few fluids (for example some starches stirred in water) appear to thicken with increasing strain rate. The science of the general flow properties of materials is called *rheology*.

We shall in this section establish the general dynamics for incompressible, isotropic and homogeneous Newtonian fluids, and postpone the analysis of the slightly more complicated compressible fluids to section 17.4 on page 300.

Isotropic viscous stress

Newton's law of viscosity (14.1) is a linear relation between the shear stress and the velocity gradient, only valid in a particular flow geometry. As for Hooke's law for elasticity (page 128) we want a definition of viscous stress which takes the same form in any flow geometry and in any Cartesian coordinate system, leaving us free to choose our own reference frame.

Most fluids are not only Newtonian, but also *isotropic*. Liquid crystals are anisotropic, but so special that we shall not consider them here. In an isotropic fluid at rest there are no internal directions at all and the stress tensor is determined by the pressure, $\sigma_{ij} = -p\,\delta_{ij}$. When such a fluid is set in motion, the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ defines a direction in every point of space, but the velocity in a particular point cannot itself provoke stress in the fluid. It is the variation in velocity from point to point that causes stress. Viscous stress must in other words be determined by the velocity gradient tensor, $\nabla_i v_j$. In an incompressible fluid, the trace of this tensor vanishes, $\sum_i \nabla_i v_i = \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, so the most general symmetric tensor one can construct from the velocity gradients is of the form,

$$\boxed{\sigma_{ij} = -p\,\delta_{ij} + \eta\left(\nabla_i v_j + \nabla_j v_i\right).} \tag{14.19}$$

This is the natural generalization of Newton's law of viscosity (14.1) for incompressible flow to arbitrary Cartesian coordinate systems. Inserting the field of a steady planar flow $\boldsymbol{v} = (v_x(y), 0, 0)$, the only non-vanishing shear stress components are $\sigma_{xy} = \sigma_{yx} = \eta\nabla_y v_x(y)$, demonstrating that the coefficient η is indeed the shear viscosity introduced before.

The Navier–Stokes equations

The right-hand side of Cauchy's general equation of motion (12.26) on page 198 equals the effective density of force $f_i^* = f_i + \sum_j \nabla_j\sigma_{ij}$. Inserting the stress tensor (14.19) and using again $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, we find

$$\sum_j \nabla_j\sigma_{ij} = -\nabla_i p + \eta\left(\sum_j \nabla_i\nabla_j v_j + \sum_j \nabla_j^2 v_i\right) = -\nabla_i p + \eta\nabla^2 v_i.$$

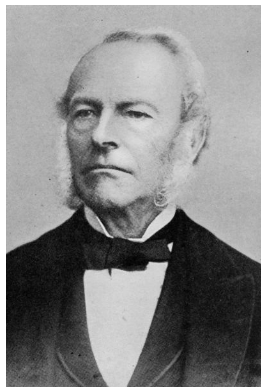
Here we have also assumed that the fluid is *homogeneous* such that the shear viscosity, like the density, does not depend on $\boldsymbol{x}$.

Inserting this expression into Cauchy's equation of motion and converting to ordinary vector notation we finally obtain the fundamental equations for incompressible, isotropic and homogenous fluids, due to Navier (1822) and Stokes (1845),

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{v}, \qquad \nabla\cdot\boldsymbol{v} = 0, \tag{14.20}$$

where ρ_0 is the constant density, $\nu = \eta/\rho_0$ is the constant kinematic viscosity and $\boldsymbol{g} = \boldsymbol{f}/\rho_0$ is the acceleration field of the volume forces (normally due to gravity). Given the acceleration field $\boldsymbol{g}$, we now have four equations for the four fields, v_x, v_y, v_z, and p. Note, however, that whereas the three components of the velocity field are truly dynamic fields, for which the time derivatives are specified, this is not the case for the pressure which is only determined indirectly through the divergence condition, as discussed in section 13.1 on page 207.

Relatively simple to look at, the Navier–Stokes equations contain all the complexity of real fluid flow, including that of Niagara Falls! It is therefore clear that one cannot in general expect to find simple solutions. Exact solutions are only found in strongly restricted geometries and under simplifying assumptions concerning the nature of the flow, as in the planar laminar flow examples in the preceding section and the examples to be studied in chapter 15.



