Exercises

Exercise 28.1 **Who ordered $\sqrt{\pi}$?** *Derive the Gaussian integral*

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2a}} = \sqrt{a}\,, \qquad a > 0\,.$$

assuming only that you know to integrate the exponential function e^{-x}. Hint, hint: x^2 is a radius-squared of something. π is related to the area or circumference of something.

Exercise 28.2 **Dirac delta function, Gaussian representation.** *Consider the Gaussian distribution function*

$$\delta_\sigma(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/2\sigma^2}\,.$$

Show that in $\sigma \to 0$ limit this is the Dirac delta function

$$\int_{\mathcal{M}} dx\, \delta(x) = 1 \text{ if } 0 \in \mathcal{M}\,, \text{ zero otherwise}\,.$$

Exercise 28.3 **D-dimensional Gaussian integrals.** *Show that the Gaussian integral in D-dimensions is given by*

$$\frac{1}{(2\pi)^{d/2}} \int d^d x e^{-\frac{1}{2}x^T \cdot M^{-1} \cdot x + x \cdot J} \quad = \quad |\det M|^{\frac{1}{2}}\, e^{\frac{1}{2}J^T \cdot M \cdot J}\,, \tag{28.53}$$

where M is a real positive definite $[d \times d]$ matrix, that is a matrix with strictly positive eigenvalues. x, J are D-dimensional vectors, and x^T is the transpose of x.

Exercise 28.4 **Stationary phase approximation in higher dimensions.** *All semiclassical approximations are based on saddlepoint evaluations of integrals of type*

$$I = \int d^D x A(x) e^{i\Phi(x)/\hbar} \tag{28.54}$$

for small values of $\hbar$. Obtain the stationary phase estimate

$$I \approx \sum_n A(x_n) e^{i\Phi(x_n)/\hbar} \frac{(2\pi i\hbar)^{D/2}}{\sqrt{\det \mathbf{D}^2\Phi(x_n)}},$$

where $\mathbf{D}^2\Phi(x_n)$ denotes the second derivative matrix.