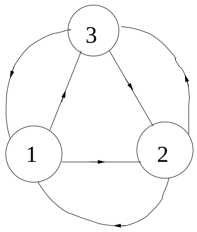
Chapter 11

Solution 11.1: A transition matrix for 3-disk pinball. *a) As the disk is convex, the transition to itself is forbidden. Therefore, the Markov diagram is*



,

with the corresponding transition matrix

$$\mathbb{T} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note that $\mathbb{T}^2 = \mathbb{T} + 2$. *Suppose that* $\mathbb{T}^n = a_n\mathbb{T} + b_n$, *then*

$$\mathbb{T}^{n+1} = a_n\mathbb{T}^2 + b_n\mathbb{T} = (a_n + b_n)\mathbb{T} + 2a_n .$$

So $a_{n+1} = a_n + b_n\,,\ b_{n+1} = 2a_n$ *with* $a_1 = 1\,,\ b_1 = 0$.

b) From a) we have $a_{n+1} = a_n + 2a_{n-1}$. *Suppose that* $a_n \propto \lambda^n$. *Then* $\lambda^2 = \lambda + 2$. *Solving this equation and using the initial condition for* $n = 1$, *we obtain the general formula*

$$\begin{aligned} a_n &= \frac{1}{3}(2^n - (-1)^n)\,, \\ b_n &= \frac{2}{3}(2^{n-1} + (-1)^n)\,. \end{aligned}$$

c) $\mathbb{T}$ *has eigenvalue* 2 *and* -1 *(degeneracy* 2*). So the topological entropy is* $\ln 2$, *the same as in the case of the binary symbolic dynamics.* *(Yueheng Lan)*

Solution 11.2: Sum of A_{ij} is like a trace. *Suppose that* $A\phi_k = \lambda_k\phi_k$, *where* $\lambda_k\,,\phi_k$ *are eigenvalues and eigenvectors, respectively. Expressing the vector* $v = (1, 1, \cdots, 1)^t$ *in terms of the eigenvectors* ϕ_k, *that is,* $v = \Sigma_k d_k\phi_k$, *we have*

$$\begin{aligned} \Gamma_n &= \Sigma_{ij}[A^n]_{ij} = v^t A^n v \ = \ \Sigma_k v^t A^n d_k\phi_k \ = \ \Sigma_k d_k\lambda_k^n(v^t\phi_k) \\ &= \Sigma_k c_k\lambda_k^n\,, \end{aligned}$$

where $c_k = (v^t\phi_k)d_k$ *are constants.*

a) As $\mathrm{tr}\, A^n = \Sigma_k \lambda_k^n$*, it is easy to see that both* $\mathrm{tr}\, A^n$ *and* Γ_n *are dominated by the largest eigenvalue* λ_0*. That is*

$$\frac{\ln |\mathrm{tr}\, A^n|}{\ln |\Gamma_n|} = \frac{n \ln |\lambda_0| + \ln |\Sigma_k (\frac{\lambda_k}{\lambda_0})^n|}{n \ln |\lambda_0| + \ln |\Sigma_k d_k (\frac{\lambda_k}{\lambda_0})^n|} \to 1 \ \text{ as } n \to \infty \,.$$

b) The nonleading eigenvalues do not need to be distinct, as the ratio in a) is controlled by the largest eigenvalues only.

(Yueheng Lan)

Solution 11.4: Transition matrix and cycle counting. *a) According to the definition of* $\mathbb{T}_{ij}$*, the transition matrix is*

$$\mathbb{T} = \left(\begin{array}{cc} a & c \\ b & 0 \end{array} \right) .$$

b) All walks of length three $0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010$ *(four symbols!) with weights* $aaa, aac, acb, cba, cbc, baa, bac, bcb$ *. Let's calculate* $\mathbb{T}^3$*,*

$$\mathbb{T}^3 = \left(\begin{array}{ll} a^3 + 2abc & a^2 c + bc^2 \\ a^2 b + b^2 c & abc \end{array} \right) .$$

There are altogether 8 *terms, corresponding exactly to the terms in all the walks.*

c) Let's look at the following equality

$$\mathbb{T}^n_{ij} = \Sigma_{k_1, k_2, \cdots, k_{n-1}} \mathbb{T}_{ik_1} \mathbb{T}_{k_1 k_2} \cdots \mathbb{T}_{k_{n-1} j} \,.$$

