Chapter 4

Solution 4.1: Trace-log of a matrix. *1) one method is to first check that this is true for any Hermitian matrix M. Then write an arbitrary complex matrix as sum $M = A + zB$, A, B Hermitian, Taylor expand in z and prove by analytic continuation that the identity applies to arbitrary M.*

(David Mermin)

to be fleshed out

2) another method: evaluate $\frac{d}{dt}\det\left(e^{t\ln M}\right)$ by definition of derivative in terms of infinitesimals.

(Kasper Juel Eriksen)

3) check appendix M.1

4) This identity makes sense for a matrix $M \subset \mathbb{C}^{n\times n}$, if $|\prod_{i=1}^{n}\lambda_i| < \infty$ and $\{|\lambda_i| > 0, \forall i\}$, where $\{\lambda_i\}$ is a set of eigenvalues of M. Under these conditions there exist a nonsingular $O : M = ODO^{-1}$, $D = \mathrm{diag}[\{\lambda_i, i = 1, \ldots, n\}]$. If $f(M)$ is a matrix valued function defined in terms of power series then $f(M) = Of(D)O^{-1}$, and $f(D) = \mathrm{diag}[\{f(\lambda_i)\}]$. Using these properties and cyclic property of the trace we obtain

$$\exp(\mathrm{tr}\,(\ln M)) = \exp\left(\sum_i \ln\lambda_i\right) = \prod_i \lambda_i = \det(M)$$

(Rytis Paškauskas)

5) Consider $M = \exp A$.

$$\det M = \det \lim_{n\to\infty}\left(\mathrm{Id} + \frac{A}{n}\right)^n = \lim_{n\to\infty}(1 + \frac{\mathrm{tr}\,A}{n} + \ldots)^n = \exp(\mathrm{tr}\,(\ln M))$$

(Rytis Paškauskas)

Solution 4.2: Stability, diagonal case. *The relation (4.14) can be verified by noting that the defining product (4.10) can be rewritten as*

$$\begin{aligned} e^{t\mathbf{A}} &= \left(\mathbf{U}\mathbf{U}^{-1} + \frac{t\mathbf{U}\mathbf{A}_D\mathbf{U}^{-1}}{m}\right)\left(\mathbf{U}\mathbf{U}^{-1} + \frac{t\mathbf{U}\mathbf{A}_D\mathbf{U}^{-1}}{m}\right)\cdots \\ &= \mathbf{U}\left(I + \frac{t\mathbf{A}_D}{m}\right)\mathbf{U}^{-1}\mathbf{U}\left(I + \frac{t\mathbf{A}_D}{m}\right)\mathbf{U}^{-1}\cdots = \mathbf{U}e^{t\mathbf{A}_D}\mathbf{U}^{-1}\,. \quad \text{(R.4)} \end{aligned}$$