Appendix N

Solutions

Chapter 1

Solution 1.1: 3-disk symbolic dynamics. *As explained in sect. 1.4, each orbit segment can be characterized by either of the two symbols* 0 *and* 1*, differentiating topologically bouncing back or going onto the third disk. So, there are* 2^n *topologically different orbits starting from each disk. Altogether, the 3-disk pinball has* $3 \cdot 2^n$ *itineraries of length* n*.*
Periodic orbits (prime cycles in fundamental domain)

- *Of length* 2*: 12,13,32; or (0).*
- *Of length* 3*: 123,321; or (1).*
- *Of length* 4*: 1213,2321,3231; or (01).*
- *Of length* 5*: 12123,13132,23231,21213,32321,31312; or (00111).*

Some of the cycles are listed in table 11.2 and drawn in figure 22.3.

(Yuheng Lan)

Solution 1.2: Sensitivity to initial conditions. *To estimate the pinball sensitivity we consider a narrow beam of point particles bouncing between two disks, figure N.1(a). Or if you find this easier to visualize, think of a narrow ray of light. We*

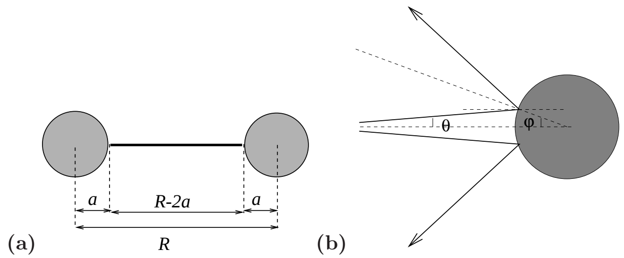


Figure N.1: The 2-disk pinball (a) geometry, (b) defocusing of scattered rays.

assume that the ray of light is focused along the axis between the two points. This is where the least unstable periodic orbit lies, so its stability should give us an upper bound on the number of bounces we can expect to achieve. To estimate the stability we assume that the ray of light has a width $w(t)$ *and a "dispersion angle"* $\theta(t)$ *(we assume both are small), figure N.1(b). Between bounces the dispersion angle stays constant while the width increases as*

$$w(t) \approx w(t') + (t - t')\theta$$

At each bounce the width stays constant while the angle increases by

$$\theta_{n+1} = \theta_n + 2\phi \approx \theta_n + w(t)/a.$$

where θ_n *denotes the angle after bounce* n*. Denoting the width of the ray at the* n*th bounce by* w_n *then we obtain the pair of coupled equations*

$$w_{n+1} = w_n + \left(R - 2a\right)\theta_n \tag{N.1}$$

$$\theta_n = \theta_{n-1} + \frac{w_n}{a} \tag{N.2}$$

where we ignore corrections of order w_n^2 *and* θ_n^2*. Solving for* θ_n *we find*

$$\theta_n = \theta_0 + \frac{1}{a}\sum_{j=1}^{n} w_n.$$

Assuming $\theta_0 = 0$ *then*

$$w_{n+1} = w_n + \frac{R-2a}{a}\sum_{j=1}^{n} w_n$$

Plugging in the values in the question we find the width at each bounce in Ångstrøms grows as 1, 5, 29, 169, 985, etc. To find the asymptotic behavior for a large number of bounces we try an solution of the form $w_n = ax^n$*. Substituting this into the equation above and ignoring terms that do not grow exponentially we find solutions*

$$w_n \approx a w_n^{asym} = a(3 \pm 2\sqrt{2})^n$$

The solution with the positive sign will clearly dominate. The constant a *we cannot determine by this local analysis although it is clearly proportional to* w_0*. However, the asymptotic solution is a good approximation even for quite a small number of bounces. To find an estimate of* a *we see that* w_n/w_n^{asym} *very rapidly converges to 0.146447, thus*

$$w_n \approx 0.146447 w_0 (3 + 2\sqrt{2})^n \approx 0.1 \times w_0 \times 5.83^n$$

The outside edges of the ray of light will miss the disk when the width of the ray exceeds 2 cm; this occurs after 11 bounces.

(Adam Prügel-Bennett)

Solution 1.2: Sensitivity to initial conditions, another try. *Adam's estimate is not very good - do you have a better one? The first problem with it is that the instability is very underestimated. As we shall check in exercise 8.1, the exact formula for the 2-cycle stability is* $\Lambda = R - 1 + R\sqrt{1-2/R}$*. For* $R = 6$*,* $a = 1$ *this yields* $w_n/w_0 \approx (5+2\sqrt{6})^n = 9.898979^n$*, so if that were the whole story, the pinball would be not likely to make it much beyond 8 bounces.*

The second problem is that local instability overestimates the escape rate from an enclosure; trajectories are reinjected by scatterers. In the 3-disk pinball the particle leaving a disk can be reinjected by hitting either of other 2 disks, hence $w_n/w_0 \approx (9.9/2)^n$*. This interplay between local instability and global reinjection will be cast into the exact formula involving "Lyapunov exponent" and "Kolmogorov entropy". In order to relate this estimate to our best continuous time escape rate estimate* $\gamma = 0.4103\ldots$ *(see table 18.2), we will have to also compute the mean free flight time (18.20). As a crude estimate, we take the shortest disk-to-disk distance,* $\langle T\rangle = R - 2 = 4$*. The continuous time escape rate result implies that* $w_n/w_0 \approx e^{(R-2)\gamma n} = (5.16)^n$*, in the same ballpark as the above expansion-reinjection estimate.*

(Predrag Cvitanović)

Chapter 2

Solution 2.1: Trajectories do not intersect. *Suppose that two trajectories* C_x *and* C_y *intersect at some point* z*. We claim that any points* $\tilde{x}$ *on* C_x *is also a point on* C_y *and vice versa. We only need to prove the first part of the statement. According to the definition of* C_x*, there exist* $t_x, t_y, t_1 \in \mathbb{R}$ *such that* $f^{t_x}(x) = z, f^{t_y}(y) = z, f^{t_1}(x) = \tilde{x}$*. It is easy to check that* $f^{t_y - t_x + t_1}(y) = \tilde{x}$*. So,* $\tilde{x} \in C_y$*. Therefore, if two trajectories intersect, then they are the same trajectory.*

(Yueheng Lan)

Solution 2.2: Evolution as a group. *Let's check the basic defining properties of a group. The members of the set are* $f^t, t \in \mathbb{R}$ *and the "product law" is given by '*$\circ$*'.*

