Cavity scattering in elastodynamics: wave versus ray dynamics

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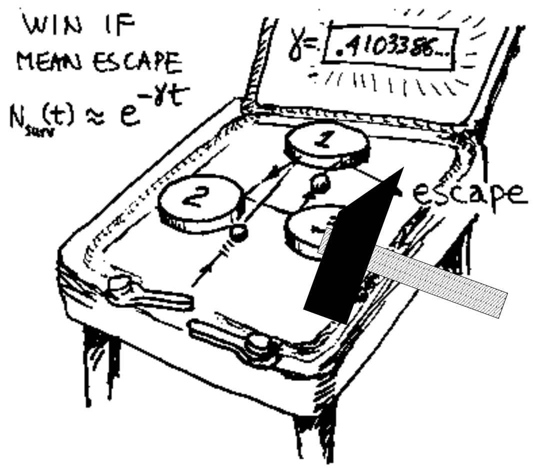

1. Setup of the problem.
2. Wave equation and boundary condition.
3. One-cavity problem.
4. Setup of the multi-cavity problem.
5. Numerical results.

Collaborators:

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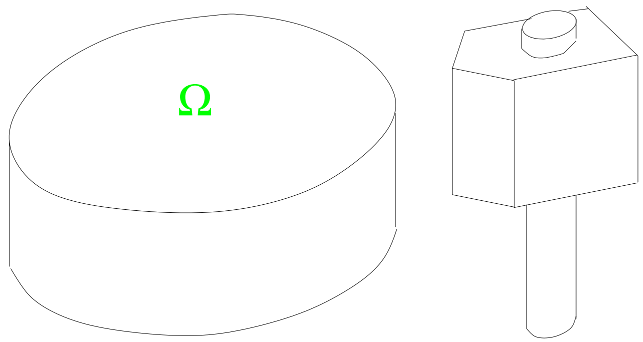
Setup of the problem:

Recipé:

1. Take a slab of an elastic (isotropic) material (e.g. quartz, polyethelene, . . .) with parallel top and bottom faces.
2. Assume planar strain:
   $\exists$ Pressure (longitudinal) and Shear (transverse) bulk propagating waves.
   No SV. Thus quasi-2-dimensional.
3. Drill out cylindrical cavities ($\perp$ faces)
4. Scattering resonances and phase shifts?



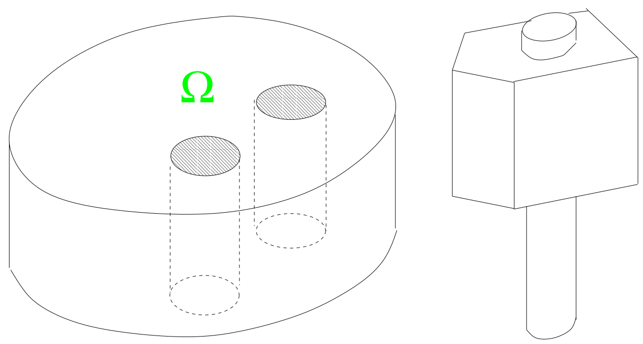
Features and virtues:

a. Simple and cheap experiments.

b. Very high Q-values ($\sim 10^6$) at lab temperatures !

c. New phenomena:
- vectorial dynamics versus scalar dynamics in QM
- ray-splitting of P- and S-modes at the boundaries,
- and surface waves without attenuation $\Rightarrow$ radically different periodic orbits.

d. Similarities and differences to quantum mechanics and semiclassical p.o. theory.

