Advection of vector fields by chaotic flows[1]

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"The high average vorticity that is known to exist in turbulent motion is caused by the extension of vortex filaments in an eddying fluid."
— G.I. Taylor (1938)

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ABSTRACT

We derive a Fredholm determinant for a smooth deterministic flow that relates the rate of fast dynamo action to the periodic orbits of the flow. We test the results on a smooth flow generated by a model ordinary differential equation as well as a flow described by a discrete map.

THE PROBLEM

When the particles of a fluid are endowed with some scalar property, its value per unit mass, S, evolves in time according to

$$\partial_t S + \mathbf{u} \cdot \nabla S \equiv \frac{DS}{Dt} = \text{thermal noise.} \tag{1}$$

The right-hand side represents the microscopic spreading of S on the molecular level, and can be thought of as noise added onto the fluid velocity $\mathbf{u}$. It is normally described by a term like $\kappa \boldsymbol{\nabla}^2 S$ where κ is a diffusivity. Extensive work on (1), especially for chaotic and turbulent flows, has been carried out for many decades.[1,2]

Fluid motions also move vector fields around. An evolving vector field $\mathbf{V}$ is governed by an equation of the form

$$\partial_t \mathbf{V} + \mathbf{u} \cdot \nabla \mathbf{V} - \mathbf{V} \cdot \nabla \mathbf{u} \equiv \frac{\mathcal{D}\mathbf{V}}{\mathcal{D}t} = \text{thermal noise.} \tag{2}$$

The advective derivative in (1) is replaced by a Lie derivative in this evolution operator. The extra term enters because the evolution of a vector field involves not only being carried about by a fluid particle but also being turned and stretched by the motion of neighboring particles. The thermal noise on the right side is usually a simple diffusion term, at least for the case of constant density. Density variations bring in some unappetizing complications that we shall ignore.

If the right sides of (1) and (2) are zero (perfect fluid motion) then S and $\mathbf{V}$ are frozen-in properties, and the fluid motions can distort any $\mathbf{V}$ in a complex fashion. In particular, when the dynamical system

$$\dot{\mathbf{x}} = \mathbf{u} \tag{3}$$

produces chaotic motion, the effect on the distribution of the advective fields can be rich and surprising, giving rise to intense local concentrations and lacunae of fine structure.

In real fluid settings the situation may be more complicated than even this description suggests. If either S or $\mathbf{V}$ can feed back on the dynamics of $\mathbf{u}$ itself, the equations lose even their superficially linear appearances. For ordinary fluid motions, we have $\rho \mathbf{V} = \boldsymbol{\omega}$, where ρ is the fluid density and the problem is no longer kinematic. Rather, $\mathbf{u}$ satisfies

$$\rho(\partial_t \mathbf{u} + \boldsymbol{\omega} \times \mathbf{u}) = -\nabla p - \frac{1}{2} \rho \nabla \mathbf{u}^2 + \mathbf{F} + \text{thermal viscous effects,} \tag{4}$$

where $\mathbf{F}$ is an external force density. So when $\rho\mathbf{V}$ is vorticity, we have coupled equations for $\mathbf{u}$ and $\mathbf{V}$.

G.I. Taylor, who made early contributions to the study of (1) for turbulent $\mathbf{u}$, observed that vorticity is concentrated by turbulence.[2] To learn what properties of the motion favor this effect, we may begin with the study of (2) without worrying, at first, whether these motions correspond to solutions of (4). That leads to the search for a field $\mathbf{u}$ that may produce local vorticity enhancement when introduced into (2). We call this the *kinematic* turbulence problem, after the usage of dynamo theory, where (2) applies when $\rho\mathbf{V}$ is the magnetic field.

Batchelor and others have sought analogies between the vorticity and magnetic field, since they are both controlled by equation (2). However, this viewpoint has been belittled because $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$, while no analogous relation exists between $\mathbf{B}$ and u. Yet this relation is implied by (2) and (4), and it need not, as a result, be considered explicitly. When $\mathbf{B}$ is the field in question, we couple it to (4) through a nonlinear term representing the Lorentz force. That is,

$$\rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p - \frac{1}{4\pi}\mathbf{B}\times\nabla\times\mathbf{B} + \mathbf{F} + \text{thermal viscous effects.} \tag{5}$$

Typically the kinematic aspects of the dynamo problem are stressed over those of the analogous vorticity problem. But if the analogy is good, it is not clear whether this should raise interest in the kinematic turbulence problem or throw the kinematic dynamo problem in a bad light. In any case, there does seem to be considerable interest in studying the effect of fluid motion on an immersed vector field through (2), for prescribed $\mathbf{u}$.

