Chapter 9

Solution 9.1: Binary symbolic dynamics. *Read the text.*

Solution 9.2: 3-disk fundamental domain symbolic dynamics. *Read sect. 1.4.*

Solution 9.5: Reduction of 3-disk symbolic dynamics. *The answer is given in sect. 20.6. Some remarks concerning part (c):*

If an orbit does not have any spatial symmetry, its length in the fundamental domain is equal to that in the full space. One fundamental domain orbit corresponds to six copies of the orbit in the full space related to each other by symmetries. If a periodic orbit does have a spatial symmetry, then its fundamental domain image is a fraction of that in the whole space, and the orbit (and its symmetry pratnenrs) in the full space is tiled by copies of the irreducible segment, corresponding to an orbit in the fundamental domain. The higher symmetry an orbit has, the shorter the irreducible segment.

Another way to construct a fundamental domain orbit: put a periodic orbit and all its spatial symmetry relatives simultaneously in the full space. The segments that fall into a fundamental domain constitute the orbit in the fundamental domain.

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Solution 9.6: Unimodal map symbolic dynamics. *Hint: write down an arbitrary binary number such as $\gamma = .1101001101000\ldots$ and generate the future itinerary S^+ by checking whether $f^n(\gamma)$ is greater or less than 1/2. Then verify that (10.4) recovers γ.*

Solution 9.7: "Golden mean" pruned map.
(a) Consider the 3-cycle drawn in the figure. Denote the lengths of the two horizontal intervals by a and b. We have

$$\frac{a}{b} = \frac{b}{a+b}\,,$$

so the slope is given by the golden mean, $\Lambda = \frac{b}{a} = \frac{1+\sqrt{5}}{2}$, and the piece-wise linear map is given by

$$f(x) = \begin{cases} \Lambda x\,,\ x \in [0, 1/2] \\ \Lambda(1-x)\,,\ x \in [1/2, 1] \end{cases}$$

(b) Evaluate

$$f\left(\frac{1}{2}\right) \;=\; \frac{1+\sqrt{5}}{4}\,, \quad f\left(\frac{1+\sqrt{5}}{4}\right) = \frac{-1+\sqrt{5}}{4}\,, \quad f\left(\frac{-1+\sqrt{5}}{4}\right) = \frac{1}{2}\,.$$

Once a point enters the region covered by the interval $\mathcal{M}$ *of length* $a+b$*, bracketed by the 3-cycle, it will be trapped there forever. Outside* $\mathcal{M}$*, all points on unit interval will be mapped to* $(0, 1/2]$*, except for* 0*. The points in the interval* $(0, \frac{-1+\sqrt{5}}{4})$ *approach* $\mathcal{M}$ *monotonically.*

(c) It will be in $(\frac{1}{2}, \frac{1+\sqrt{5}}{4})$.

(d) From (b), we know that except for the origin 0*, all periodic orbits should be in* $\mathcal{M}$*. By (c), we cannot have the substring* 00 *in a periodic orbit (except for the fixed point at* 0*). Hence* 00 *is the only* pruning block*, and the symbolic dynamics is a finite subshift, with alphabet* $\{0, 1\}$ *and only one grammar rule: a consecutive repeat of symbol* 0 *is inadmissible).*

(e) Yes. 0 *is a periodic orbit with the symbol sequence* $\overline{0}$*. It is a repeller and no point in its neighborhood will return. So it plays no role in the asymptotic dynamics.*

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