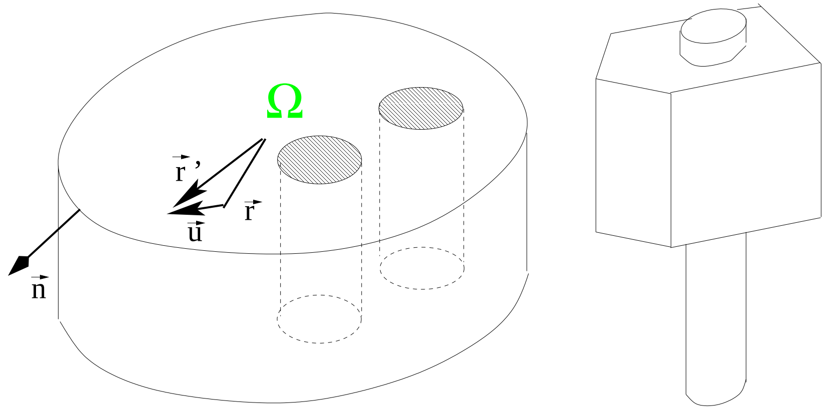


Isotropic elasticity

The PDE: *The Navier-Cauchy equation*

$$\underbrace{\mu\boldsymbol{\nabla}^2\mathbf{u} + (\lambda+\mu)\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{u}\right)}_{\boldsymbol{\Delta}^*(\mathbf{u})} + \rho\omega^2\mathbf{u} = \mathbf{0}$$

$\mathbf{u}(\mathbf{r},t)$:	displacement vector
λ and μ:	Lamé constants
ρ:	density
ω:	frequency



The BC: *free* boundaries / vanishing traction

$$\mathbf{0} = \mathbf{t}(\mathbf{u}) \equiv \left[\lambda\Big(\boldsymbol{\nabla}\cdot\mathbf{u}\Big)\,\mathbb{1} + \mu\Big\{\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^t\Big\}\right]\cdot\mathbf{n}$$

Two wave equations in one

1. Separation into scalar and vector *elastic potentials*: $\mathbf{u} = \underbrace{\boldsymbol{\nabla}\phi}_{\mathbf{u}_L} + \underbrace{\boldsymbol{\nabla} \times \boldsymbol{\Psi}}_{\mathbf{u}_T}$.

2. Insert and find $scalar$ and $vectorial$ Helmholtz equations:

$$\Delta\phi + k_L^2\phi = 0 \qquad (k_L = \omega/c_L : \text{ longitudinal})$$

$$\Delta\boldsymbol{\Psi} + k_T^2\boldsymbol{\Psi} = 0 \qquad (k_T = \omega/c_T : \text{ transverse})$$

3. Planar strain / Reduction to two dimensions:
   Assume polarization entirely in scattering plane

$$\boldsymbol{\Psi} := \hat{\mathbf{z}}\psi \ .$$

   Thus general ansatz for *displacement vector*

$$\mathbf{u} = \mathbf{u}_L + \mathbf{u}_T = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \hat{\mathbf{z}}\ \psi.$$

4. Scattering states formed from partial-wave decompositon of the elastic potentials (as in QM$_{2D}$):

$$\phi_l, \psi_l \in \{J_l(kr), H_l^{(+)}(kr), H_l^{(-)}(kr)\}e^{il\theta} \quad \text{where } l = 0, \pm1, \pm2, \pm3, \ldots$$

(*regular*$^{(\wedge)}$, *out-*$^{(+)}$ or *in-going*$^{(-)}$ partial waves)

e.g.: $\mathbf{u}_{lL}^{(\wedge)}(\mathbf{r}) = \boldsymbol{\nabla}\phi_l^{(\wedge)} = \boldsymbol{\nabla}J_l(kr)e^{il\theta},$

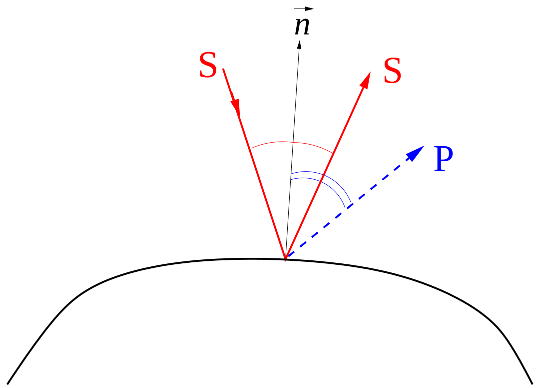
$\mathbf{u}_{lT}^{(+)}(\mathbf{r}) = \boldsymbol{\nabla} \times \hat{\mathbf{z}}\,\psi_l^{(+)} = \boldsymbol{\nabla} \times \hat{\mathbf{z}}\,H_l^{(+)}(kr)e^{il\theta},$ etc.