These general thoughts lead into our discussion of the effect of a chaotic motion on a vector field satisfying (2). We describe in general terms the procedures that we have been following, omit the most strenuous portions of the calculations and refer to the related work on the mixing of a passive scalar by a continuous flow[3], and to the study by Aurell and Gilbert[4] of fast dynamos in discrete maps.

THE FORMAL SOLUTION

We consider velocity fields $\mathbf{u}$ that are steady. In this autonomous case, the problem can be simplified by writing $\mathbf{V}(\mathbf{x},t) = \mathbf{V}_0(\mathbf{x})\exp(\lambda t)$. Then we obtain, for the diffusionless case,

$$\mathcal{D}\mathbf{V}/\mathcal{D}t \equiv (\mathbf{u}\cdot\nabla\mathbf{V}_0 - \mathbf{V}_0\cdot\nabla\mathbf{u})e^{\lambda t} + -\lambda\mathbf{V}_0 e^{\lambda t} = 0. \tag{6}$$

We may look for a solution of the form $\mathbf{V}_0 = q(\mathbf{x})\mathbf{u}(\mathbf{x})$, where

$$\mathbf{u}\cdot\nabla q = -\lambda q \quad . \tag{7}$$

This shows how the kinematic problem for vector fields may be related to the more extensively studied problem of passive scalar transport[1,3]; if we set $S(\mathbf{x},t) =$

$q(\mathbf{x})\exp(\lambda t)$ in (1), we get (7). Moreover, if q is constant on streamlines, then we must have $\lambda = 0$. These special solutions arise in the very restricted conditions of steady flow without dispersive effects, and they illustrate the kind of degeneracy that we encounter when the conditions are too simple.

More generally, we can write the solution of (2) formally, as shown by Cauchy. Let $\mathbf{x}(t, \mathbf{a})$ be the position of the fluid particle that was at the point $\mathbf{a}$ when $t = 0$. Then the field evolves according to

$$\mathbf{V}(\mathbf{x}, t) = \mathcal{J}(\mathbf{a}, t)\mathbf{V}(\mathbf{a}, 0) \quad , \tag{8}$$

where $\mathcal{J}(\mathbf{a}, t) = \partial(\mathbf{x})/\partial(\mathbf{a})$ is the Jacobian matrix of the transformation that moves the fluid into itself with time with $\mathbf{x} = \mathbf{x}(\mathbf{a}, t)$.

We write $\mathbf{x} = \boldsymbol{\varphi}^t\mathbf{a}$, where $\boldsymbol{\varphi}^t$ is the flow that maps the initial positions of the fluid particles into their positions at time t. Its inverse, $\mathbf{a} = \boldsymbol{\varphi}^{-t}\mathbf{x}$, maps particles at time t and position $\mathbf{x}$ back to their initial positions. Then we can write (8) in the seemingly complicated, but quite useful form,

$$\mathbf{V}(\mathbf{x}, t) = \int \delta(\mathbf{a} - \boldsymbol{\varphi}^{-t}\mathbf{x})\mathcal{J}(\mathbf{a}, t)\, \mathbf{V}(\mathbf{a}, 0)\, d^3\mathbf{a} \quad , \tag{9}$$

where the integral operator introduced here is the counterpart of the Perron-Frobenius operator[5] which arises in the case of scalar advection. Having turned the differential equation (2) into an integral equation, we may use analogues of Fredholm's methods[6] to solve it.

If we were to include the effects of diffusion by using a term of the form $\eta\nabla^2\mathbf{V}$ on the right hand side of (2), with η constant, we would need to replace $\mathcal{D}/\mathcal{D}t$ by $\mathcal{D}/\mathcal{D}t - \eta\nabla^2$ in (6) and modify (9), but for now we shall speak only of the noise-free case.

STRUCTURES OF CHAOTIC FLOWS

To describe the evolution of a frozen-in magnetic field, we need a suitable means to characterize the flow. For the chaotic flows on which we concentrate here, the Lagrangian orbits determined by (3) densely contain unstable periodic orbits. Their union forms what we may call a chaotic set. If this object is attracting, the term *strange attractor* is appropriate while in the opposite case it becomes a strange *repeller,* from which representative points depart as time proceeds. Even though all periodic orbits within such an ensemble are unstable, trajectories of the flow often spend extended periods tracking these paths. The strange set provides a delicate, skeletal structure embedded within the flow that can be used to approximate systematically such properties as its local stability and geometry.[7] This is the basic idea underlying *cycle expansions*[8], the method that we apply here to the problem of vector advection.

