Exercises

Exercise 26.1 **Period doubling in your pocket:** *Take a programmable pocket calculator or Matlab or whatever makes you feel good and program the function*

$$f(x) = \lambda - x^2 \, .$$

The game consists in staring at the display, and looking for regularities in the sequences of iterates.

(a) *(no thinking) Try to determine fixed point $x_* = f(x_*)$ by blind iteration. Chose some value of λ a bit bigger than 0, and initial x between -1 and 1. Enter the initial x_0 and read off the next x_1. Start again, with x_1 as input. The number x_2 appears on the display. Is it a fixed point? Press the button again, and again, until $x_n = x_*$ to desired accuracy.*

(f) *(no thinking) Increase λ in small steps, as long as the trajectory does not blow up, let transients die, and then plot few hundred consecutive x_n. Generate a figure to replace the hand drawn figure 26.14.*

(c) *(thinking) Determine the smallest positive λ for which almost any initial x_0 iterates to $-\infty$.*

(b) *(no thinking) Try $\lambda = 3/4$. How's the convergence now?*

(c) *(thinking) Determine λ for which the fixed point x_* goes unstable.*

(d) *(no thinking) Try also λ: 1, 1.31070274134, 1.38154748443, 1.3979453597.*

(e) *(thinking) Compute the next number in this series. Estimate Feigenbaum δ.*

(g) *(thinking) Determine numerically scaling factors α_m which overlay (approximately) neighborhood of $x = 0$ for superstable $f^{2^{(m-1)}}(x)$ over the neighborhood for $\alpha_m f^{2^m}(f^{2^m}(x/\alpha_m))$ for 4, 8, 16, $\cdots$ superstable cycles. Draw a figure to replace the hand drawn figure 26.26.*

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⇓PRELIMINARY

F. Vivaldi: period doubling problem set

⇑PRELIMINARY

(P. Cvitanović)

Exercise 26.2 **Period doubling in a 3-d flow:** *This is a more time consuming problem, but it gives you a feel for how numerical experiments in nonlinear dynamics are* really *done, and sets you up for doing real-life problems later on.*

Consider the nonlinear oscillator

$$\frac{d^2}{dt^2}x + k\frac{d}{dt}x - x + 4x^3 = A\cos(\omega t) \, . \tag{26.24}$$

The oscillator is driven by an external force of frequency ω, with amplitude A and the natural time unit $T_0 = 2\pi/\omega$. The dissipation is controlled by the friction coefficient k.

Given the initial displacement and velocity one can easily follow numerically (by the Runge-Kutta method, for example) the phase space trajectories of the system. Due to the dissipation it does not matter where one starts in the phase space. For a strong friction or a weak forcing a wide range of initial points converge to a fixed point or a limit cycle (trajectory loops onto itself). However, as one decreases the friction, bifurcations and chaos are observed.

(a) Rewrite as a nonautonomous system of 2 first order ODEs.

(b) Rewrite as an autonomous system of 3 first order ODEs (add an equation whose solution is $\cos(\omega t)$).

(c) Observe long-time trajectories for $k = 0.154$, $\omega = 1.2199778$ and A in some range like $A = 0.05$ to 0.2. Anything interessting happening? [Helpful tips: let the system run for sufficiently long time that the transients have had time to die out. Whenever changing the parameter (in this case, parameter A), increase parameter in small steps (adiabatically), and then restart the trajectory from the last point of the preceeding simulation. This minimizes the transients.]

(d) Construct a stroboscopic Poincaré section by recording values of $(x_n, \dot{x}_n)$ at times separated T_0, $2T_0, \cdots$, $2^n T_n, \cdots$

(e) Play a bit with plotting various combinations of x_n, x_{n+1}, $\dot{x}_n$, and $\dot{x}_{n+1}$. Often a Poincaré return map - such as (x_n, x_{n+1}) - is more informative than a Poincaré section, such as $(x_n, \dot{x}_n)$.

(f) Increase A in small steps, as long as the trajectory does not blow up, let transients die, and then plot a few hundred consecutive x_n (or whatever variable you like best). Plot a (A, x_n) period-doubling tree to replace the hand drawn figure 26.14.

(g) Observe long-time trajectories for $(k, \omega) = (0.154, 1.2199778)$ and $A = 0.1$, 0.11, 0.114, 0.11437, $\ldots$. Do you observe bifurcations?

There is nothing special about these parameter values; we give them just to help you with finding your first bifurcation sequence in a realistic nonlinear flow.

(P. Cvitanović)

Exercise 26.3 Renormalization in a 3-d flow: *Consider the nonlinear oscillator (26.24).*

(a) For $(k, \omega) = (0.154, 1.2199778)$ determine accurately amplitude values $A_1, A_2, \cdots, A_n, \cdots$, for which 2^n-cycle bifurcates into 2^{n+1}-cycle. [Tip: use the Newton method on the Poincaré section. You'll need to also integrate the linearized stability matrix.]

(b) Estimate Feigenbaum δ.

(g) Estimate Feigenbaum α.

(P. Cvitanović)