Three Kinds of Waves / Rays

Mode Conversion

Different wave speeds: Shear and Pressure waves

$$c_T = \sqrt{\frac{\mu}{\rho}} \quad < \quad c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

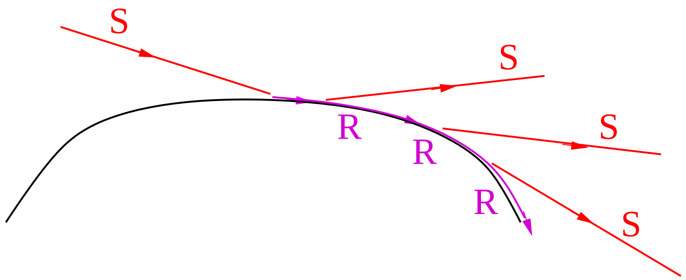


and Surface waves

Rayleigh waves: occur in earthquakes

$c_R < c_T < c_L$ with

$$0 = \left(2\left(\frac{c_T}{c_R}\right)^2 - 1\right)^2 - 4\left(\frac{c_T}{c_R}\right)^2 \sqrt{\left(\frac{c_T}{c_R}\right)^2 - \left(\frac{c_T}{c_L}\right)^2} \sqrt{\left(\frac{c_T}{c_R}\right)^2 - 1}$$



(in polyethelene $c_L \approx$ 1950 m/s, $c_T \approx$ 540 m/s, and $c_R \approx$ 513 m/s)

One-cavity scattering problem

The scattering off a single cylindrical cavity in an infinite slab is *separable*

(as the 1-disk problem of QM where $S^{(1)}_{m,m'} = -\frac{\mathrm{B.C.}\,[H^{(2)}_{m'}(ka)]}{\mathrm{B.C.}\,[H^{(1)}_{m}(ka)]}\delta_{mm'}$.)

$\Rightarrow$ one-cavity S-matrix of elastodynamics determined by the *traction matrices* :

$$S^{(1)}_{m\pi,m'\pi'} = -\mathbf{t}[\psi^{(+)\sim}_{\mathbf{m}}]^{-1}\cdot\mathbf{t}[\psi^{(-)\sim}_{\mathbf{m}'}]\,\delta_{mm'} = \delta_{mm'}\delta_{\pi\pi'} - 2\delta_{mm'}\,\mathbf{t}[\psi^{(+)\sim}_{\mathbf{m}}]^{-1}\cdot\mathbf{t}[\psi^{(\wedge)\sim}_{\mathbf{m}'}], \quad \begin{array}{l}\mathbf{m}\equiv(m,\pi),\\ \mathbf{m}'\equiv(m',\pi').\end{array}$$

Insertion of radial out- and in-going scattering states

$$\mathbf{u}^{(+/-)\sim}_{P\,m} = \boldsymbol{\nabla}\, H^{(1/2)}_m(k_L r)e^{-im\theta} \quad \text{and} \quad \mathbf{u}^{(+/-)\sim}_{S\,m} = \boldsymbol{\nabla}\times\hat{\mathbf{z}}\, H^{(1/2)}_m(k_T r)e^{-im\theta}$$

into traction $\mathbf{t}(\mathbf{u}) = \left[\lambda\left(\boldsymbol{\nabla}\cdot\mathbf{u}\right)\mathbb{1} + \mu\left\{\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^t\right\}\right]\cdot\mathbf{n}$ yields the $(P,S)\times(r,\theta)$ traction <u>*matrix*</u>

$$\mathbf{t}[\psi^{(+)\sim}_{\mathbf{m}}] = \begin{pmatrix} t(P^{(+)}_m)_r & t(S^{(+)}_m)_r \\ t(P^{(+)}_m)_\theta & t(S^{(+)}_m)_\theta \end{pmatrix} e^{-im\theta}$$

$$= \frac{2\mu}{a^2}\begin{pmatrix} (m^2-\frac{1}{2}k_T^2a^2)H^{(1)}_m(k_La) - ak_L H^{(1)'}_m(k_La) & im\{H^{(1)}_m(k_Ta) - k_T H^{(1)'}_m(k_Ta)\} \\ im\{H^{(1)}_m(k_La) - ak_L H^{(1)'}_m(k_La)\} & ak_T H^{(1)'}_m(k_Ta) - (m^2-\frac{1}{2}k_T^2a^2)H^{(1)}_m(k_Ta) \end{pmatrix} e^{-im\theta}$$

and similarly for $\mathbf{t}[\psi^{(-)\sim}_{\mathbf{m}}]$ and $\mathbf{t}[\psi^{(\wedge)\sim}_{\mathbf{m}}]$.