- *As* $f^{t+s} = f^t \circ f^s$*, the set is closed, i.e., the product of any two members generates another member of the set.*
- *It is associative, as* $(f^t \circ f^s) \circ f^r = f^{t+s+r} = f^t \circ (f^s \circ f^r)$*.*
- $I = f^0$ *is the identity, as* $f^t \circ f^0 = f^t$*.*
- f^{-t} *is the inverse of* f^t*, as* $f^{-t} \circ f^t = I$*.*

So, $\{f^t, \circ\}_{t\in\mathbb{R}}$ *forms a group. As* $f^t \circ f^s = f^{t+s} = f^s \circ f^t$*, it is a commutative (Abelian) group.*

Any Abelian group can replace the continuous time. For example, $\mathbb{R}$ *can be replaced by* $\mathbb{Z}_6$*. To mess things up try a non-commutative group.*

(Yueheng Lan)

Solution 2.3: Almost ODE's. *What is an ODE on* $\mathbb{R}$ *? An ODE is an equality which reveals explicitly the relation between function* $x(t)$ *and its time derivatives* $\dot{x}, \ddot{x}, \cdots$*, i.e.,* $F(t, x, \dot{x}, \ddot{x}, \cdots) = 0$ *for some given function* F*. Let's check the equations given in the exercise.*
(a) $\dot{x} = \exp(\dot{x})$ *is an ODE.*
(b) $\dot{x} = x(x(t))$ *is not an ODE, as* $x(x(t))$ *is not a known function acting on* $x(t)$*.*
(c) $\dot{x} = x(t+1)$ *is not an ODE, as* $x(t+1)$ *is not a value at current time. Actually, it is a difference-differential equation.*

(Yueheng Lan)

Solution 2.4: All equilibrium points are fixed points. *Given a vector field* $v(x)$*, the phase space dynamics is defined by*

$$\frac{d}{dt}x(t) = v(x(t))\,. \tag{N.3}$$

An equilibrium point a *of* v *is defined by* $v(a) = 0$*, so* $x(t) = a$ *is a constant solution of (N.3). For the flow* f^t *defined by (N.3), this solution satisfies* $f^t(a) = a\,, t \in \mathbb{R}$*. So, it is a fixed point of the dynamics* f^t*.*

(Yueheng Lan)

Solution 2.5: Gradient systems.

1. *The directional derivative*

   $$\frac{d}{dn}\phi = n \cdot \nabla\phi$$

   produces the increasing rate along the unit vector n*. So, along the gradient direction* $\nabla\phi/|\nabla\phi|$*,* ϕ *has the largest increasing rate. The velocity of the particle has the opposite direction to the gradient, so* ϕ *deceases most rapidly in the velocity direction.*

2. *An extremum* a *of* ϕ *satisfies* $\nabla\phi(a) = 0$*. According to exercise 2.4,* a *is a fixed point of the flow.*

3. *Two arguments lead to the same conclusion here.*

   First, near an equilibrium point, the equation is always linearizable. For gradient system, after orthogonal transformation it is even possible to write the linearized equation in diagonal form so that we need only to consider one eigendirection. The corresponding scalar equation is $\dot{x} = \lambda x$*. Notice that we moved the origin to the equilibrium point. The solution of this equation is* $x(t) = x(0)\exp(\lambda t)$*, for* $\lambda \neq 0$*. if* $x(0) \neq 0$*, it will take infinite amount of time (positive or negative) for* $x(t) \to 0$*. For* $\lambda = 0$*, the approach to zero is even slower as then only higher orders of* x *take effect.*

   The second argument seems easier. We know that the solution curve through an equilibrium point is the point itself. According to exercise 2.1, no other solution curve will intersect it, which means that if not starting from the equilibrium point itself, other point can never reach it.

4. *On a periodic orbit, the velocity is bounded away from zero. So* ϕ *is always decreasing on a periodic orbit, but in view of the periodicity, we know that this can not happen (at each point, there is only one value of* ϕ*.). So, there is no periodic orbit in a gradient system.*

(Yueheng Lan)

Solution 2.10: Classical collinear helium dynamics. *An example of a solution are A. Prügel-Bennett's programs, available at* `ChaosBook.org/extras`.

Solution 2.8: Equilibria of the Rössler system.

1. *Solve* $\dot{x} = \dot{y} = \dot{z} = 0$, *to get* $x = az$, $y = -z$ *and* $x^2 - cx + ab = 0$. *There are two solutions of a quadratic equation, hence there are two equilibrium points:*

$$x^{\pm} = az^{\pm} = -ay^{\pm} = (c \pm \sqrt{c^2 - 4ab})/2\,. \tag{N.4}$$

2. *That above expressions are exact. However, it pays to think of* $\epsilon = a/c$ *as a small parameter in the problem. By subsitution from (2.23),*

$$x^{\pm} = cp^{\pm}, \quad y^{\pm} = -p^{\pm}/\epsilon, \quad z^{\pm} = p^{\pm}/\epsilon. \tag{N.5}$$

Expanding $\sqrt{D}$ *in* ϵ *yields* $p^- = \epsilon^2 + o(\epsilon^3)$, *and* $p^+ = 1 - \epsilon^2 + o(\epsilon^3)$. *Hence*

$$\begin{array}{ll} x^- = a^2/c + o(\epsilon^3), & x^+ = c - a^2/c + o(\epsilon^3), \\ y^- = -a/c + o(\epsilon^2), & z^+ = c/a + a/c + o(\epsilon^2), \\ z^- = a/c + o(\epsilon^2), & z^+ = c/a - a/c + o(\epsilon^2). \end{array} \tag{N.6}$$

For $a = b = 0.2$, $c = 5.7$ *in (2.14),* $\epsilon \approx 0.035$, *so*

$$\begin{array}{l} (x^-, y^-, z^-) = (\,0.0070,\ -0.0351,\ 0.0351\,)\,, \\ (x^+, y^+, z^+) = (\,5.6929,\ -28.464,\ 28.464\,)\,. \end{array} \tag{N.7}$$

(Rytis Paškauskas)

Chapter 3

(No solutions available.)