To be explicit, we consider a trajectory $x(t)$ generated by the ordinary differential equation

$$\ddot{x} + \dot{x} - cx + x^3 = 0 \quad , \tag{10}$$

with parameter c. This is a special case of equations arising in multiply diffusive convection[9]. The absence of a second derivative in (10) ensures that the flow is solenoidal, in line with the kinds of flow most commonly studied in dynamo theory, in spite of the fact that every putative dynamo is compressible. For certain values of c, homoclinic orbits exist and we set c to be close to one of these values and such that, in the neighborhood of the origin, the flow satisfies Shil'nikov's criterion for the existence of an infinity of unstable periodic orbits.[10] Then, an infinite sequence of intertwined saddle-node and period-doubling bifurcations creates a dense chaotic set, and the motion of particles suspended in the flow is chaotic.

Details of the structure of this flow are presented elsewhere;[11] here we mention only that $x(t)$, under the conditions mentioned, is a sequence of pulses. Moreover, the separation of the k^{th} and $(k-1)^{st}$ pulses, Δ_k, may be expressed in terms of the two previous spacings. This provides us with a two-dimensional *timing map*, which formally resembles the Hénon map[12]. Thus, to good approximation,

$$\Delta_k = \bar{T} + \alpha\tau_k \quad , \tag{11}$$

where $\bar{T}$ is a mean period and τ_k an irregular timing fluctuation satisfying,

$$\tau_{k+1} = 1 - a\tau_k^2 - \tau_{k-1} \quad , \tag{12}$$

which is the orientation and area preserving form of the Hénon map. The quantity α is determined by the particular pulse sequence under consideration, but here we treat it as an appropriately fitted parameter.

The form of any particular pulse is quite close to that of the homoclinic orbit, which can be computed at the outset, and the map (12) determines a sequence of pulse positions. Once these are known, we can generate a complete and reasonably accurate solution for the velocity field. Thus we can reconstruct the entire *flow* from this simple map. Moreover, this map also contains the invariant information about the periodic orbits of the chaotic set (cycle topology and stability eigenvalues) and is useful in the construction of cycle expansions.

In the following sections, we describe the technique of cycle expansions for the problem at hand. This needs no explicit specification of the velocity field $\mathbf{u}$, but in the concluding section we report some numerical results obtained for the particular flow modeled by (12). For comparison, we also study the dynamo action of a two-dimensional flow which is governed by the discrete Hénon mapping.

EVOLUTION AND TRANSFER OPERATORS

A frozen-in vector field is stretched, squeezed and swept around by the chaotic flow, evolving as described by (8). To compute the large-scale evolution of the field, we rewrite (9) as

$$V_i(\mathbf{x}, t) = \int_\Sigma d^3\mathbf{a}\, \mathcal{L}^t_{ij}(\mathbf{x}, \mathbf{a}) V_j(\mathbf{a}, 0) \quad , \tag{13}$$

with

$$\mathcal{L}^t_{ij}(\mathbf{x},\mathbf{a}) = \delta(\mathbf{a}-\boldsymbol{\varphi}^{-t}\mathbf{x})\frac{\partial x_i}{\partial a_j} \quad , \tag{14}$$

where summation over repeated indices is understood. The kernel $\mathcal{L}^t_{ij}$ controls the evolution of the embedded vector field. This *transfer operator* is linear and typically possesses a sequence of eigenvalues, $e^{-\nu_0 t}, e^{-\nu_1 t}, e^{-\nu_2 t}, \ldots$ (for certain hyperbolic systems, the so-called Axiom A flows, this can be rigorously established). For large times, the effect of $\mathcal{L}^t$ is dominated by its leading eigenvalue, $e^{-\nu_0 t}$ with $Re(\nu_0) < Re(\nu_i)$, $i = 1, 2, 3, \ldots$. In this way the transfer operator furnishes the fast dynamo rate, $\nu \equiv -\nu_0$.

The operator $\mathcal{L}^t_{ij}$ was introduced in ref. 8 in order to study the "stability of strange sets", and applied to discrete map models of fast dynamos in ref. 4. Here we apply it to continuous time flows.