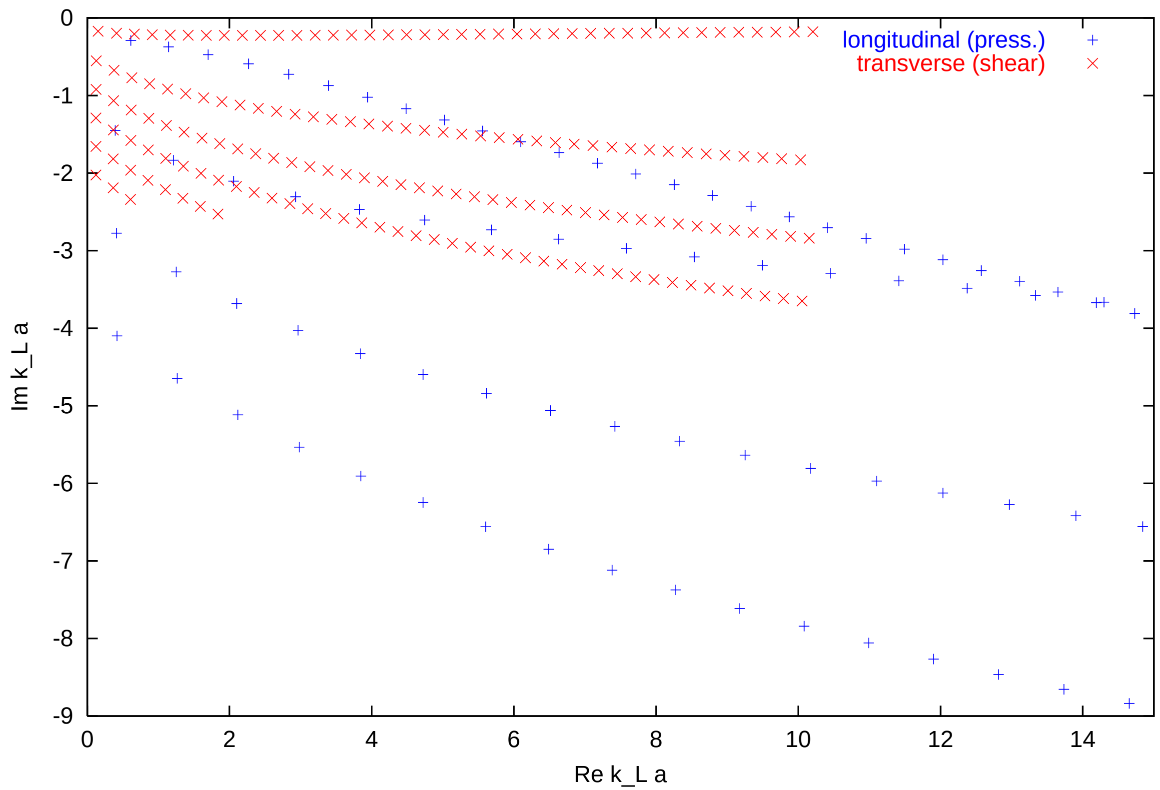
One-cavity scattering resonances

Resonances of elastodynamical 1-disk system with traction-free boundary cond.'s determined by:

$$0 = \mathrm{Det}\left(\mathbf{t}\left[\psi^{(+)\sim}_{\mathbf{m}}\right]\right) \qquad \mathbf{m} = \{m,\pi\} \text{ with } m = 0, \pm1, \pm2, \cdots \text{ and } \pi \in \{P, S\}$$

$$= \mathrm{Det}\begin{pmatrix} \left(m^2 - \frac{1}{2}k_T^2 a^2\right) H_m^{(1)}(k_L a) - a k_L H_m^{(1)'}(k_L a) & im\left\{H_m^{(1)}(k_T a) - k_T H_m^{(1)'}(k_T a)\right\} \\ im\left\{H_m^{(1)}(k_L a) - a k_L H_m^{(1)'}(k_L a)\right\} & a k_T H_m^{(1)'}(k_T a) - \left(m^2 - \frac{1}{2}k_T^2 a^2\right) H_m^{(1)}(k_T a) \end{pmatrix}$$