Chapter 4

Solution 4.1: Trace-log of a matrix. *1) one method is to first check that this is true for any Hermitian matrix* M. *Then write an arbitrary complex matrix as sum* $M = A + zB$, A, B *Hermitian, Taylor expand in* z *and prove by analytic continuation that the identity applies to arbitrary* M. *(David Mermin)*

2) another method: evaluate $\frac{d}{dt}\det\left(e^{t \ln M}\right)$ *by definition of derivative in terms of infinitesimals. (Kasper Juel Eriksen)*

3) check appendix K.1

4) This identity makes sense for a matrix $M \subset \mathbb{C}^{n\times n}$, *if* $|\prod_{i=1}^n \lambda_i| < \infty$ *and* $\{|\lambda_i| > 0, \forall i\}$, *where* $\{\lambda_i\}$ *is a set of eigenvalues of* M. *Under these conditions there exist a nonsingular* $O : M = ODO^{-1}$, $D = \mathrm{diag}[\{\lambda_i, i = 1, \ldots, n\}]$. *If* $f(M)$ *is a*

matrix valued function defined in terms of power series then $f(M) = Of(D)O^{-1}$, *and* $f(D) = \mathrm{diag}[\{f(\lambda_i)\}]$. *Using these properties and cyclic property of the trace we obtain*

$$\exp(\mathrm{tr}\,(\ln M)) = \exp\left(\sum_i \ln \lambda_i\right) = \prod_i \lambda_i = \det(M)$$

5) Consider $M = \exp A$.

$$\det M = \det \lim_{n\to\infty}\left(\mathbf{1} + \frac{1}{n}A\right)^n = \lim_{n\to\infty}(\mathbf{1} + \frac{1}{n}\mathrm{tr}\,A + \ldots)^n = \exp(\mathrm{tr}\,(\ln M))$$

Solution 4.2: Stability, diagonal case. *The relation (4.16) can be verified by noting that the defining product (4.11) can be rewritten as*

$$\begin{aligned} e^{t\mathbf{A}} &= \left(\mathbf{U}\mathbf{U}^{-1} + \frac{t\mathbf{U}\mathbf{A}_D\mathbf{U}^{-1}}{m}\right)\left(\mathbf{U}\mathbf{U}^{-1} + \frac{t\mathbf{U}\mathbf{A}_D\mathbf{U}^{-1}}{m}\right)\cdots \\ &= \mathbf{U}\left(I + \frac{t\mathbf{A}_D}{m}\right)\mathbf{U}^{-1}\mathbf{U}\left(I + \frac{t\mathbf{A}_D}{m}\right)\mathbf{U}^{-1}\cdots = \mathbf{U}e^{t\mathbf{A}_D}\mathbf{U}^{-1}\,. \end{aligned} \tag{N.8}$$

Solution 4.3: Topology of the Rössler flow.

1. *The characteristic determinant of the matrix of variations that yields the equilibrium point stability (4.24) yields*

$$\begin{vmatrix} -\lambda & -1 & -1 \\ 1 & a - \lambda & 0 \\ z^{\pm} & 0 & x^{\pm} - c - \lambda \end{vmatrix} = 0$$

$$\lambda^3 + \lambda^2(-a - x^{\pm} + c) + \lambda(a(x^{\pm} - c) + 1 + x^{\pm}/a) + c - 2x^{\pm} = 0\,.$$

Equation (4.42) follows after noting that $x^{\pm} - c = c(p^{\pm} - 1) = -cp^{\mp}$ *and* $2x^{\pm} - c = c(2p^{\pm} - 1) = \pm c\sqrt{D}$, *see (2.8).*

2. *Approximate solutions of (4.42) are obtained by expanding* $p^{\pm}$ *and* $\sqrt{D}$ *and substituting into this equation. Namely,*

$$\begin{array}{ll} \sqrt{D} & = 1 - 2\epsilon^2 - 2\epsilon^4 - 4\epsilon^6 - \ldots \\ p^- & = \epsilon^2 + \epsilon^4 + 2\epsilon^6 + \ldots \\ p^+ & = 1 - \epsilon^2 - \epsilon^4 - 2\epsilon^6 + \ldots \end{array}$$

In case of the equilibrium "–", close to the origin expansion of (4.42) results in

$$(\lambda^2 + 1)(\lambda + c) = -\epsilon\lambda(1 - c^2 - c\lambda) + \epsilon^2 c(\lambda^2 + 2) + o(\epsilon^2)$$

The term on the left-hand side suggests the expansion for eigenvalues as

$$\lambda_1 = -c + \epsilon a_1 + \ldots\,, \quad \lambda_2 + i\theta_2 = \epsilon b_1 + i + \ldots \quad .$$

after some algebra one finds the first order correction coefficients $a_1 = c/(c^2+1)$ *and* $b_1 = (c^3 + i)/(2(c^2 + 1))$. *Numerical values are* $\lambda_1 \approx -5.694$, $\lambda_2 + i\theta_2 \approx 0.0970 + i1.0005$.

In case of p^+, the leading order term in (4.42) is $1/\epsilon$. Set $x = \lambda/\epsilon$, then expansion of (4.42) results in

$$x = c - \epsilon x - \epsilon^2(2c - x) - \epsilon^3(x^3 - cx^2) - \epsilon^4(2c - x(1+c^2) + cx^2) + o(\epsilon^4)$$

Solve for real eigenvalue first. Set $x = c + \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 + \epsilon^4 a_4 + \ldots$. The subtle point here is that leading order correction term of the real eigenvalue is ϵa_1, but to determine leading order of the real part of complex eigenvalue, one needs all terms a_1 through a_4.

Collecting powers of ϵ results in

$$\begin{array}{lll} \epsilon: & a_1 + c = 0 & a_1 = -c \\ \epsilon^2: & c + a_1 + a_2 = 0 & a_2 = 0 \\ \epsilon^3: & a_1 - a_2 - a_3 = 0 & a_3 = -c \\ \epsilon^4: & c + c^2 a_1 - a_2 + a_3 + a_4 = 0 & a_4 = c^3 . \end{array}$$

hence

$$\lambda_1 = \epsilon x = a - a^2/c + o(\epsilon^3) \approx 0.192982 .$$

To calculate the complex eigenvalue, one can make use of identities $\det A = \prod \lambda = 2x^+ - c$, and $\mathrm{tr}\, A = \sum \lambda = a + x^+ - c$. Namely,

$$\lambda_2 = \frac{1}{2}\left(a - cp^- - \lambda_1\right) = -\frac{a^5}{2c^2} + o(\epsilon^5) \approx -0.49 \times 10^{-6} ,$$

$$\theta_2 = \sqrt{\frac{2x^+ - c}{\lambda_1} - \lambda_2^2} = \sqrt{\frac{a+c}{a}}\left(1 + o(\epsilon)\right) \approx 5.431 .$$

(Rytis Paškauskas)

Chapter 5

(No solutions available.)