A TRACE FORMULA

To calculate the leading eigenvalue of $\mathcal{L}^t$, we evaluate the trace

$$\mathrm{tr}(\mathcal{L}^t) = \int_\Sigma d^3\mathbf{a}\,\mathcal{L}^t_{ii}(\mathbf{a},\mathbf{a}) = \int_\Sigma d^3\mathbf{a}\,\delta(\mathbf{a}-\boldsymbol{\varphi}^{-t}\mathbf{a})\frac{\partial \varphi^t_i}{\partial a_i} \quad , \tag{15}$$

which asymptotes to $e^{\nu t}$ for long times. We evaluate this integral by means of explicit periodic orbit expansions. Each cycle within the fabric of the flow contributes to the overall trace of the operator, and each contribution is obtained by integration along the periodic orbit in suitable local coordinates. We omit the details of this technical operation (see ref. 3) and report that each prime cycle p, together with its repeats, contributes a term

$$T_p \sum_{r=1}^{\infty} \frac{\mathrm{tr}\,\mathbf{J}_p^r}{|\det(\mathbf{1}-\mathbf{J}_p^{-r})|}\delta(t - rT_p) \tag{16}$$

to the integral. T_p is the cycle period and $\mathbf{J}_p$ is the transverse stability matrix $\hat{\mathbf{u}}(t+T_p) = \mathbf{J}_p\hat{\mathbf{u}}(t)$ for a two-vector in the tangent plane transverse to the flow. The $\mathbf{J}_p$ eigenvalues $\Lambda_{p,1}$, $\Lambda_{p,2}$ are independent of the position along the orbit and the choice of transverse coordinates. Note that in this formula we have omitted a term arising from the Jacobian transformation along the orbit which would give $1 + \mathrm{tr}\,\mathbf{J}_p^r$ in the numerator rather than just the trace of $\mathbf{J}_p^r$. Since the extra term corresponds to advection along the orbit, and this does not evolve the magnetic field, we have chosen to ignore it. An alternative way of viewing this omission is that the neglected term describes evolution on closed streamlines which, as we have already mentioned, is physically uninteresting.

The trace of the transfer operator is the sum over all periodic orbit contributions, with each cycle weighted by its intrinsic stability

$$\mathrm{tr}(\mathcal{L}^t) = \sum_p T_p \sum_{r=1}^{\infty} \frac{\mathrm{tr}\,\mathbf{J}_p^r}{|\det(\mathbf{1}-\mathbf{J}_p^{-r})|}\delta(t - rT_p) \tag{17}$$

$$= \frac{i}{2\pi} \int_0^\infty dk \; e^{ikt} \frac{\partial}{\partial k} \sum_p \sum_r \frac{1}{r} \frac{\mathrm{tr}\; \mathbf{J}_p^r}{|\det(\mathbf{1} - \mathbf{J}_p^{-r}|} e^{-ikrT_p} \quad , \tag{18}$$

where we have introduced the Fourier representation of Dirac delta functions.

A FREDHOLM DETERMINANT

We rotate the axes by the Wick transformation, $k \to is$, and observe that the summations in (18) correspond to the logarithmic derivative of the function

$$F(s) = \exp\left[-\sum_p \sum_{r=1}^\infty \frac{1}{r} \frac{\mathrm{tr}\; \mathbf{J}_p^r}{|\det(\mathbf{1} - \mathbf{J}_p^{-r})|} e^{srT_p} \right] \quad , \tag{19}$$

related to the above trace by

$$\mathrm{tr}(\mathcal{L}^t) = \frac{i}{2\pi} \int_{-i\infty}^{i\infty} ds \; e^{-st} F'(s)/F(s) \quad . \tag{20}$$

The function $F(s)$ is a *Fredholm determinant*. Its zeros produce singularities in the integrand of (20). If it is an *entire function* (defined over the whole complex plane), then the contour of integration can be suitably deformed so as to encircle the various poles of $F'(s)/F(s)$, and the trace becomes a sum over the residues of the integrand:

$$\mathrm{tr}(\mathcal{L}^{\mathbf{t}}) = \sum_{n=0}^\infty m_n e^{-\nu_n t} \quad , \tag{21}$$

where ν_n is a pole of multiplicity m_n. Hence, the spectrum of the transfer operator is determined by the zeros of the Fredholm determinant $F(s)$.

