Exercises

Exercise 27.1 **Schrödinger equation in the Madelung form.** *Verify the decomposition of Schrödinger equation into real and imaginary parts, eqs. (27.3) and (27.4).*

Exercise 27.2 **Transport equations.** *Write the wave-function in the asymptotic form*

$$\psi(q,t) = e^{\frac{i}{\hbar}R(x,t)+\frac{i}{\hbar}\varepsilon t}\sum_{n\geq 0}(i\hbar)^n A_n(x,t)\,.$$

Derive the transport equations for the A_n by substituting this into the Schrödinger equation and then collecting terms by orders of $\hbar$. Notice that equation for $\dot{A}_n$ only requires knowledge of A_{n-1} and R.

Exercise 27.3 **Easy examples of the Hamilton's principal function.** *Calculate $R(q,q',t)$ for*

a) a D-dimensional free particle

b) a 3-dimensional particle in constant magnetic field

c) a 1-dimensional harmonic oscillator.

(Continuation: exercise 27.10.)

Exercise 27.4 **Dirac delta function, Gaussian representation.** *Consider the Gaussian distribution function*

$$\delta_\sigma(z) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-z^2/2\sigma^2}\,.$$

Show that in $\sigma \to 0$ limit this is the Dirac delta function (27.24).

Exercise 27.5 **D-dimensional Gaussian integrals.** *Show that the Gaussian integral in D-dimensions is given by*

$$\frac{1}{(2\pi)^{d/2}}\int d^dx e^{-\frac{1}{2}x^T\cdot M^{-1}\cdot x + x\cdot J} \;=\; |\det M|^{\frac{1}{2}}\, e^{\frac{1}{2}J^T\cdot M\cdot J}\,, \qquad (27.52)$$

where M is a real positive definite $[d\times d]$ matrix, that is a matrix with strictly positive eigenvalues. x, J are D-dimensional vectors, and x^T is the transpose of x.

Exercise 27.6 **Stationary phase approximation in higher dimensions.** *All semiclassical approximations are based on saddlepoint evaluations of integrals of type*

$$I = \int d^D x A(x) e^{i\Phi(x)/\hbar} \tag{27.53}$$

for small values of $\hbar$. Obtain the stationary phase estimate

$$I \approx \sum_n A(x_n) e^{i\Phi(x_n)/\hbar} \frac{(2\pi i\hbar)^{D/2}}{\sqrt{\det \mathbf{D}^2\Phi(x_n)}},$$

where $\mathbf{D}^2\Phi(x_n)$ denotes the second derivative matrix.

Exercise 27.7 **D-dimensional free particle propagator.** *Verify the results in sect. 27.2.2; show explicitely that (27.33), the semiclassical Van Vleck propagator in D dimensions, solves the Schrödinger's equation.*

Exercise 27.8 **Propagator, charged particle in constant magnetic field.** *Calculate the semiclassical propagator for a charged particle in constant magnetic field in 3 dimensions. Verify that the semiclassical expression coincides with the exact solution.*

Exercise 27.9 **1-dimensional harmonic oscillator propagator.** *Calculate the semiclassical propagator for a 1-dimensional harmonic oscillator and verify that it is identical to the exact quantum propagator.*

Exercise 27.10 **Free particle action.** *Calculate the energy dependent action for a free particle, a charged particle in a constant magnetic field and for the harmonic oscillator.*

Exercise 27.11 **Zero length orbits.** *Derive the classical trace (12.1) rigorously and either add the $t \to 0_+$ zero length contribution to the trace formula, or show that it vanishes. Send us a reprint of* Phys. Rev. Lett. *with the correct derivation.*

Exercise 27.12 **A usefull determinant identity.** *Show that the following two determinants equal each other:*
$A[(n+1)\times(n+?)]determinant$

$$\det\left(M'_n\right) = \begin{matrix} \\ \\ \\ \\ n+1 \end{matrix} \overbrace{\begin{bmatrix} x_{1,1} & \dots & x_{1,n} & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{n,1} & \dots & x_{n,n} & y_n \\ z_1 & \dots & z_n & E \end{bmatrix}}^{n+1} \tag{27.54}$$

and the $[n \times n]$ *determinant:*

$$E \det(M_n) = E \begin{matrix} \\ n \end{matrix} \overbrace{\begin{bmatrix} x_{1,1} - y_1 z_1 E^{-1} & \dots & x_{1,n} - y_1 z_n E^{-1} \\ \vdots & \ddots & \vdots \\ x_{n,1} - y_n z_1 E^{-1} & \dots & x_{n,n} - y_n z_n E^{-1} \end{bmatrix}}^{n} \quad (27.55)$$

Exercise 27.13 Free particle semiclassical Green's functions. *Calculate the semiclassical Green's functions for the systems of exercise 27.10.*