Chapter 6

Solution 6.1: A pinball simulator. *Examples of pretty pinballs are A. Prügel-Bennett's xpinball.c and W. Benfold's java programs, available at* `ChaosBook.org/extras`

Solution 6.4: Billiard exercises. *Korsch and Jodl [1.9] have a whole book of numerical exercises with billiards, including 3-disks.*

Chapter 7

Solution 7.2: Linearization for maps. *(difficulty: medium) The first few terms of the map h that conjugates f to αz*

$$f(z) = h^{-1}(\alpha h(z)) .$$

are determined many places, for example in ref. [9.5].

There are conditions on the derivative of f at the origin to assure that the conjugation is always possible. These conditions are formulated in ref. [1.14], among others.

Chapter 8

(No solutions available.)

Chapter 9

Solution 9.1: Integrating over Dirac delta functions. *(a) Whenever $h(x)$ crosses 0 with a nonzero velocity ($\det \partial_x h(x) \neq 0$), the delta function contributes to the integral. Let $x_0 \in h^{-1}(0)$. Consider a small neighborhood V_0 of x_0 so that $h : V_0 \to V_0$ is a one-to-one map, with the inverse function $x = x(h)$. By changing variable from x to h, we have*

$$\begin{aligned} \int_{V_0} dx\, \delta(h(x)) &= \int_{h(V_0)} dh\, |\det \partial_h x|\, \delta(h) = \int_{h(V_0)} dh\, \frac{1}{|\det \partial_x h|} \delta(h) \\ &= \frac{1}{|\det \partial_x h|_{h=0}} . \end{aligned}$$

Here, the absolute value $|\cdot|$ is taken because delta function is always positive and we keep the orientation of the volume when the change of variables is made. Therefore all the contributions from each point in $h^{-1}(0)$ add up to the integral

$$\int_{\mathbb{R}^d} dx\, \delta(h(x)) = \Sigma_{x \in h^{-1}(0)} \frac{1}{|\det \partial_x h|} .$$

Notice that if $\det \partial_x h = 0$, then the delta function integral is not well defined.

(b) The formal expression can be written as the limit

$$F := \int_{\mathbb{R}} dx\, \delta(x^2) = \lim_{\sigma \to 0} \int_{\mathbb{R}} dx\, \frac{e^{-\frac{x^4}{2\sigma}}}{\sqrt{2\pi\sigma}} ,$$

by invoking the approximation given in the exercise. The change of variable $y = x^2/\sqrt{\sigma}$ gives

$$F = \lim_{\sigma \to 0} \sigma^{-3/4} \int_{\mathbb{R}^+} dy\, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi y}} = \infty ,$$

where $\mathbb{R}^+$ represents the positive part of the real axis. So, the formal expression does not make sense. Notice that x^2 has a zero derivative at $x = 0$, which invalidates the expression in (a).

(Yueheng Lan)

Solution 9.2: Derivatives of Dirac delta functions. *We do this problem just by direct evaluation. We denote by Ω_y a sufficiently small neighborhood of y.*
(a)

$$\begin{aligned}\int_{\mathbb{R}} dx\, \delta'(y) &= \Sigma_{x\in y^{-1}(0)} \int_{\Omega_y} dy \det(\frac{dy}{dx})^{-1}\delta'(y) \\ &= \Sigma_{x\in y^{-1}(0)} \frac{\delta(y)}{|y'|}|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \frac{\delta(y)}{y'^2}(-y'')\frac{1}{y'} \\ &= \Sigma_{x\in y^{-1}(0)} \frac{y''}{|y'|y'^2}\,,\end{aligned}$$

where the absolute value is taken to take care of the sign of the volume.

(b)

$$\begin{aligned}\int_{\mathbb{R}} dx\, \delta^{(2)}(y) &= \Sigma_{x\in y^{-1}(0)} \int_{\Omega_y} dy\, \frac{\delta^{(2)}(y)}{y'} \\ &= \Sigma_{x\in y^{-1}(0)} \frac{\delta'(y)}{|y'|}|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \frac{\delta'(y)}{y'^2}(-y'')\frac{1}{y'} \\ &= \Sigma_{x\in y^{-1}(0)} \frac{y''\delta(y)}{|y'|y'^2}|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \delta(y)\frac{d}{dx}(\frac{y''}{y'^3})\frac{1}{y'} \\ &= \Sigma_{x\in y^{-1}(0)} - \int_{\Omega_y} dy\, \delta(y)\left(\frac{y'''}{y'^3} - 3\frac{y''^2}{y'^4}\right)\frac{1}{y'} \\ &= \Sigma_{x\in y^{-1}(0)} \left(3\frac{y''^2}{y'^4} - \frac{y'''}{y'^3}\right)\frac{1}{|y'|}\,.\end{aligned}$$

(c)

$$\begin{aligned}\int_{\mathbb{R}} dx\, b(x)\delta^{(2)}(y) &= \Sigma_{x\in y^{-1}(0)} \int_{\Omega_y} dy\, b(x)\frac{\delta^{(2)}(y)}{y'} \\ &= \Sigma_{x\in y^{-1}(0)} \frac{b(x)\delta'(y)}{|y'|}|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \delta'(y)\frac{d}{dx}(\frac{b}{y'})\frac{1}{y'} \\ &= \Sigma_{x\in y^{-1}(0)} - \delta(y)\frac{d}{dx}(\frac{b}{y'})\frac{1}{y'}|_{-\epsilon}^{\epsilon} + \int_{\Omega_y} dy\, \delta(y)\frac{d}{dx}(\frac{d}{dx}(\frac{b}{y'})\frac{1}{y'})\frac{1}{y'} \\ &= \Sigma_{x\in y^{-1}(0)} \frac{1}{|y'|}\frac{d}{dx}(\frac{b'}{y'^2} - \frac{by''}{y'^3})) \\ &= \Sigma_{x\in y^{-1}(0)} \frac{1}{|y'|}\left[\frac{b''}{y'^2} - \frac{b'y''}{y'^3} - 2\frac{b'y''}{y'^3} + b(3\frac{y''^2}{y'^4} - \frac{y'''}{y'^3})\right] \\ &= \Sigma_{x\in y^{-1}(0)} \frac{1}{|y'|}\left[\frac{b''}{y'^2} - 3\frac{b'y''}{y'^3} + b(3\frac{y''^2}{y'^4} - \frac{y'''}{y'^3})\right]\,.\end{aligned}$$