Every term in the sum is a possible path from i *to* j*, though the weight could be zero. The summation is over all possible intermediate points (*$n-1$ *of them). So,* $\mathbb{T}^n_{ij}$ *gives the total weight (probability or number) of all the walks from* i *to* j *in n steps.*

d) We take $a = b = c = 1$ *to just count the number of possible walks in* n *steps. This is the crudest description of the dynamics. Taking* a, b, c *as transition probabilities would give a more detailed description. The eigenvlues of* $\mathbb{T}$ *is* $(1 \pm \sqrt{5})/2$*, so we get* $N(n) \propto (\frac{1+\sqrt{5}}{2})^n$*.*

e) The topological entropy is then $\ln \frac{1+\sqrt{5}}{2}$*.* *(Yueheng Lan)*

Solution 11.6: "Golden mean" pruned map. *It is easy to write the transition matrix* $\mathbb{T}$

$$\mathbb{T} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right) .$$

The eigenvalues are $(1 \pm \sqrt{5})/2$. *The number of periodic orbits of length* n *is the trace*

$$\mathbb{T}^n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n} \,.$$

(Yueheng Lan)

Solution 11.5: 3-disk prime cycle counting. *The formula for arbitrary length cycles is derived in sect. 11.4.*

Solution 11.45: Alphabet $\{0,1\}$, prune $\_1000\_$, $\_00100\_$, $\_01100\_$.

step 1. $\_1000\_$ *prunes all cycles with a* $\_000\_$ *subsequence with the exception of the fixed point* $\overline{0}$; *hence we factor out* $(1 - t_0)$ *explicitly, and prune* $\_000\_$ *from the rest. Physically this means that* x_0 *is an isolated fixed point - no cycle stays in its vicinity for more than 2 iterations. In the notation of exercise 11.18, the alphabet is* $\{1,\, 2,\, 3;\ \overline{0}\}$, *and the remaining pruning rules have to be rewritten in terms of symbols 2=10, 3=100:*

step 2. *alphabet* $\{1,\, 2,\, 3;\ \overline{0}\}$, *prune* $\_33\_$, $\_213\_$,$\_313\_$. *Physically, the 3-cycle* $\overline{3} = \overline{100}$ *is pruned and no long cycles stay close enough to it for a single* $\_100\_$ *repeat. As in exercise 11.7, prohibition of* $\_33\_$ *is implemented by dropping the symbol "3" and extending the alphabet by the allowed blocks 13, 23:*

step 3. *alphabet* $\{1,\, 2,\, \underline{13},\, \underline{23};\ \overline{0}\}$, *prune* $\_2\underline{13}\_$, $\_\underline{23}\,\underline{13}\_$, $\_\underline{13}\,\underline{13}\_$, *where* $\underline{13} = 13$, $\underline{23} = 23$ *are now used as single letters. Pruning of the repetitions* $\_\underline{13}\,\underline{13}\_$ *(the 4-cycle* $\overline{13} = \overline{1100}$ *is pruned) yields the*

Result: *alphabet* $\{1,\, 2,\, \underline{23},\, \underline{113};\ \overline{0}\}$, *unrestricted 4-ary dynamics. The other remaining possible blocks* $\_\underline{213}\_$, $\_\underline{2313}\_$ *are forbidden by the rules of step 3. The topological zeta function is given by*

$$1/\zeta = (1 - t_0)(1 - t_1 - t_2 - t_{23} - t_{113}) \tag{O.6}$$

for unrestricted 4-letter alphabet $\{1,\, 2,\, \underline{23},\, \underline{113}\}$.

Solution 11.11: Whence Möbius function? *Written out* $f(n)$ *line-by-line for a few values of* n, *(11.38) yields*

$$\begin{aligned}
f(1) &= g(1) \\
f(2) &= g(2) + g(1) \\
f(3) &= g(3) + g(1) \\
f(4) &= g(4) + g(2) + g(1) \\
&\cdots \\
f(6) &= g(6) + g(3) + g(2) + g(1) \\
&\cdots
\end{aligned} \tag{O.7}$$

Now invert recursively this infinite tower of equations to obtain

$$\begin{aligned}
g(1) &= f(1) \\
g(2) &= f(2) - f(1) \\
g(3) &= f(3) - f(1) \\
g(4) &= f(4) - [f(2) - f(1)] - f(1) = f(4) - f(2) \\
&\cdots \\
g(6) &= f(6) - [f(3) - f(1)] - [f(2) - f(1)] - f(1) \\
&\cdots
\end{aligned}$$

We see that $f(n)$ *contributes with factor* -1 *if* n *prime, and not at all if* n *contains a prime factor to a higher power. This is precisely the raison d'etre for the Möbius function, with whose help the inverse of (11.38) can be written as the Möbius inversion formula [22.29] (11.39).*