Dimensional Analysis with $\hbar = c = 1$

(Answer questions (a)-(c))

We have set $\hbar = c = 1$. This allows us to convert a time $T$ to a length $L$ via $T = L/c$, and a length $L$ to a mass $M$ via $M = \hbar c^{-1}/L$. Thus any quantity $A$ can be thought of as having units of mass to some power (positive, negative, or zero) that we will call $[A]$. For example,

$$[m] = +1,$$  \hspace{1cm} (289)

$$[x^\mu] = -1,$$ \hspace{1cm} (290)

$$[\partial^\mu] = +1,$$ \hspace{1cm} (291)

$$[d^d x] = -d.$$ \hspace{1cm} (292)

In the last line, we have generalized our considerations to theories in $d$ spacetime dimensions.

Let us now consider a scalar field in $d$ spacetime dimensions with lagrangian density

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \sum_{n=3}^N \frac{1}{n!} g_n \varphi^n.$$ \hspace{1cm} (293)

The action is

$$S = \int d^d x \mathcal{L},$$ \hspace{1cm} (294)

and the path integral is

$$Z(J) = \int \mathcal{D}\varphi \exp\left[i \int d^d x (\mathcal{L} + J\varphi)\right].$$ \hspace{1cm} (295)

From eq. (295), we see that the action $S$ must be dimensionless, because it appears as the argument of the exponential function. Therefore

$$[S] = 0.$$ \hspace{1cm} (296)
If you got the dimensions right, for $\phi^3$ theory

$$[g_\phi] = \frac{\xi}{\delta} \quad \Rightarrow \quad \xi = 6 - d$$

So in general the coupling constant is "dimensionfull"

$$g_\phi = g \frac{\Lambda}{M}$$

"mass" scale

"dimensionless number"

At high energies, $p^2 >> m$ - rest mass

the only scale is $p^2$, so

$$g_\phi(p^2) \propto g |p^2|^{\xi/2}$$

what happens is

(d) $\xi < 0$ ?

(e) $\xi = 0$ 

(f) $\xi > 0$ ?

If you got this right, you want to study in more detail:

$\phi^3$ in 6 dimensions
1- Loop Corrections to the Propagator

The (connected) propagator is related to the 1-P-I propagator by P.C. "Field Theory" eq. (2.32):

\[ \tilde{\Delta}(k^2) = \frac{1}{k^2 + m^2 - \Pi(k^2)} \]

If in doubt, remember:

\[ m^2 \rightarrow m^2 - i\varepsilon \quad (306) \]

Physical mass-shell condition

\[ k^2 = -m^2 \] pole \( g \tilde{\Delta}(k^2) \)

consistent with eq. (306) if and only if

\[ \Pi(-m^2) = 0, \quad (307) \]

\[ \Pi'(-m^2) = 0, \quad (308) \]

One-loop contributions

\[ i\Pi(k^2) = \frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d} \tilde{\Delta}(\ell + k) \tilde{\Delta}(\ell) \]

\[ -i(Ak^2 + Bm^2) + O(g^4). \quad (303) \]

Here we have written the integral appropriate for \( d \) spacetime dimensions; for now we will leave \( d \) arbitrary, but later we will want to focus on \( d = 6 \), where the coupling \( g \) is dimensionless.
Prove the *Feynman's formula* to combine denominators,

\[ \frac{1}{a_1 \ldots a_n} = \int dF_n (x_1 a_1 + \ldots + x_n a_n)^{-n}, \quad (309) \]

where the integration measure over the *Feynman parameters* \( x_i \) is

\[ \int dF_n = (n-1)! \int_0^1 dx_1 \ldots dx_n \delta(x_1 + \ldots + x_n - 1). \quad (310) \]

Verify that the measure is normalized so that

\[ \int dF_n 1 = 1. \quad (311) \]

Eq. (309) can be proven by direct evaluation for \( n = 2 \), and by induction for \( n > 2 \).

(I prefer going to the Schwinger-exponential-parametric representation, where this formula is a triviality.)