data for polyethelene with c_L / c_T = 1950 / 540

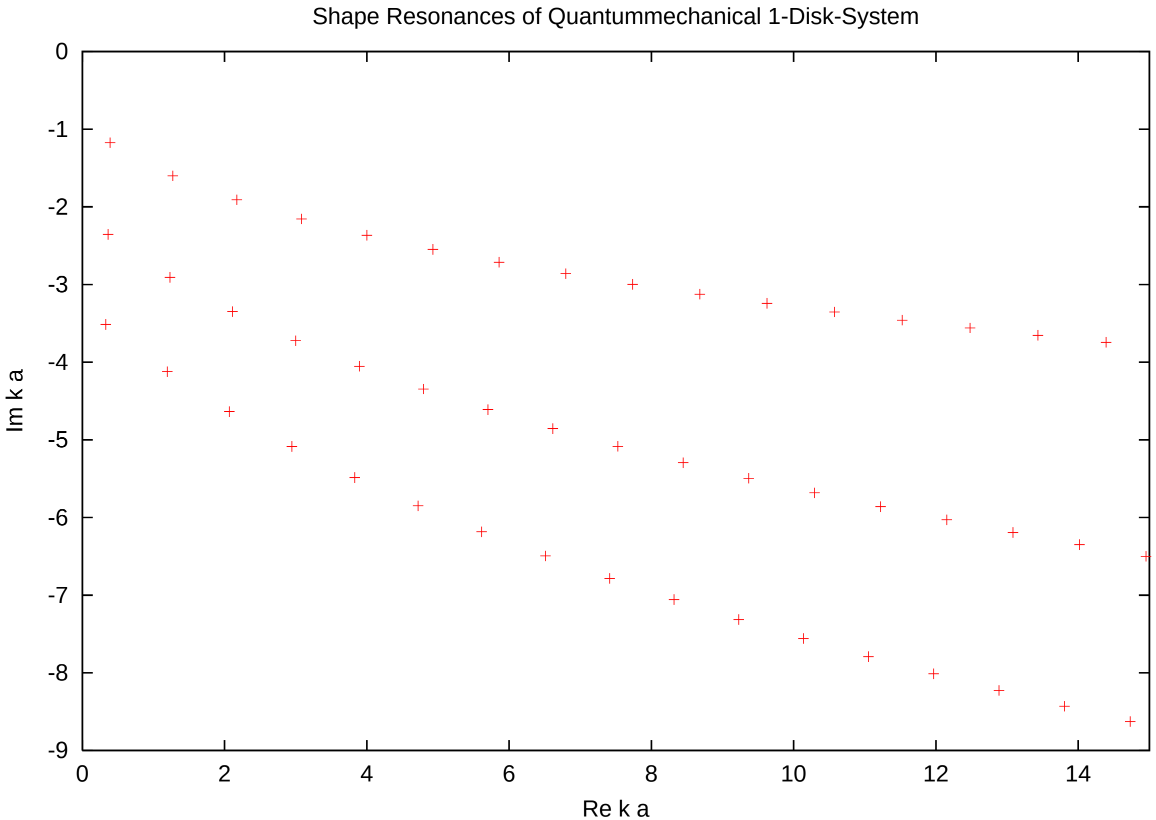


(see also J.L. Izbicki et al., Wave Motion **28**, 227-239 (1998).)

1-disk scattering resonances in QM

Resonances of q.m. 1-disk system with Dirichlet boundary conditions determined by:

$$H_m^{(1)}(ka) = 0, \qquad m = 0, \pm 1, \pm 2, \ldots$$

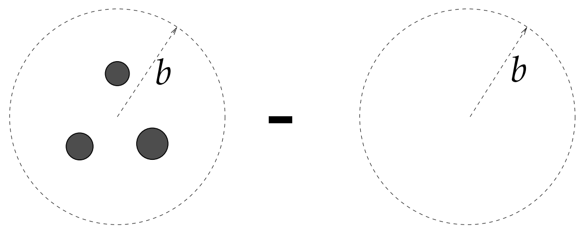


Semiclassical Interpretation of 1-disk S-matrix

$$\det \mathbf{S}^{(1)}(ka) \xrightarrow{\text{s.c.}} \left\{e^{-i\pi N(k)}\right\}^2 \frac{\prod_{\ell=1}^{\infty}\left[1-e^{i2\pi\nu_\ell(k^*)^*}\right]^2}{\prod_{\ell=1}^{\infty}\left[1-e^{i2\pi\nu_\ell(k)}\right]^2}$$