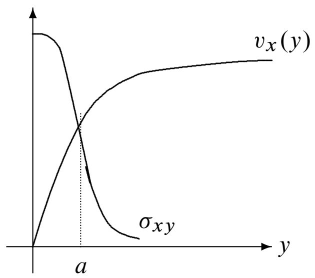
George Gabriel Stokes (1819–1903). British mathematician and physicist. Contributed to the development of field calculus, fluid dynamics, optics and heat conduction.

Millenium Prize Problem: Among the seven Millenium Prize Problems set out by the Clay Mathematics Institute of Cambridge, Massachusetts, one concerns the existence of smooth, nonsingular solutions to the Navier–Stokes equations (even for the simpler case of incompressible flow). The prize money of one million dollars illustrates how little we know and how much we would like to know about the general features of these equations which appear to defy the standard analytic methods for solving partial differential equations.

Boundary conditions

The Navier-Stokes equation for the velocity field is of first order in time, and needs initial values for the fields and their spatial derivatives in order to calculate the field values at later times. But what about physical boundaries, the solid containers of fluids, or even internal boundaries between different fluids? How do the fields behave there? Let us discuss the various fields that we have met one by one.

Density: The density is easy to dispose of, since it is allowed to be discontinuous and jump at a boundary between two materials, so this provides us with no condition at all.



Sketch of strongly varying velocity and shear stress in a region of size a near a boundary. For $a \to 0$ the velocity develops an abrupt jump and the stress becomes infinite. From the Navier-Stokes equation, it follows that the strong decrease in shear stress away from the discontinuity leads to rapid momentum diffusion and smoothing out of the discontinuity.

Velocity: The normal component of the velocity field, $v_n = \boldsymbol{v}\cdot\boldsymbol{n}$, must always be continuous across an interface between incompressible materials. If this were not the case, the materials on the two sides of the interface would not move in unison. The tangential velocity component $\boldsymbol{v}_t = \boldsymbol{n}\times(\boldsymbol{v}\times\boldsymbol{n})$ must also be continuous but for a different reason: Although a nearly discontinuous tangential velocity variation may be created close to a wall, for example by hitting the fluid container with a hammer, it cannot remain for long but is quickly smoothed out by viscous momentum diffusion. A viscous fluid never slips along material boundaries but adjusts it velocity to match the velocity of the material on the other side without any discontinuity. This is the previously mentioned *no-slip condition*.

The complete velocity field may thus always be assumed to be continuous across any interface between incompressible materials,

$$\Delta\boldsymbol{v} = \boldsymbol{0}. \tag{14.21}$$

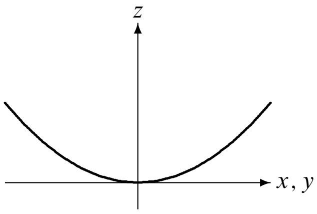
Only if the continuum approximation breaks down, cavitation and shear slippage may occur. A notable exception to this rule is an open interface between a liquid and vacuum, where the velocity field will always be discontinuous (vacuum is not a material!).

Stress: Newton's third law only requires the stress vector $\boldsymbol{\sigma} \cdot \boldsymbol{n}$ to be continuous across any material interface (in the absence of surface tension),

$$\Delta\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} = \boldsymbol{0}. \tag{14.22}$$

This only concerns three of the six components of the stress tensor. The other three are free to jump at the interface (see example 6.7 on page 105). At an interface between a fluid and vacuum the material stress vector must vanish, $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} = 0$, because the vacuum cannot exert any force on any material object.

Pressure: Pressure is not necessarily continuous across a physical interface between two materials — even in the absence of surface tension (see page 105). For general fluids at rest and ideal fluids in motion, pressure is the only stress component, and must necessarily be continuous because of the continuity of the stress vector. We shall now see that the pressure must also be continuous for an incompressible viscous fluid moving along a solid wall, so that the pressure in the fluid *near* the wall is identical to the pressure acting *on* the wall. In all other cases the pressure will in general be discontinuous.



Local coordinate system at a particular point of the interface.

The solid wall may without loss of generality be assumed to be at rest. Let us select any point on the wall and choose a local coordinate system with origin in this point and with the z-axis along the normal to the wall. Near the origin of this coordinate system, the wall may be described by a quadratic function, $z = Ax^2 + By^2 + 2Cxy$. Expanding to first order in the coordinates the velocity field becomes $\boldsymbol{v}(x, y, z) = x\nabla_x \boldsymbol{v}|_0 + y\nabla_y \boldsymbol{v}|_0$. As the velocity field has to vanish everywhere along the wall it follows that all the tangential derivatives must vanish at the origin, $\nabla_x\boldsymbol{v} = \nabla_y\boldsymbol{v} = \boldsymbol{0}$ for $x = y = z = 0$. From the divergence condition, $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, we find that the normal derivative of the normal component, $\nabla_z v_z = -\nabla_x v_x - \nabla_y v_y = 0$, must also vanish at the origin. Finally, using the stress tensor (14.19) we get $\sigma_{zz} = -p + 2\eta\nabla_z v_z = -p$ at the origin. The continuity of the normal component, σ_{zz}, at the wall consequently guarantees the continuity of the pressure.

