Appendix O

Solutions

Chapter 1

Solution 1.1: 3-disk symbolic dynamics. *As explained in sect. 1.4, each orbit segment can be characterized by either of the two symbols* 0 *and* 1*, differentiating topologically bouncing back or going onto the third disk. So, there are* 2^n *topologically different orbits starting from each disk. Altogether, the 3-disk pinball has* $3 \cdot 2^n$ *itineraries of length* n*.*
Periodic orbits (prime cycles in fundamental domain)

- *Of length* 2*: 12,13,32; or (0).*
- *Of length* 3*: 123,321; or (1).*
- *Of length* 4*: 1213,2321,3231; or (01).*
- *Of length* 5*: 12123,13132,23231,21213,32321,31312; or (00111).*

Some of the cycles are listed in table 9.1 and drawn in fig. 19.3.

(Yuheng Lan)

Solution 1.2: Sensitivity to initial conditions. *To estimate the pinball sensitivity we consider a narrow beam of point particles bouncing between two disks, fig. O.1(a). Or if you find this easier to visualize, think of a narrow ray of light. We assume that the ray of light is focused along the axis between the two points. This is where the least unstable periodic orbit lies, so its stability should give us an upper bound on the number of bounces we can expect to achieve. To estimate the stability we assume that the ray of light has a width* $w(t)$ *and a "dispersion angle"* $\theta(t)$ *(we assume both are small), fig. O.1(b). Between bounces the dispersion angle stays constant while the width increases as*

$$w(t) \approx w(t') + (t - t')\theta$$

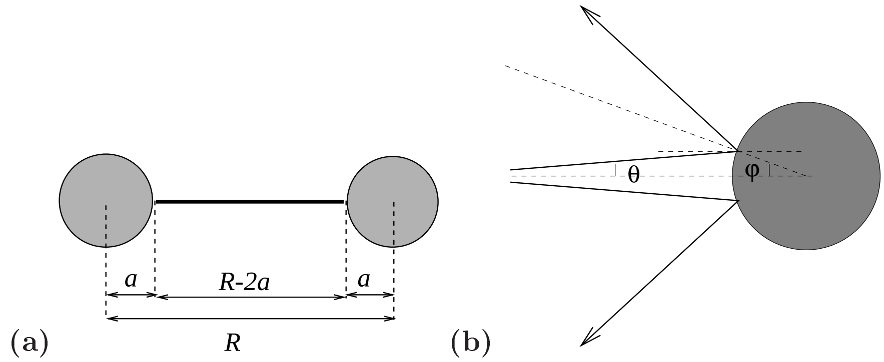

Figure O.1: The 2-disk pinball (a) geometry, (b) defocusing of scattered rays.

At each bounce the width stays constant while the angle increases by

$$\theta_{n+1} = \theta_n + 2\phi \approx \theta_n + w(t)/a.$$

where θ_n denotes the angle after bounce n. Denoting the width of the ray at the nth bounce by w_n then we obtain the pair of coupled equations

$$w_{n+1} = w_n + \left(R - 2a\right)\theta_n \tag{O.1}$$

$$\theta_n = \theta_{n-1} + \frac{w_n}{a} \tag{O.2}$$

where we ignore corrections of order w_n^2 and θ_n^2. Solving for θ_n we find

$$\theta_n = \theta_0 + \frac{1}{a}\sum_{j=1}^{n} w_n.$$

Assuming $\theta_0 = 0$ then

$$w_{n+1} = w_n + \frac{R-2a}{a}\sum_{j=1}^{n} w_n$$

Plugging in the values in the question we find the width at each bounce in Ångstrøms grows as 1, 5, 29, 169, 985, etc. To find the asymptotic behavior for a large number of bounces we try an solution of the form $w_n = ax^n$. Substituting this into the equation above and ignoring terms that do not grow exponentially we find solutions

$$w_n \approx a w_n^{asym} = a(3 \pm 2\sqrt{2})^n$$

The solution with the positive sign will clearly dominate. The constant a we cannot determine by this local analysis although it is clearly proportional to w_0. However, the *asymptotic solution is a good approximation even for quite a small number of bounces. To find an estimate of a we see that w_n/w_n^{asym} very rapidly converges to 0.146447, thus*

$$w_n \approx 0.146447 w_0 (3 + 2\sqrt{2})^n \approx 0.1 \times w_0 \times 5.83^n$$

The outside edges of the ray of light will miss the disk when the width of the ray exceeds 2 cm; this occurs after 11 bounces.

(Adam Prügel-Bennett)

Solution 1.2: Sensitivity to initial conditions, another try. *Adam's estimate is not very good - do you have a better one? The first problem with it is that the instability is very underestimated. As we shall check in exercise 5.5, the exact formula for the 2-cycle stability is $\Lambda = R - 1 + R\sqrt{1 - 2/R}$. For $R = 6$, $a = 1$ this yields $w_n/w_0 \approx (5 + 2\sqrt{6})^n = 9.898979^n$, so if that were the whole story, the pinball would be not likely to make it much beyond 8 bounces.*

The second problem is that local instability overestimates the escape rate from an enclosure; trajectories are reinjected by scatterers. In the 3-disk pinball the particle leaving a disk can be reinjected by hitting either of other 2 disks, hence $w_n/w_0 \approx (9.9/2)^n$. This interplay between local instability and global reinjection will be cast into the exact formula involving "Lyapunov exponent" and "Kolmogorov entropy". In order to relate this estimate to our best continuous time escape rate estimate $\gamma = 0.4103\ldots$ (see table 15.2), we will have to also compute the mean free flight time (15.20). As a crude estimate, we take the shortest disk-to-disk distance, $\langle T \rangle = R - 2 = 4$. The continuous time escape rate result implies that $w_n/w_0 \approx e^{(R-2)\gamma n} = (5.16)^n$, in the same ballpark as the above expansion-reinjection estimate.