Show that

\[ \Delta(k+\ell)\Delta(\ell) = \frac{1}{(\ell^2 + m^2)((\ell + k)^2 + m^2)} \]

\[ = \int_0^1 dx \left[ q^2 + D \right]^{-2}. \quad (312) \]

In the last line we have defined \( q \equiv \ell + xk \) and

\[ D \equiv x(1-x)k^2 + m^2. \quad (313) \]
Wick rotation:
evaluate \int_{-\infty}^{\infty}dq_0 \text{ integral by drawing a contour in the complex } q_0 \text{ plane avoiding the } m^2 - i\epsilon \text{ pole and closing it along } \oint dq_0 \text{. Define } q_0 = \bar{q}_d, \quad q_i = \bar{q}_i \text{ otherwise and a Euclidean vector } 
\left[ \bar{q}_1, \bar{q}_2, \ldots, \bar{q}_d \right] 
such that } q^2 = \bar{q}^2, \quad \text{and } dq = i dq', \text{ Check that (303) becomes}

(4)

\[ \Pi(k^2) = \frac{1}{2} g^2 I(k^2) - A k^2 - B m^2 + O(g^4), \quad (316) \]

where

\[ I(k^2) \equiv \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2}. \quad (317) \]
The overall result (generalized slightly for later use) is
\[
\int \frac{d^d \vec{q}}{(2\pi)^d} \frac{(\vec{q}^2)^a}{(\vec{q}^2 + D)^b} = \frac{\Gamma(b-a-\frac{1}{2}d)\Gamma(a+\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{1}{2}d)} D^{-(b-a-d/2)}.
\]

In the case of interest, eq. (317), we have \(a = 0\) and \(b = 2\).

### Useful Integrals:

You will find useful to know the following integrals:
\[
\frac{1}{A^n B^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx \frac{x^{n-1}(1-x)^{m-1}}{(xA + (1-x)B)^{n+m}} = \frac{1}{2} S_D \frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)} (m_0^2 - q^2)^{\frac{D}{2} - n}
\]

where \(S_D\) is the volume of the \(D\)-dimensional unit hypersphere
\[
S_D = \left[2^{D-1} \pi^{D/2} \frac{D}{2}\right]^{-1}
\]
and \(\Gamma(s)\) is the \(\Gamma\)-function
\[
\Gamma(s) = \int_0^\infty dt \, t^{s-1} e^{-t}
\]
For \(s \to 0\), the \(\Gamma\)-function behaves like
\[
\Gamma(s) = \frac{\Gamma(s+1)}{s} \approx \frac{1}{s} + \text{finite terms}
\]

\(\Gamma(x)\) is the Euler gamma function; for a nonnegative integer \(n\) and small \(x\),
\[
\Gamma(n+1) = n!,
\]
\[
\Gamma(n + \frac{1}{2}) = \frac{(2n)!}{n!2^n \sqrt{\pi}},
\]
\[
\Gamma(-n + x) = \frac{(-1)^n}{n!} \left[ \frac{1}{x} - \gamma + \sum_{k=1}^n \frac{1}{k} + O(x) \right],
\]
where \(\gamma = 0.5772\ldots\) is the Euler-Mascheroni constant.
We now return to eq. (317), use eq. (324), and set \( d = 6 - \varepsilon \); we get

\[
I(k^2) = \frac{\Gamma(-1+\frac{\varepsilon}{2})}{(4\pi)^{3/2}} \int_0^1 dx \frac{D}{D} \left( \frac{4\pi \mu^2}{\varepsilon D} \right)^{\varepsilon/2}.
\] (328)

Show that in \( \varepsilon \to 0 \) limit (\( \alpha = \frac{g^2}{(4\pi)^3} \)),

\[
\Pi(k^2) = -\frac{1}{2}\alpha \left[ \left( \frac{2}{3} + 1 \right) \left( \frac{1}{6}k^2 + m^2 \right) + \int_0^1 dx \frac{D}{D} \ln \frac{4\pi \mu^2}{\varepsilon D} \right] \]
\[
- Ak^2 - Bm^2 + O(\alpha^2).
\] (332)

we must still impose the conditions \( \Pi(-m^2) = 0 \) and \( \Pi'(-m^2) = 0 \). The easiest way to do this is to first note that, schematically,

\[
\Pi(k^2) = \frac{1}{2}\alpha \int_0^1 dx \ln D + \text{linear in } k^2 \text{ and } m^2 + O(\alpha^2).
\] (338)

We can then impose \( \Pi(-m^2) = 0 \) via

\[
\Pi(k^2) = \frac{1}{2}\alpha \int_0^1 dx \frac{D}{D_0} \ln \frac{D}{D_0} + \text{linear in } (k^2 + m^2) + O(\alpha^2).
\] (339)

where

\[
D_0 \equiv D \bigg|_{k^2=-m^2} = [1-x(1-x)]m^2.
\] (340)

Show that:

that \( \Pi'(-m^2) \) vanishes for

\[
\Pi(k^2) = \frac{1}{2}\alpha \int_0^1 dx \frac{D}{D_0} \ln \frac{D}{D_0} - \frac{1}{12}\alpha (k^2 + m^2) + O(\alpha^2).
\] (341)

This is our final formula for the \( O(\alpha) \) term in \( \Pi(k^2) \).