* Viscous dissipation

When you gently and steadily stir a pot of soup, the fluid will after some time settle down into a nearly steady flow. The fact that you still have to perform work while you stir steadily, shows that there must be viscous friction forces at play in the soup. The friction forces between the sides of the pot and the soup cannot perform any work because the fluid is at rest there, due to the no-slip condition. All the work you perform must for this reason be spent against the *internal* friction forces in the soup, the shear stresses acting between the moving layers of the fluid. If you stop stirring, the soup quickly comes to rest and its kinetic energy is *dissipated* into heat. We shall return to dissipation in chapters 20 and 22.

To calculate the dissipative rate of work, we turn back to the discussion of deformation work resulting in eq. (7.38) on page 120. Since a fluid particle is displaced by $\delta\boldsymbol{u} = \boldsymbol{v}\,\delta t$ in a small time interval δt, fluid motion may be seen as a continuous sequence of infinitesimal deformations with strain tensor, $\delta u_{ij} = \frac{1}{2}\left(\nabla_i \delta u_j + \nabla_j \delta u_i\right) = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right)\delta t$. The symmetrized velocity gradients $v_{ij} \equiv \delta u_{ij}/\delta t = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right)$ may thus be understood as the *rate of deformation* or *rate of strain* of the fluid material. The rate of work $\dot{W} = \delta W/\delta t$ performed against the internal stresses is consequently,

$$\dot{W}_{\text{int}} = \int_V \sum_{ij} \sigma_{ij} \nabla_j v_i \, dV = \int_V \sum_{ij} \sigma_{ij} v_{ij} \, dV = \int_V 2\eta \sum_{ij} v_{ij}^2 \, dV. \tag{14.23}$$

In the last step we have inserted the Newtonian stress tensor (14.19) and used that $\sum_i v_{ii} = \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$. Evidently, the rate of work against internal shear stresses is always positive. It always costs work to keep things moving against friction forces.

14.4 Classification of flows

The most interesting phenomena in fluid dynamics arise from the competition between inertia and viscosity, represented in the Navier–Stokes equation (14.20) by the advective acceleration $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ and the viscous diffusion term $\nu\nabla^2\boldsymbol{v}$. Inertia attempts to continue the motion of a fluid once it is started whereas viscosity acts as a brake. If inertia is dominant we may leave out the viscous term, arriving again at Euler's equations (13.1) describing lively, *non-viscous* or *ideal* flow, analyzed in chapter 13. If on the other hand viscosity is dominant, we may drop the advective term, and obtain the basic equations for sluggish *creeping* flow to be analyzed in chapter 16.

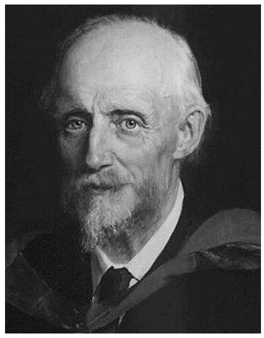
The Reynolds number

As a measure of how much an actual flow is lively or sluggish, one may make a rough estimate, called the *Reynolds number*, for the ratio of the advective to the viscous terms. To get a simple expression we assume that the velocity is of typical size $|\boldsymbol{v}| \approx U$ and that it changes by a similar amount over a region of size L. The order of magnitude of the first-order spatial derivatives of the velocity will then be of magnitude $|\nabla\boldsymbol{v}| \approx U/L$, and the second-order derivatives will be $\left|\nabla^2\boldsymbol{v}\right| \approx U/L^2$. Consequently, the Reynolds number becomes,

$$\mathsf{Re} \approx \frac{|(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}|}{|\nu\nabla^2\boldsymbol{v}|} \approx \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu}. \tag{14.24}$$

As we saw on page 232, the Reynolds number may also be understood as the ratio $\mathsf{Re} \approx t_{\text{diff}}/t_{\text{flow}}$ of the typical diffusion time scale $t_{\text{diff}} \sim L^2/\nu$ to the typical flow time scale $t_{\text{flow}} \sim L/U$.