(Predrag Cvitanović)

Chapter 2

Solution 2.1: Trajectories do not intersect. *Suppose that two trajectories C_x and C_y intersect at some point z. We claim that any points $\tilde{x}$ on C_x is also a point on C_y and vice versa. We only need to prove the first part of the statement. According to the definition of C_x, there exist $t_x, t_y, t_1 \in \mathbb{R}$ such that $f^{t_x}(x) = z, f^{t_y}(y) = z, f^{t_1}(x) = \tilde{x}$. It is easy to check that $f^{t_y - t_x + t_1}(y) = \tilde{x}$. So, $\tilde{x} \in C_y$. Therefore, if two trajectories intersect, then they are the same trajectory.*

(Yueheng Lan)

Solution 2.2: Evolution as a group. *Let's check the basic defining properties of a group. The members of the set are $f^t, t \in \mathbb{R}$ and the "product law" is given by '$\circ$'.*

- *As $f^{t+s} = f^t \circ f^s$, the set is closed,* i.e., *the product of any two members generates another member of the set.*

- *It is associative, as* $(f^t \circ f^s) \circ f^r = f^{t+s+r} = f^t \circ (f^s \circ f^r)$.
- $I = f^0$ *is the identity, as* $f^t \circ f^0 = f^t$.
- f^{-t} *is the inverse of* f^t*, as* $f^{-t} \circ f^t = I$.

So, $\{f^t, \circ\}_{t \in \mathbb{R}}$ *forms a group. As* $f^t \circ f^s = f^{t+s} = f^s \circ f^t$*, it is a commutative (Abelian) group.*
Any Abelian group can replace the continuous time. For example, $\mathbb{R}$ *can be replaced by* $\mathbb{Z}_6$*. To mess things up try a non-commutative group.*

(Yueheng Lan)

Solution 2.3: Almost ODE's. *What is an ODE on* $\mathbb{R}$ *? An ODE is an equality which reveals explicitly the relation between function* $x(t)$ *and its time derivatives* $\dot{x}, \ddot{x}, \cdots$, i.e., $F(t, x, \dot{x}, \ddot{x}, \cdots) = 0$ *for some given function* F*. Let's check the equations given in the exercise.*
(a) $\dot{x} = \exp(\dot{x})$ *is an ODE.*
(b) $\dot{x} = x(x(t))$ *is not an ODE, as* $x(x(t))$ *is not a known function acting on* $x(t)$*.*
(c) $\dot{x} = x(t+1)$ *is not an ODE, as* $x(t+1)$ *is not a value at current time. Actually, it is a difference-differential equation.*

(Yueheng Lan)

Solution 2.4: All equilibrium points are fixed points. *Given a vector field* $v(x)$*, the phase space dynamics is defined by*

$$\frac{d}{dt}x(t) = v(x(t))\,. \tag{O.3}$$

An equilibrium point a *of* v *is defined by* $v(a) = 0$*, so* $x(t) = a$ *is a constant solution of (O.3). For the flow* f^t *defined by (O.3), this solution satisfies* $f^t(a) = a\,, t \in \mathbb{R}$*. So, it is a fixed point of the dynamics* f^t*.*

(Yueheng Lan)

Solution 2.5: Gradient systems. *(a) The directional derivative*

$$\frac{d}{dn}\phi = n \cdot \nabla\phi$$

produces the increasing rate along the unit vector n*. So, along the gradient direction* $\nabla\phi/|\nabla\phi|$*,* ϕ *has the largest increasing rate. The velocity of the particle has the opposite direction to the gradient, so* ϕ *deceases most rapidly in the velocity direction.*

(b) An extremum a *of* ϕ *satisfies* $\nabla\phi(a) = 0$*. According to Exercise 2.4,* a *is a fixed point of the flow.*

(c) Two arguments lead to the same conclusion here.
First, near an equilibrium point, the equation is always linearizable. For gradient system, after orthogonal transformation it is even possible to write the linearized equation in diagonal form so that we need only to consider one eigendirection. The corresponding scalar equation is $\dot{x} = \lambda x$. Notice that we moved the origin to the equilibrium point. The solution of this equation is $x(t) = x(0)\exp(\lambda t),$ for $\lambda \neq 0.$ if $x(0) \neq 0$, it will take infinite amount of time (positive or negative) for $x(t) \to 0.$ For $\lambda = 0$, the approach to zero is even slower as then only higher orders of x take effect.
The second argument seems easier. We know that the solution curve through an equilibrium point is the point itself. According to Excercise 2.1, no other solution curve will intersect it, which means that if not starting from the equilibrium point itself, other point can never reach it.

(d) On a periodic orbit, the velocity is bounded away from zero. So ϕ is always decreasing on a periodic orbit, but in view of the periodicity, we know that this can not happen (at each point, there is only one value of ϕ.). So, there is no periodic orbit in a gradient system.

(Yueheng Lan)

Solution 2.9: Classical collinear helium dynamics. *An example of a solution are A. Prügel-Bennett's programs, available at* `www.nbi.dk/ChaosBook/extras`.