(Yueheng Lan)

Solution 9.3: $\mathcal{L}^t$ generates a semigroup. *Every "sufficiently good" transformation f^t in phase space $\mathcal{M}$ is associated with a Perron-Frobenius operator $\mathcal{L}^t$ which is when acting on a function $\rho(x)$ in $\mathcal{M}$*

$$\mathcal{L}^t \cdot \rho(x) = \int_{\mathcal{M}} dy\, \delta(x - f^t(y))\rho(y)\,.$$

In some proper function space $\mathcal{F}$ on $\mathcal{M}$, the one parameter family of operators $\{\mathcal{L}^t\}_{t\in\mathbb{R}^+}$ generate a semigroup. Let's check this statement. For any $t_1, t_2 > 0$ and $\rho \in \mathcal{F}$, the product "$\circ$" of two operators is defined as usual

$$(\mathcal{L}^{t_1} \circ \mathcal{L}^{t_2}) \cdot \rho(y) = \mathcal{L}^{t_1} \cdot (\mathcal{L}^{t_2} \cdot \rho)(y)\,.$$

So, we have

$$\begin{aligned}(\mathcal{L}^{t_1} \circ \mathcal{L}^{t_2})(y, x) &= \int_{\mathcal{M}} dz\, \mathcal{L}^{t_1}(y, z)\mathcal{L}^{t_2}(z, x) \\ &= \int_{\mathcal{M}} dz\, \delta(y - f^{t_1}(z))\delta(z - f^{t_2}(x)) \\ &= \delta(y - f^{t_1}(f^{t_2}(x))) \\ &= \delta(y - f^{t_1+t_2}(x)) \\ &= \mathcal{L}^{t_1+t_2}(y, x)\,,\end{aligned}$$

where the semigroup property $f^{t_1}(f^{t_2}(x)) = f^{t_1+t_2}(x)$ of f^t has been used. This proves the claim in the title.

(Yueheng Lan)

Solution 9.5: Invariant measure. *Hint: Compare the second map to the construction of Exercise 13.6. We do (a),(b),(c),(d) for the first map and (e) for the second.*
(a) The partition point is in the middle of $[0, 1]$. If the density on the two pieces are two constants ρ_0^A and ρ_0^B, respectively, the Perron-Frobenius operator still leads to the piecewise constant density

$$\rho_1^A = \frac{1}{2}(\rho_0^A + \rho_0^B)\,,$$
$$\rho_1^B = \frac{1}{2}(\rho_0^A + \rho_0^B)\,.$$

Notice that in general if a finite Markov partition exists and the map is affine (linear on each partition cell), a finite-dimensional invariant subspace consisting of piecewise constant function can always be identified in the function space.

(b) From the discussion of (a), any constant function on $[0, 1]$ is an invariant measure. If we consider the invariant probability measure, then the constant has to be 1.

(c) As the map is invariant in $[0, 1]$ (there is no escaping), the leading eigenvalue of $\mathcal{L}$ is always 1 due to the "mass" conservation.

(d) Take a typical point on $[0,1]$ and record its trajectory under the first map for some time (10^5 steps). Plot the histogram...ONLY 0 is left finally!! This happens because of the finite accuracy of the computer arithmetics. A small trick is to change the slope 2 to 1.99999999. You will find a constant measure on $[0,1]$ which is the natural measure. Still, the finite presicion of the computer will make every point eventually periodic and strictly speaking the measure is defined only on some fine lattice points. But when the resolution improves, the computer-generated measure will steadily approach the natural measue. For the first map, any small deviation from the constant profile will be stretched and smeared out. So, the natural measure has to be constant.

(e) Simple calculation shows that α is the partition point. We may use A, B to mark the left and right part of the partition, respectively. A maps to B and B maps to the whole interval $[0,1]$. As the magnitude of the slope $\Lambda = (\sqrt{5}+1)/2$ is greater than 1, we may expect the natural measure is still piecewise constant with eigenvalue 1. The determining equation is

$$\begin{pmatrix} 0 & 1/\Lambda \\ 1/\Lambda & 1/\Lambda \end{pmatrix} \begin{pmatrix} \rho^A \\ \rho^B \end{pmatrix} = \begin{pmatrix} \rho^A \\ \rho^B \end{pmatrix},$$

which gives $\rho^B/\rho^A = \Lambda$.

(Yueheng Lan)

Solution 9.7: Eigenvalues of the skew Ulam tent map Perron-Frobenius operator. *If we have density $\rho_n(x)$, the action of the Perron-Frobenius operator associated with $f(x)$ gives a new density*

$$\rho_{n+1}(x) = \frac{1}{\Lambda_0}\rho_n(x/\Lambda_0) + \frac{1}{\Lambda_1}\rho_n(1 - x/\Lambda_1),$$

where $\Lambda_1 = \frac{\Lambda_0}{\Lambda_0 - 1}$. The eigenvalue equation is given by

$$\rho_{n+1}(x) = \lambda \rho_n(x). \qquad \text{(N.9)}$$

We may solve it by assuming that the eigenfunctions are N-th order polynomials $P(N)$ (check it). Indeed, detailed calculation gives the following results:

- *$P(0)$ gives $\lambda = 1$, corresponding to the expected leading eigenvalue.*
- *$P(1)$ gives $\lambda = \frac{1}{\Lambda_0^2} - \frac{1}{\Lambda_1^2} = \frac{2}{\Lambda_0} - 1$,*
- *$P(2)$ gives $\lambda = \frac{1}{\Lambda_0^3} + \frac{1}{\Lambda_1^3}$,*
- *$P(3)$ gives $\lambda = \frac{1}{\Lambda_0^4} - \frac{1}{\Lambda_1^4}$,*
- *The guess is that $P(N)$ gives $\lambda = \frac{1}{\Lambda_0^{N+1}} + (-1)^N \frac{1}{\Lambda_1^{N+1}}$.*

The final solution is that the piecewise linear function $\rho^A = -\Lambda_0, \rho^B = \Lambda_1$ gives the eigenvalue 0. If only the continuous functions are considered, this kind of eigenfunction of course should not be included.