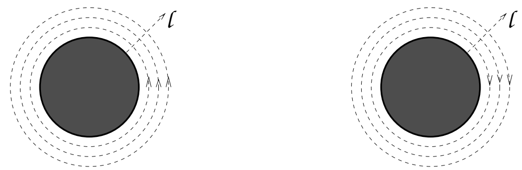
Interpretation:

1. Weyl-term $N(k) = \overbrace{\frac{\pi a^2}{4\pi}}^{\text{surf.}} k^2 + \overbrace{\frac{2\pi a}{4\pi}}^{\text{circ.}} k + \cdots$ according to Krein formula for d.o.s:



Excluded "volume" of n disks $\longrightarrow$ $N(k)$ with *minus* sign and *Neumann* subleading term (although *Dirichlet* conditions $\psi(\mathbf{r})=0$ on disk boundaries!)

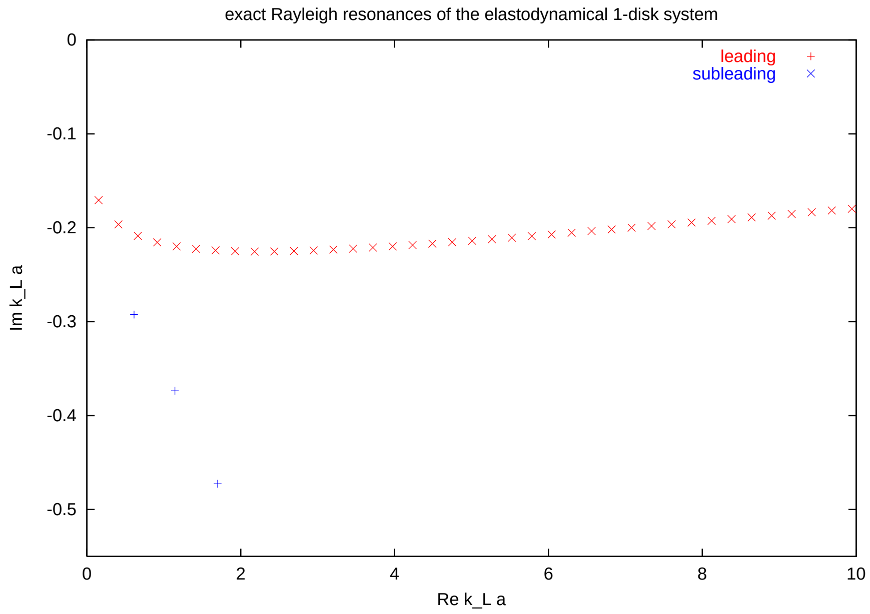
2. Creeping poles $\nu_\ell(k) \sim ka - e^{i\frac{\pi}{3}}\left(\frac{ka}{2}\right)^{\frac{1}{3}} q_\ell$ as creeping periodic orbits: [q_ℓ: ℓ-th zero of Ai(q)]