In order to simplify $F(s)$, we factor the denominator cycle stability determinants into products of expanding and contracting eigenvalues. The example at hand is a 3-dimensional hyperbolic flow with cycles possessing one expanding eigenvalue Λ_p (with $|\Lambda_p| > 1$), and one contracting eigenvalue λ_p (with $|\lambda_p| < 1$). Then the determinant may be expanded as follows:

$$|\det\left(\mathbf{1} - \mathbf{J}_p^{-r}\right)|^{-1} = |(1 - \Lambda_p^{-r})(1 - \lambda_p^{-r})|^{-1} = |\lambda_p|^r \sum_{j=0}^\infty \sum_{k=0}^\infty \Lambda_p^{-jr} \lambda_p^{kr} \quad . \tag{22}$$

With this decomposition we can rewrite the exponent in (19) as

$$\sum_p \sum_{r=1}^\infty \frac{1}{r} \frac{(\lambda_p^r + \Lambda_p^r) e^{srT_p}}{|\det\left(\mathbf{1} - \mathbf{J}_p^{-r}\right)|} = \sum_p \sum_{j,k=0}^\infty \sum_{r=1}^\infty \frac{1}{r} \left(|\lambda_p| \Lambda_p^{-j} \lambda_p^k e^{sT_p} \right)^r (\lambda_p^r + \Lambda_p^r) \quad , \tag{23}$$

which has the form of the expansion of a logarithm:

$$\sum_p \sum_{j,k} \left[\log\left(1 - e^{sT_p} |\lambda_p| \Lambda_p^{1-j} \lambda_p^k\right) + \log\left(1 - e^{sT_p} |\lambda_p| \Lambda_p^{-j} \lambda_p^{1+k}\right) \right] \quad . \tag{24}$$

The Fredholm determinant is therefore of the form,

$$F(s) = F_e(s) F_c(s) \quad , \tag{25}$$

where

$$F_e(s) = \prod_p \prod_{j,k=0}^{\infty} \left(1 - t_p^{(jk)} \Lambda_p\right) \quad , \tag{26}$$

$$F_c(s) = \prod_p \prod_{j,k=0}^{\infty} \left(1 - t_p^{(jk)} \lambda_p\right) , \tag{27}$$

with

$$t_p^{(jk)} = e^{sT_p} |\lambda_p| \frac{\lambda_p^k}{\Lambda_p^j} \quad . \tag{29}$$

The two factors present in $F(s)$ correspond to the two Floquet exponents of the periodic orbits, the expanding and contracting exponents. (Haad we not neglected a term in (16), there would be a third factor corresponding to the translating Floquet exponent.) In form, they are the Selberg-type products known in other areas of mathematical physics.[13]

DYNAMOS FOR MAPS AND FLOWS

For 2-d Hamiltonian volume preserving systems, $\lambda = 1/\Lambda$ and (26) reduces to

$$F_e(s) = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{t_p}{\Lambda_p^{k-1}}\right)^{k+1} , \quad t_p = \frac{e^{sT_p}}{|\Lambda_p|} \quad . \tag{30}$$

With $\sigma_p = \Lambda_p / |\Lambda_p|$, the Hamiltonian zeta function (the $j = k = 0$ part of the product (27)) is given by

$$1/\zeta_{dyn}(s) = \prod_p \left(1 - \sigma_p e^{sT_p}\right) . \tag{31}$$

This is a curious formula — the zeta function depends only on the return times, not on the eigenvalues of the cycles. Furthermore, the identity,

$$\frac{\Lambda + 1/\Lambda}{|(1-\Lambda)(1-1/\Lambda)|} = \sigma + \frac{2}{|(1-\Lambda)(1-1/\Lambda)|} \quad ,$$

when substituted into (25), leads to a relation between the vector and scalar advection Fredholm determinants:

$$F_{dyn}(s) = F_0(s)^2 / \zeta_{dyn}(s) . \tag{32}$$

The Fredholm determinants in this equation are entire for hyperbolic (axiom A) systems, since both of them correspond to multiplicative operators[14,15].