(Yueheng Lan)

Solution 9.7: Eigenvalues of the skew Ulam tent map Perron-Frobenius operator. *The first few eigenvalues are*

$$\begin{aligned} e^{s_0} &= 1, \qquad e^{s_1} = \frac{2}{\Lambda_0} - 1 \\ e^{s_2} &= \frac{1}{4} + \frac{3}{4}\left(\frac{2}{\Lambda_0} - 1\right)^2, \qquad e^{s_3} = \frac{1}{2}\left(\frac{2}{\Lambda_0} - 1\right) + \frac{1}{2}\left(\frac{2}{\Lambda_0} - 1\right)^3 \ldots \end{aligned}$$

For eigenvectors (invariant densities for skew tent maps), see for example Invariant densities for skew tent maps by L. Billings and E.M. Bolt.

Solution 9.10: $\mathcal{A}$ as a generator of translations. *If v is a constant in space, Taylor series expansion gives*

$$a(x+tv) = \Sigma_{k=0}^{\infty} \frac{1}{k!}(tv\frac{\partial}{\partial x})^k a(x) = e^{tv\frac{\partial}{\partial x}} a(x).$$

(Yueheng Lan)

Chapter 10

Solution 10.1: How unstable is the Hénon attractor?

1. *Evaluate numerically the Lyapunov exponent by iterating the Hénon map; For $a = 1.4$, $b = 0.3$ the answer should be close to $\lambda = 0.41$. If you have a good estimate and a plot of the convergence of your estimate with t, please send us your results for possible inclusion into this text.*
2. *Both Lyapunov exponents for $a = 1.39945219$, $b = 0.3$ are negative, roughly $\lambda_1 = -0.2712, \lambda_2 = -0.9328$ (check that these values respect the constant volume contraction condition (4.37) for the Hénon map). Why? Because after a long transient exploration of the Hénon map's non–wandering set, on average after some 11,000 iterates, almost every inital point falls into a stable 13-cycle. You can check its existence by starting at one of its periodic points $(x_p, y_p) = (-0.2061, -0.3181)$.*
   If you missed the stable 13-cycle (as all students in one of the courses did), you should treat your computer experiments with great deal of scepticism.

As the product of eigenvalues is the constant $-b$, you need to evaluate only the expanding eigenvalue. There are many ways to implement this calculation - here are a few:

1. *The most naive way - take the log of distance of two nearby trajectories, iterate until you run out of accuracy. Tray this many times, estimate an average.*
2. *Slighly smarter: as above, but keep rescaling the length of the vector connecting neighboring points so it remains small, average over the sum of logs of rescaling factors. You can run this forever.*
3. *Keep multiplying the [2×2] Jacobian stability matrix (4.36) until you run out of accuracy. Compute the log of the leading eigenvalue (4.18), try this many times, estimate an average.*

4. *Slighly smarter still: as above, but start with an arbitrary initial tangent space vector, keep multiplying it with the Jacobian stability matrix, and rescaling the length of the vector so it remains small. You can run this forever.*
5. *There is probably no need to use the QR decomposition method or any other such numerical method for this 2-dimensional problem.*

(Yueheng Lan and P. Cvitanović)

Chapter 11

Solution 11.1: Binary symbolic dynamics. *Read the text.*

Solution 11.2: 3-disk fundamental domain symbolic dynamics. *Read sect. 1.4.*

Solution 11.5: Reduction of 3-disk symbolic dynamics. *The answer is given in sect. 22.6. Some remarks concerning part (c):*

If an orbit does not have any spatial symmetry, its length in the fundamental domain is equal to that in the full space. One fundamental domain orbit corresponds to six copies of the orbit in the full space related to each other by symmetries. If a periodic orbit does have a spatial symmetry, then its fundamental domain image is a fraction of that in the whole space, and the orbit (and its symmetry pratnenrs) in the full space is tiled by copies of the irreducible segment, corresponding to an orbit in the fundamental domain. The higher symmetry an orbit has, the shorter the irreducible segment.

Another way to construct a fundamental domain orbit: put a periodic orbit and all its spatial symmetry relatives simultaneously in the full space. The segments that fall into a fundamental domain constitute the orbit in the fundamental domain.

(Yueheng Lan)

Solution 11.6: Unimodal map symbolic dynamics. *Hint: write down an arbitrary binary number such as $\gamma = .1101001101000\ldots$ and generate the future itinerary S^+ by checking whether $f^n(\gamma)$ is greater or less than 1/2. Then verify that (??) recovers γ.*

Solution 11.7: "Golden mean" pruned map.
(a) Consider the 3-cycle drawn in the figure. Denote the lengths of the two horizontal intervals by a and b. We have

$$\frac{a}{b} = \frac{b}{a+b}\,,$$

so the slope is given by the golden mean, $\Lambda = \frac{b}{a} = \frac{1+\sqrt{5}}{2}$, and the piece-wise linear map is given by

$$f(x) = \begin{cases} \Lambda x\,,\ x \in [0,1/2] \\ \Lambda(1-x)\,,\ x \in [1/2,1] \end{cases}$$

(b) Evaluate

$$f\left(\frac{1}{2}\right) = \frac{1+\sqrt{5}}{4}\,,\quad f\left(\frac{1+\sqrt{5}}{4}\right) = \frac{-1+\sqrt{5}}{4}\,,\quad f\left(\frac{-1+\sqrt{5}}{4}\right) = \frac{1}{2}\,.$$

Once a point enters the region covered by the interval $\mathcal{M}$ of length $a+b$, bracketed by the 3-cycle, it will be trapped there forever. Outside $\mathcal{M}$, all points on unit interval will be mapped to $(0,1/2]$, except for 0. The points in the interval $(0,\frac{-1+\sqrt{5}}{4})$ approach $\mathcal{M}$ monotonically.

(c) It will be in $(\frac{1}{2},\frac{1+\sqrt{5}}{4})$.

(d) From (b), we know that except for the origin 0, all periodic orbits should be in $\mathcal{M}$. By (c), we cannot have the substring 00 in a periodic orbit (except for the fixed point at 0). Hence 00 is the only pruning block, *and the symbolic dynamics is a finite subshift, with alphabet $\{0,1\}$ and only one grammar rule: a consecutive repeat of symbol 0 is inadmissible.*

(e) Yes. 0 is a periodic orbit with the symbol sequence $\overline{0}$. It is a repeller and no point in its neighborhood will return. So it plays no role in the asymptotic dynamics.