In table 14.2 the Reynolds number is estimated for a number of flows, covering many orders of magnitude. For small values of the Reynolds number, $\mathsf{Re} \ll 1$, advection plays only little role and the flow just oozes along, while for large values, $\mathsf{Re} \gg 1$, viscosity does not have much influence and the flow tends to be lively. The streamline pattern of creeping flow is always orderly and layered, also called *laminar*, well known from the kitchen when mixing cocoa into dough to make a chocolate cake (although dough is hardly Newtonian!). The laminar flow pattern continues quite far beyond $\mathsf{Re} \simeq 1$, but depending on the flow geometry and other circumstances, there will be a Reynolds number, typically in the region of thousands, where *turbulence* sets in with its characteristic tumbling and chaotic behavior.



Osborne Reynolds (1842–1912). British engineer and physicist. Contributed to fluid mechanics in general, and to the understanding of lubrication, turbulence and tidal motion in particular.

Example 14.2 [Bathtub turbulence]: Getting out of a bathtub you create flows with speeds of say $U \approx 1$ m s^{-1} over a distance of $L \approx 1$ m. The Reynolds number becomes $\mathsf{Re} \approx 10^6$ and you are definitely creating visible turbulence in the water. Similarly, when jogging you create air flows with $U \approx 3$ m s^{-1} and $L \approx 1$ m, leading to a Reynolds number around 2×10^5, and you know that you must leave all kinds of little invisible turbulent eddies in the air behind you. The fact that the Reynolds number is smaller in air than in water despite the higher velocity is a consequence of the kinematic viscosity being larger for air than for water.

Example 14.3 [Curling]: For planar flow between two plates (section 14.1), the velocity scale is set by the velocity difference U between the plates whereas the length scale is set by the distance d between the plates. In the curling example 14.1 on page 231 we found $U \approx 3$ m s^{-1} and $d \approx 43$ μm, leading to a Reynolds number $\mathsf{Re} = Ud/\nu \approx 140$. Although not truly creeping flow, it is definitely laminar and not turbulent.

Example 14.4 [Water pipe]: A typical 1/2-inch water pipe has diameter $d \approx 1.25$ cm and that sets the length scale. If the volume flux of water is $Q = 100$ cm^3 s^{-1}, the average water speed becomes $U = Q/\pi a^2 \approx 0.8$ m s^{-1} and we get a Reynolds number $\mathsf{Re} = Ud/\nu \approx 10^4$ which brings the flow well into the turbulent regime. For olive oil under otherwise identical conditions we get $\mathsf{Re} \approx 0.15$, and the flow would be creeping.

Table 14.2. Table of Reynolds numbers for some moving objects calculated on the basis of typical values of lengths and speeds. Viscosities are taken from table 14.1 on page 228. It is perhaps surprising that a submarine operates at a Reynolds number that is larger than that of a passenger jet at cruising speed, but this is mainly due to the kinematic viscosity of air being 15 times larger than that of water.

	Fluid	Size L [m]	Velocity U [ms^{-1}]	Reynolds number
Ship (Queen Mary II; fig. 3.3)	water	345	15	5.2×10^9
Submarine (Ohio class; nuclear)	water	170	12	2.0×10^9
Jet airplane (Boeing 747-400)	air	71	250	1.2×10^9
Blue whale	water	33	10	3.3×10^8
Car	air	5	30	9.7×10^6
Swimming human	water	2	1	2.0×10^6
Jogging human	air	1	3	2.0×10^5
Herring	water	0.3	1	3.8×10^5
Golf ball	air	0.043	40	2.2×10^5
Ping-pong ball	air	0.040	10	5×10^4
Fly	air	0.01	1	600
Flea	air	0.001	3	190
Gnat	air	0.001	0.1	6
Bacterium	water	10^{-6}	10^{-5}	10^{-5}

Hydrodynamic similarity

What does it mean if two flows have the same Reynolds number? A stone of size $L = 1$ m sitting in a steady water flow with velocity $U = 2$ m s^{-1} has the same Reynolds number as another stone of size $L = 2$ m in a steady water flow with velocity $U = 1$ m s^{-1}. It even has the same Reynolds number as a stone of size $L = 4$ m in a steady airflow with velocity $U = 8$ m s^{-1}, because the kinematic viscosity of air is about 15 times larger than of water (at normal temperature and pressure). We shall now see that provided the stones are geometrically similar, i.e. have congruent geometrical shapes, flows with the same Reynolds numbers are also *hydrodynamically similar* and only differ by their overall length and velocity scales, so that their flow patterns visualized by streamlines will look identical.