In the case of a flow governed by a map, we can adapt the formulae (30) and (31) for the dynamo determinants by simply making the substitution

$$z^{n_p} = e^{sT_p} \quad , \tag{33}$$

where n_p is the integer order of the cycle. Then we find the Fredholm determinant $F_e(z)$ given by equation (30) but with

$$t_p = \frac{z^{n_p}}{|\Lambda_p|} \tag{34}$$

for the weights, and

$$1/\zeta_{dyn}(z) = \Pi_p \left(1 - \sigma_p z^{n_p}\right) \tag{35}$$

for the zeta-function (*cf.* Aurell and Gilbert[4]).

For *maps* with finite Markov partition the inverse zeta function (35) reduces to a polynomial for z since curvature terms[8] in the cycle expansion vanish. For example, for maps with complete binary partition, and with the fixed point stabilities of opposite signs, the cycle expansion reduces to

$$1/\zeta_{dyn}(s) = 1. \tag{36}$$

For such *maps* the dynamo Fredholm determinant is simply the square of the scalar advection Fredholm determinant, and therefore all its zeros are double. In other words, for flows governed by such discrete maps, the fast dynamo rate equals the scalar advection rate[16].

In contrast, for three-dimensional *flows*, the dynamo effect is distinct from the scalar advection. For example, for flows with finite symbolic dynamical grammars, (32) implies that the dynamo zeta function is a ratio of two entire determinants:

$$1/\zeta_{dyn}(s) = F_{dyn}(s)/F_0^2(s) \, . \tag{37}$$

This relation implies that for *flows* the zeta function has double poles at the zeros of the scalar advection Fredholm determinant, with zeros of the dynamo Fredholm determinant no longer coinciding with the zeros of the scalar advection Fredholm determinant; the leading zero of the dynamo Fredholm determinant is larger than the scalar advection rate, and the rate of decay of the magnetic field is no longer governed by the scalar advection.

NUMERICAL INVESTIGATIONS

For our numerical investigations of the kinematic dynamo effect we have used (11) with $\overline{T} = 1$, $a = 6$, and $\alpha \in [0, 1]$. For $a = 6$ the Hénon map has a repeller with complete binary grammar. We have computed all periodic orbits up to length 13. The period of the prime cycle p is obtained by summing up the contributions (11):

$$T_p = n_p + \alpha \sum_{i=0}^{n_p - 1} x_{p,i} \, . \tag{38}$$

where $x_{p,i}$ are the periodic points in the prime cycle p. In table 1 we list the stabilities and $\sum x_{p,i}$ for the cycles up to length $n_p \leq 6$. From such a listing one can generate the prime periods for any parameter value α, feed this data into the cycle expansions, and extract a set of leading eigenvalues by determining the zeros of $F(s)$ in the complex plane by standard root finding routines. A visualization of the complex function $F(s)$ is afforded by a contour plot of $\ln|F(s)|$. In such a plot the eigenvalues correspond to the minima of $\ln|F(s)|$, and the range of validity of a cycle expansion is indicated by absence of fine structure beyond some $\mathrm{Re}(s)$, which reveals the effects of truncation.

Figs. 1(a) and 1(b) show contour plots of Fredholm determinants for scalar and vector advection, respectively for a discrete-time two-dimensional flow. The example shown here is described by the Hénon map; other examples are given by Aurell and Gilbert[4]. For this case the dynamo Fredholm determinant has double zeros coinciding with the zeros of the scalar advection Fredholm determinant, as we already indicated. In this computation, we used all prime cycles up to topological length 12, with polynomial truncation yielding the 12 leading zeros in both cases. Since the zeros of the dynamo Fredholm determinant are double zeros, in the polynomial truncation only the first few are accurate, with the nonleading terms converging poorly.

The result for maps should be contrasted with the result for flows: Figs. 2(a) and 2(b) show the corresponding Fredholm determinants for a flow with large dispersion of return times, $\alpha = 1$. Here the zeros of the two Fredholm determinants do not coincide, and the leading zero of the dynamo Fredholm determinant yields the dynamo rate. Parenthetically, this lifting of eigenvalue degeneracy is very noticeable already for return time dispersion as small as $\alpha = 0.01$. Fig. 3 shows the contour plot corresponding to the dynamo zeta function (31) for this value of α. The leading zeros coincide with the leading zero of the dynamo Fredholm determinant, but this zeta has a double pole at the escape rate zero of $F_0(s)$ (see eq. (34)) which is very apparent in this plot. In contrast, the zeta function (33) for a discrete time map is trivial. Aurell and Gilbert's[4] study based on the baker's map does not show this because they introduced phase shifts into the flow.