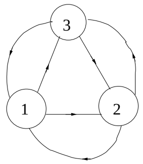
(Yueheng Lan)

Chapter 12

(No solutions available.)

Chapter 13

Solution 13.1: A transition matrix for 3-disk pinball. *a) As the disk is convex, the transition to itself is forbidden. Therefore, the Markov diagram is*



,

with the corresponding transition matrix

$$\mathbb{T} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note that $\mathbb{T}^2 = \mathbb{T} + 2$. Suppose that $\mathbb{T}^n = a_n\mathbb{T} + b_n$, then

$$\mathbb{T}^{n+1} = a_n\mathbb{T}^2 + b_n\mathbb{T} = (a_n+b_n)\mathbb{T} + 2a_n\,.$$

So $a_{n+1} = a_n + b_n\,,\ b_{n+1} = 2a_n$ *with* $a_1 = 1\,,\ b_1 = 0$.

b) From a) we have $a_{n+1} = a_n + 2a_{n-1}$. *Suppose that* $a_n \propto \lambda^n$. *Then* $\lambda^2 = \lambda + 2$. *Solving this equation and using the initial condition for* $n = 1$, *we obtain the general formula*

$$\begin{aligned} a_n &= \frac{1}{3}(2^n - (-1)^n)\,, \\ b_n &= \frac{2}{3}(2^{n-1} + (-1)^n)\,. \end{aligned}$$

c) $\mathbb{T}$ *has eigenvalue* 2 *and* -1 *(degeneracy* 2*). So the topological entropy is* $\ln 2$, *the same as in the case of the binary symbolic dynamics.* *(Yueheng Lan)*

Solution 13.2: Sum of A_{ij} is like a trace. *Suppose that* $A\phi_k = \lambda_k\phi_k$, *where* $\lambda_k\,,\phi_k$ *are eigenvalues and eigenvectors, respectively. Expressing the vector* $v = (1, 1, \cdots, 1)^t$ *in terms of the eigenvectors* ϕ_k, *that is,* $v = \Sigma_k d_k\phi_k$, *we have*

$$\begin{aligned} \Gamma_n &= \Sigma_{ij}[A^n]_{ij} = v^t A^n v \;=\; \Sigma_k v^t A^n d_k\phi_k \;=\; \Sigma_k d_k \lambda_k^n (v^t\phi_k) \\ &= \Sigma_k c_k \lambda_k^n\,, \end{aligned}$$

where $c_k = (v^t\phi_k)d_k$ *are constants.*

a) As $\mathrm{tr}\, A^n = \Sigma_k \lambda_k^n$, *it is easy to see that both* $\mathrm{tr}\, A^n$ *and* Γ_n *are dominated by the largest eigenvalue* λ_0. *That is*

$$\frac{\ln|\mathrm{tr}\, A^n|}{\ln|\Gamma_n|} = \frac{n\ln|\lambda_0| + \ln|\Sigma_k(\frac{\lambda_k}{\lambda_0})^n|}{n\ln|\lambda_0| + \ln|\Sigma_k d_k(\frac{\lambda_k}{\lambda_0})^n|} \to 1 \ \ \textit{as } n \to \infty\,.$$

b) The nonleading eigenvalues do not need to be distinct, as the ratio in a) is controlled by the largest eigenvalues only.

(Yueheng Lan)

Solution 13.4: Transition matrix and cycle counting. *a) According to the definition of* $\mathbb{T}_{ij}$, *the transition matrix is*

$$\mathbb{T} = \begin{pmatrix} a & c \\ b & 0 \end{pmatrix}.$$

b) All walks of length three $0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010$ *(four symbols!) with weights* $aaa, aac, acb, cba, cbc, baa, bac, bcb$. *Let's calculate* $\mathbb{T}^3$,

$$\mathbb{T}^3 = \begin{pmatrix} a^3 + 2abc & a^2c + bc^2 \\ a^2b + b^2c & abc \end{pmatrix}.$$

There are altogether 8 *terms, corresponding exactly to the terms in all the walks.*

c) Let's look at the following equality

$$\mathbb{T}^n_{ij} = \Sigma_{k_1,k_2,\cdots,k_{n-1}} \mathbb{T}_{ik_1}\mathbb{T}_{k_1k_2}\cdots\mathbb{T}_{k_{n-1}j}\,.$$

Every term in the sum is a possible path from i *to* j, *though the weight could be zero. The summation is over all possible intermediate points* $(n-1$ *of them). So,* $\mathbb{T}^n_{ij}$ *gives the total weight (probability or number) of all the walks from* i *to* j *in n steps.*

d) We take $a = b = c = 1$ *to just count the number of possible walks in* n *steps. This is the crudest description of the dynamics. Taking* a, b, c *as transition probabilities would give a more detailed description. The eigenvlues of* $\mathbb{T}$ *is* $(1 \pm \sqrt{5})/2$, *so we get* $N(n) \propto (\frac{1+\sqrt{5}}{2})^n$.

e) The topological entropy is then $\ln\frac{1+\sqrt{5}}{2}$. *(Yueheng Lan)*

Solution 13.6: "Golden mean" pruned map. *It is easy to write the transition matrix* $\mathbb{T}$

$$\mathbb{T} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues are $(1 \pm \sqrt{5})/2$. *The number of periodic orbits of length* n *is the trace*

$$\mathbb{T}^n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n}\,.$$

(Yueheng Lan)

Solution 13.5: 3-disk prime cycle counting. *The formula for arbitrary length cycles is derived in sect. 13.4.*

Solution 13.45: Alphabet {0,1}, prune $\_1000\_$, $\_00100\_$, $\_01100\_$.