In the absence of volume forces, steady incompressible flow is determined by (14.20) with $\boldsymbol{g} = \boldsymbol{0}$ and $\partial \boldsymbol{v}/\partial t = \boldsymbol{0}$, or

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \boldsymbol{v}. \qquad (14.25)$$

Let us rescale all the variables by means of the overall scales ρ_0, U, and L, writing

$$\boldsymbol{v} = U\tilde{\boldsymbol{v}}, \quad \boldsymbol{x} = L\tilde{\boldsymbol{x}}, \quad p = \rho_0 U^2 \tilde{p}, \quad \nabla = \frac{1}{L}\tilde{\nabla}, \qquad (14.26)$$

where the "tilded" symbols are all dimensionless. Inserting these variables, the steady flow equation takes the form,

$$(\tilde{\boldsymbol{v}} \cdot \tilde{\nabla})\tilde{\boldsymbol{v}} = -\tilde{\nabla}\tilde{p} + \frac{1}{\mathsf{Re}}\tilde{\nabla}^2\tilde{\boldsymbol{v}}. \qquad (14.27)$$

The only parameter appearing in this equation is the Reynolds number which may be interpreted as the inverse of the dimensionless kinematic viscosity. The pressure is as mentioned not an independent dynamic variable and its scale is here fixed by the velocity scale,

$P = \rho_0 U^2$. If the flow instead is driven by external pressure differences of magnitude P rather than by velocity, the equivalent flow velocity scale is given by $U = \sqrt{P/\rho_0}$.

In congruent flow geometries, the no-slip boundary conditions will also be the same, so that any solution of the dimensionless equation can be scaled back to a solution of the original equation by means of (14.26) . The three different flows around stones mentioned at the beginning of this subsection may thus all be obtained from the same dimensionless solution if the stones are geometrically similar and the Reynolds numbers identical.

Even if the flows are similar in air and water, the forces exerted on the stones will, however, not be the same. The shear stress magnitude may be estimated as $\sigma \approx \eta |\nabla \boldsymbol{v}| \sim \eta U/L$. The viscous drag on an object of size L will then be of magnitude $\mathcal{D} \approx \sigma L^2 \sim \eta U L = \eta \nu \mathsf{Re}$. The Reynolds numbers are assumed to be the same in the two cases, making the ratio of the viscous drag on the stone in air to that in water about $\mathcal{D}_{\text{air}}/\mathcal{D}_{\text{water}} \approx (\eta\nu)_{\text{air}}/(\eta\nu)_{\text{water}} \approx 0.27$.

Example 14.5 [Flight of the robofly]: The similarity of flows in congruent geometries can be exploited to study the flow around tiny insects by means of enlarged slower moving models, immersed in another fluid. It is, for example, hard to study the air flow around the wing of a hovering fruit fly, when the wing flaps $f = 50$ times per second. For a wing size of $L \approx 4$ mm flapping through $180°$ the average velocity becomes $U \approx \pi L f \approx 1.3 \text{ m s}^{-1}$ and the corresponding Reynolds number $\mathsf{Re} \approx UL/\nu \approx 160$. The same Reynolds number can be obtained from a 19 cm plastic wing of the same shape, flapping once every 6 s in mineral oil with kinematic viscosity $\nu = 1.15 \text{ cm}^2 \text{ s}^{-1}$, allowing for easy recording of the flow around the wing [BD01] .

Example 14.6 [High-pressure wind tunnels]: In the early days of flight, wind tunnels were extensively used for empirical studies of lift and drag on scaled-down models of wings and aircraft. Unfortunately, the smaller geometrical sizes of the models reduced the attainable Reynolds number below that of real aircraft in flight. A solution to the problem was obtained by operating wind tunnels at much higher than atmospheric pressure. Since the dynamic viscosity η is independent of pressure (page 229), the Reynolds number $\mathsf{Re} = \rho_0 UL/\eta$ scales with the air density and thus with pressure (at a given temperature). The famous *Variable Density Tunnel (VDT)* built in 1922 by the US National Advisory Committee for Aeronautics (NACA) operated on a pressure of 20 atm and was capable of attaining full-scale Reynolds numbers for models only 1/20th of the size of real aircraft [Anderson 1997, p. 301]. The results obtained from the VDT had great influence on aircraft design in the following 20 years.