Note that the dynamo zeta function does not require evaluation of the cycle eigenvalues Λ_p, yet the pole in the zeta function continues to yield the escape rate. This means that for an incompressible flow we can compute the escape rate simply from the periods, T_p, of the cycles. This suggests an interesting connection between the periods and stabilities of the system.

Another interesting property of the fast dynamo flows is that in these systems the magnetic field can grow exponentially although the flow is repelling. The escape rate here is positive, and particles are lost to the volume as time proceeds. Thus the scalar advection determinant has poles with positive imaginary part in contrast to the dynamo determinant's poles (see fig. 2).

CONCLUSIONS

We have analyzed the transfer operator for chaotic flows whose leading eigenvalue yields the dynamo rate of the fast kinematic dynamo. Using a cycle expansion of the Fredholm determinant of this operator, we have evaluated its spectrum in various cases. In particular, we have given the leading-order eigenvalues of the spectrum for a dicrete-time flow governed by Hènon's map, and for a volume-preserving continuous flow given by a much-studied dynamical system. For the map, the dynamo rate is the same as the escape rate of scalar quantities, and there is no fast dynamo. For the flow, a spread in Poincaré section return times lifts the degeneracy of the vector and scalar advection rates, and leads to dynamo rates that dominate over the scalar advection rates. For sufficiently large time spreads, we have even found repellers for which the magnetic field *grows* exponentially, even though the scalar densities are decaying exponentially.

These first results are encouraging and we aim to amplify them for a greater range of flows, for as yet they are merely illustrative of the possibilities. To proceed, we need to elaborate on the asymptotic scheme used to obtain the periodic orbits and to supplement it with other, more flexible ways to provide the cycles needed in the expansions. We need also to see whether the inclusion of diffusion will have a noticeable effect on the effectiveness of the cycle expansions.

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FIGURE CAPTIONS

Figure 1: Contour plot of $\log |F(s)|$ for (a) the scalar advection Fredholm determinant, and for (b) the dynamo Fredholm determinant for a discrete-time flow governed by the Hénon map using prime cycles up to topological length 12. Note that the scalar and vector advection rates coincide.

Figure 2: Contour plot of $\log |F(s)|$ for (a) the scalar advection Fredholm determinant, and (b) the dynamo Fredholm determinant for the nearly homoclinic flow described by equations (10), (11) and (12) with $\alpha = 1$. Prime cycles up to topological length 12 were used. Note that for flows the vector advection eigenvalue (fast dynamo rate) dominates the scalar advection eigenvalue.

Figure 3: Contour plot of the dynamo zeta for the nearly homoclinic flow with $\alpha = 0.01$.

period	Λ_p	$\sum x_{p,i}$	code
1	0.715167524380×10^1	-0.60762521851077	0
1	$-0.295284632592\times10^1$	0.27429188517743	1
2	$-0.989897948557\times10^1$	0.33333333333333	10
3	$-0.131907273972\times10^3$	-0.20601132958330	100
3	0.558969649960×10^2	0.53934466291663	110
4	$-0.104430107304\times10^4$	-0.81649658092773	1000
4	0.577998269891×10^4	0.00000000000000	1100
4	$-0.103688325098\times10^3$	0.81649658092773	1110
5	$-0.760653437184\times10^4$	-1.42603220657928	10000
5	0.444552400077×10^4	-0.60665407777388	11000
5	0.770202485970×10^3	0.15137550164056	10100
5	$-0.710688356166\times10^3$	0.24846322760447	11100
5	$-0.589498852840\times10^3$	0.87069547289495	11010
5	0.390994248124×10^3	1.09548541554650	11110
6	$-0.545745270604\times10^5$	-2.03413425566653	100000
6	0.322220609858×10^5	-1.21525043702153	110000
6	0.513761651093×10^4	-0.45066243593297	101000
6	$-0.478461466317\times10^4$	-0.36602540378444	111000
6	$-0.639399984360\times10^4$	0.33333333333333	110100
6	$-0.639399984360\times10^4$	0.33333333333333	101100
6	0.390193872690×10^4	0.54858377035486	111100
6	0.109490945979×10^4	1.15146335826616	111010
6	$-0.104338416941\times10^4$	1.36602540378444	111110

Table 1. All periodic orbits up to 6 bounces for the Hamiltonian Hénon mapping (11), $a = 6$. The columns list the topological length of the cycle, its expanding eigenvalue Λ_p, the fluctuation period of the orbit, and the binary code for the cycle.