What have we learned: the physical "polarization" correction is finite, and we have its \( k^2 \) dependence.
Loop Corrections to the Vertex

We can define an exact three-point vertex function \( igV_3(k_1, k_2, k_3) \) as the sum of one-particle irreducible diagrams with three external lines carrying momenta \( k_1, k_2, \) and \( k_3\), all incoming, with \( k_1 + k_2 + k_3 = 0 \) by momentum conservation. (In adopting this convention, we allow \( k_i^0 \) to have either sign; if \( k_i \) is the momentum of an external particle, then the sign of \( k_i^0 \) is positive if the particle is incoming, and negative if it is outgoing.) The original vertex \( iZg \) is the first term in this sum,

\[
\Delta(\ell-k_1)\Delta(\ell+k_2)\Delta(\ell) = \int dF_3 \left[ q^2 + D \right]^{-3}.
\]

(357)

In the last line, we have defined \( q \equiv \ell - x_1 k_1 + x_2 k_2 \), and

\[
D \equiv x_1(1-x_1)k_1^2 + x_2(1-x_2)k_2^2 + 2x_1x_2k_1\cdot k_2 + m^2
= x_3 x_1 k_1^2 + x_1 x_2 k_2^2 + x_2 x_3 k_3^2 + m^2.
\]

(358)

After making a Wick rotation of the \( q^0 \) contour, we have

\[
V_3(k_1, k_2, k_3) = Zg + g^2 \int dF_3 \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 + D)^3} + O(g^4),
\]

(359)

Derive this relation.

\[
\int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 + D)^3} = \frac{\Gamma(3-\frac{3}{2}d)}{2(4\pi)^{d/2}} D^{-(3-d/2)}.
\]

(360)

Then we set \( d = 6 - \varepsilon \). To keep \( g \) dimensionless, we make the replacement \( g \rightarrow g \tilde{\mu}^{\varepsilon/2} \). Then we have

\[
V_3(k_1, k_2, k_3) = Zg + \frac{1}{2} \alpha \Gamma(\frac{\varepsilon}{2}) \int dF_3 \left( \frac{4\pi \tilde{\mu}^2}{D} \right)^{\varepsilon/2} + O(\alpha^2),
\]

(361)
take the $\varepsilon \to 0$ limit. The result is

$$V_3(k_1, k_2, k_3) = Z_g + \frac{1}{2} \alpha \left[ \frac{2}{\varepsilon} + \int dF_3 \ln \left( \frac{4\pi \bar{\mu}^2}{e^2 D} \right) \right] + O(\alpha^2), \quad (362)$$

where we have used $\int dF_3 = 1$. We use $\mu^2 = 4\pi e^2 \bar{\mu}^2$, set

$$Z_g = 1 + C, \quad (363)$$

and rearrange to get

$$V_3(k_1, k_2, k_3) = 1 + \left\{ \alpha \left[ \frac{1}{\varepsilon} + \ln(\mu/m) \right] + C \right\} - \frac{1}{2} \alpha \int dF_3 \ln(D/m^2) + O(\alpha^2). \quad (364)$$

If we take $C$ to have the form

$$C = -\alpha \left[ \frac{1}{\varepsilon} + \ln(\mu/m) + \kappa_C \right] + O(\alpha^2), \quad (365)$$

where $\kappa_C$ is a purely numerical constant, then we get

$$V_3(k_1, k_2, k_3) = 1 - \frac{1}{2} \alpha \int dF_3 \ln(D/m^2) - \kappa_C \alpha + O(\alpha^2). \quad (366)$$

Thus this choice of $C$ renders $V_3(k_1, k_2, k_3)$ finite and independent of $\mu$, as required.

We now need a condition, analogous to $\Pi(-m^2) = 0$ and $\Pi'(-m^2) = 0$, to fix the value of $\kappa_C$. These conditions on $\Pi(k^2)$ were mandated by known properties of the exact propagator, but there is nothing directly comparable for the vertex. Different choices of $\kappa_C$ correspond to different definitions of the coupling $g$. This is because, in order to measure $g$, we would measure a cross section that depends on $g$; these cross sections also depend on $\kappa_C$. Thus we can use any value for $\kappa_C$ that we might fancy, as long as we all agree on that value when we compare our calculations with experimental measurements. It is then most convenient to simply set $\kappa_C = 0$. This corresponds to the condition

$$V_3(0, 0, 0) = 1. \quad (367)$$

This condition can then be used to fix the higher-order (in $g$) terms in $Z_g$. 

∞ have a good summer ∞