In the presence of external volume forces, for example gravity, or for flows coupled to other equations of motion, for example a heat equation, the flow patterns will depend on further dimensionless quantities besides the Reynolds number. We shall only introduce such quantities when they arise naturally in particular cases.

Flows in different geometries can only be compared in a coarse sense, even if they have the same Reynolds number. A running man has the same Reynolds number as a swimming herring, and a flying gnat the same Reynolds number as a man swimming in castor oil (which cannot be recommended). In both cases the flow geometries are quite different, leading to different streamline patterns. Here the Reynolds number can only be used to indicate the character of the flow which tends to be turbulent around the running man and the swimming herring, whereas it is laminar around the flying gnat and the man recklessly swimming in castor oil.

Problems

14.1 Calculate the temperature dependence of the kinematic viscosity for an isentropic gas. What is the exponent of the temperature for monatomic, diatomic and multiatomic gases?

14.2 A car with $M = 1000$ kg moving at $U_0 = 100$ km h^{-1} suddenly hits a patch of ice and begins to slide. The total contact area between each wheel and the water is 800 cm^2 and it is observed to slide to a full stop in about 300 m. Calculate the thickness of the water layer and discuss whether it is a reasonable value. What is the time scale for stopping the car?

14.3 Consider planar momentum diffusion (page 231). Assume that the flow of the incompressible 'river' vanishes fast at infinity, as in the Gaussian case. **(a)** Show that for any river flowing along x the total volume flux per unit of length in the z-direction is independent of time. **(b)** Show that the total momentum per unit of length in the z-direction is likewise constant. **(c)** For the Gaussian river, calculate the kinetic energy per unit of length in the x and z directions as a function of time. What happens for $t \to \infty$?

14.4 Estimate the Reynolds number for (a) an ocean current, (b) a water fall, (c) a weather cyclone, (d) a hurricane, (e) a tornado, (f) lava running down a mountainside and (g) plate tectonic motion.

* **14.5** Show that the general solution to the momentum diffusion equation (14.5) is

$$v_x(y,t) = \frac{1}{2\sqrt{\pi\nu t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-y')^2}{4\nu t}\right) v_x(y',0)\, dy'. \tag{14.28}$$

Use this to show that any bounded initial velocity distribution which is non-vanishing only for $|y| < a$ at $t = 0$ is Gaussian for $|y| \to \infty$ at any later time.

* **14.6** At an interface (with local normal along z in a given point) between two incompressible viscous fluids with different viscosities it is known (see p. 236) that there are no discontinuities in the velocity field $\boldsymbol{v}$, its tangential derivatives $\nabla_x \boldsymbol{v}$ and $\nabla_y \boldsymbol{v}$, and in the stress vector components σ_{xz}, σ_{yz} and σ_{zz}.

Show that the non-vanishing discontinuities that follow from a discontinuity in viscosity $\Delta\eta$ are

(a) $\Delta p = 2\Delta\eta\nabla_z v_z$ (proven in the text).

(b) $\Delta\nabla_z v_x = \sigma_{xz}\Delta(1/\eta)$ and similarly for $\nabla_z v_y$.

(c) $\Delta\sigma_{xy} = \Delta\eta(\nabla_x v_y + \nabla_y v_x)$.

(d) $\Delta\sigma_{xx} = -\Delta p + 2\Delta\eta\nabla_x v_x$ and similarly for $\Delta\sigma_{yy}$.

14.7 Show that the pressure discontinuity across an interface between moving incompressible homogeneous fluids is given by the jump in viscosity,

$$\Delta p = 2\Delta\eta\nabla_z v_z. \tag{14.29}$$

14.7 For the much rarer case of an interface between moving viscous fluids, we may also choose the velocity to vanish at any chosen point of the interface, but since the velocity will not in general vanish at bit away from this point, the tangential derivatives $\nabla_x \boldsymbol{v}$ and $\nabla_y \boldsymbol{v}$ will not vanish. From the continuity of the velocity field across the interface, it follows however that they are continuous. Using the divergence condition, we conclude that $\nabla_z v_z$ is also continuous but generally non-vanishing. Using this result and the continuity of $\sigma_{zz} = -p + 2\eta\nabla_z v_z$, we see that the pressure will only be discontinuous if the viscosities are different on the two sides of the interface, with a jump in pressure $\Delta p = 2\Delta\eta\nabla_z v_z$. Other discontinuities following from the discontinuity of the viscosity are determined in problem 14.6.