step 1. $\_1000\_$ *prunes all cycles with a* $\_000\_$ *subsequence with the exception of the fixed point* $\overline{0}$; *hence we factor out* $(1 - t_0)$ *explicitly, and prune* $\_000\_$ *from the rest. Physically this means that* x_0 *is an isolated fixed point - no cycle stays in its vicinity for more than 2 iterations. In the notation of exercise 13.18, the alphabet is* $\{$*1, 2, 3;* $\overline{0}\}$, *and the remaining pruning rules have to be rewritten in terms of symbols 2=10, 3=100:*

step 2. *alphabet* $\{$*1, 2, 3;* $\overline{0}\}$, *prune* $\_33\_$, $\_213\_$, $\_313\_$. *Physically, the 3-cycle* $\overline{3} = \overline{100}$ *is pruned and no long cycles stay close enough to it for a single* $\_100\_$ *repeat. As in exercise 13.7, prohibition of* $\_33\_$ *is implemented by dropping the symbol "3" and extending the alphabet by the allowed blocks 13, 23:*

step 3. *alphabet* $\{$*1, 2,* $\underline{13}$, $\underline{23}$; $\overline{0}\}$, *prune* $\_2\underline{13}\_$, $\_\underline{23}\,\underline{13}\_$, $\_\underline{13}\,\underline{13}\_$, *where* $\underline{13} = 13$, $\underline{23} = 23$ *are now used as single letters. Pruning of the repetitions* $\_\underline{13}\,\underline{13}\_$ *(the 4-cycle* $\overline{13} = \overline{1100}$ *is pruned) yields the*

Result: *alphabet* $\{$*1, 2,* $\underline{23}$, $\underline{113}$; $\overline{0}\}$, *unrestricted 4-ary dynamics. The other remaining possible blocks* $\_\underline{213}\_$, $\_\underline{2313}\_$ *are forbidden by the rules of step 3. The topological zeta function is given by*

$$1/\zeta = (1 - t_0)(1 - t_1 - t_2 - t_{23} - t_{113}) \tag{N.10}$$

for unrestricted 4-letter alphabet $\{$*1, 2,* $\underline{23}$, $\underline{113}\}$.

Solution 13.8: Spectrum of the "golden mean" pruned map.

1. *The idea is that with the redefinition $2 = 10$, the alphabet $\{1,2\}$ is unrestricted binary, and due to the piecewise linearity of the map, the stability weights factor in a way similar to (14.10).*
2. *As in (15.9), the spectral determinant for the Perron-Frobenius operator takes form (15.11)*

$$\det(1 - z\mathcal{L}) = \prod_{k=0}^{\infty} \frac{1}{\zeta_k}, \qquad \frac{1}{\zeta_k} = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|\Lambda_p^k}\right).$$

The mapping is piecewise linear, so the form of the topological zeta function worked out in (13.16) already suggests the form of the answer. The alphabet $\{1,2\}$ is unrestricted binary, so the dynamical zeta functions receive contributions only from the two fixed points, with all other cycle contributions cancelled exactly. The $1/\zeta_0$ is the spectral determinant for the transfer operator like the one in (10.19) with the $T_{00} = 0$, and in general

$$\begin{aligned} \frac{1}{\zeta_k} &= \left(1 - \frac{z}{|\Lambda_1|\Lambda_1^k}\right)\left(1 - \frac{z^2}{|\Lambda_2|\Lambda_2^k}\right)\left(1 - \frac{z^3}{|\Lambda_{12}|\Lambda_{12}^k}\right)\cdots \\ &= 1 - (-1)^k\left(\frac{z}{\Lambda^{k+1}} + \frac{z^2}{\Lambda^{2k+2}}\right). \end{aligned} \tag{N.11}$$

The factor $(-1)^k$ arises because both stabilities Λ_1 and Λ_2 include a factor $-\Lambda$ from the right branch of the map.

Solution 15.2: Dynamical zeta functions

1. *Work through section sect. 15.3.2.*
2. *Generalize the transition matrix (11.15) to a transfer operator.*

Solution 13.11: Whence Möbius function? *Written out $f(n)$ line-by-line for a few values of n, (13.38) yields*

$$\begin{aligned} f(1) &= g(1) \\ f(2) &= g(2) + g(1) \\ f(3) &= g(3) + g(1) \\ f(4) &= g(4) + g(2) + g(1) \\ &\cdots \\ f(6) &= g(6) + g(3) + g(2) + g(1) \\ &\cdots \end{aligned} \tag{N.12}$$

Now invert recursively this infinite tower of equations to obtain

$$\begin{aligned} g(1) &= f(1) \\ g(2) &= f(2) - f(1) \\ g(3) &= f(3) - f(1) \\ g(4) &= f(4) - [f(2) - f(1)] - f(1) = f(4) - f(2) \\ &\cdots \\ g(6) &= f(6) - [f(3) - f(1)] - [f(2) - f(1)] - f(1) \\ &\cdots \end{aligned}$$

We see that $f(n)$ contributes with factor -1 if n prime, and not at all if n contains a prime factor to a higher power. This is precisely the raison d'etre for the Möbius function, with whose help the inverse of (13.38) can be written as the Möbius inversion formula [24.29] (13.39).

Chapter 14

(No solutions available.)

Chapter 15

Solution 15.1: Numerical estimate of the escape rate for a 1-d repeller *The logistic map is defined by $x_{n+1} = Ax_n(1 - x_n)$. For $A \le 4$ any point in the unit interval $[0,1]$ will remain in the interval forever. For $A > 4$ almost all points starting in the unit interval will eventually escape towards $-\infty$.*

The rate of escape can be easily measured by numerical experiment. We define the fraction of initial conditions that leave the interval after n iterations to be Γ_n. Figure N.2 shows a plot of $\log(\Gamma_n)$ versus n, computed by starting with $10\,000\,000$ random initial points. Asymptotically the escape rate falls off exponentially as

$$\Gamma(n) = Ce^{-\gamma n}.$$

Figure N.2 suggests that this formula is very accurate even for relatively small n. We estimate γ by measuring the slope of the curve in figure N.2. To avoid errors due to rounding and transients only the points $5 \le n \le 10$ were used. A linear regression fit yields the escape rate for $A = 6$:

$$\gamma = 0.8315 \pm 0.0001,$$

where the error is from statistical fluctuations (there may be systematic errors either due to rounding or because we are not in the true asymptotic regime).

(Adam Prügel-Bennet)

Solution 15.2: Dynamical zeta functions

1. *Work through section sect. 15.3.2.*
2. *Generalize the transition matrix (11.15) to a transfer operator.*

Solution 15.5: Dynamical zeta functions as ratios of spectral determinants. *Try inserting a factor equal to one in the zeta function and then expanding it. The problem is solved in sect. 15.5.*

Solution 15.8: Dynamical zeta functions for Hamiltonian maps. *Read sect. 15.7.*