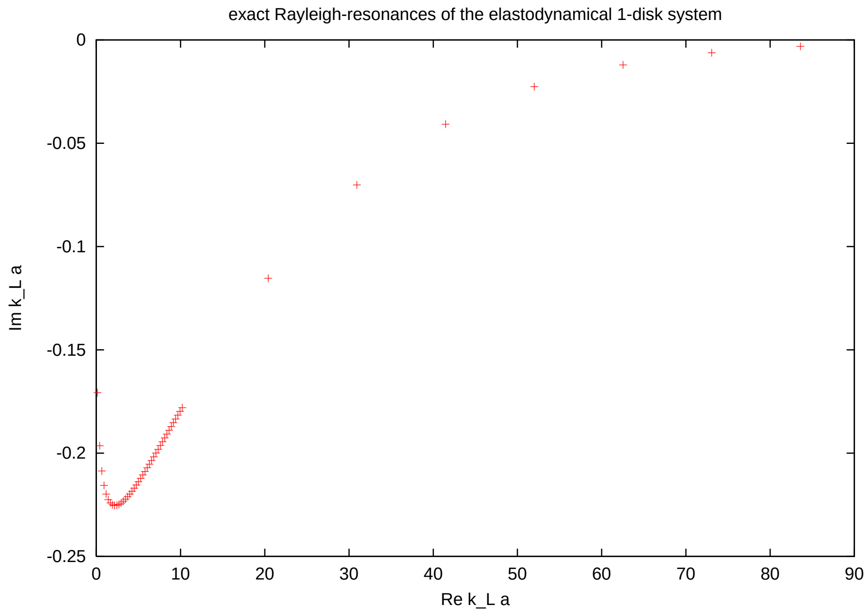


Note: *left-handed* & *right-handed* creeping p.o.'s $\Rightarrow$ power of 2

Rayleigh resonances in elastodynamics

The Rayleigh resonances have no counterpart in QM



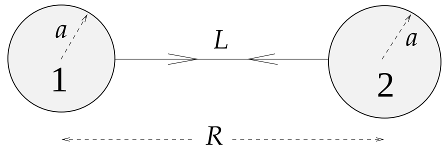


$\mathrm{Re}\,\nu_R(k) = 1.0525597 k_T a + 2.1436297 + 26.845142/k_T a + 1847.9982/k_T^2 a^2$ for polyethelene

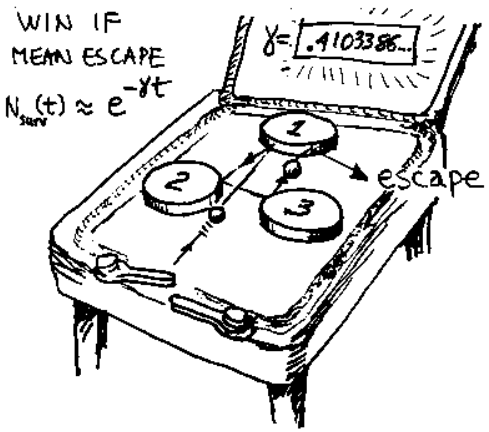
no simple semiclassical expression for $\mathrm{Im}\,\nu_R(k)$

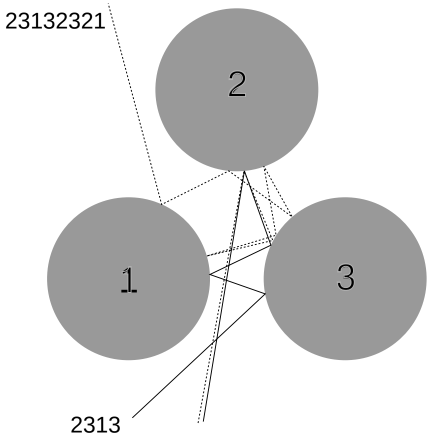
- Not separable for $n \geq 2$.
- The scattering of a *point particle* from $1 < n < \infty$ non-overlapping disks that are fixed in the two-dimensional plane contains some of the simplest scattering problems:

1. *2-disk scattering problem* is classically hyperbolic, but non-chaotic:
   only one periodic orbit $\overline{12}$ which is unstable because of defocusing nature of reflection



2. *3-disk scattering problem* is classically hyperbolic *and* chaotic:
   exponentially proliferating number of unstable, isolated p.o.'s, labelled by a 3-letter symbolic dynamics (successive letters must be different)





References:

B. Eckhardt, *J. Phys.* A20 (1987),

P. Cvitanović & B. Eckhardt, *PRL* 63 (1989),

P. Gaspard & S. Rice, *J. Chem. Phys.* 90 ('89).

Programme for solving the n-cavity scattering problem

- Construction of the *outgoing vectorial(!) wave* in terms of the incoming vectorial wave in order to read off the *S-matrix*:
  - either via a direct construction of the *cavity dyadic Green's function*
  - or via a construction of the *free* dyadic Green's fct. plus application of *Green's theorem*.
- Generation of the *determinant* of the S-matrix:
  - only sensitive to *spectral* properties of the problem and not to wave-function aspects,
  - independent of coordinate system;
  - linked to physical observables as resonances, phase shifts, Wigner time-delays;
  - existence linked to trace-class property of T-matrix and multi-scattering kernels.
- Separation of the one-cavity contributions from the n-cavity contributions:
  - doable since one-cavity scattering itself is separable,
  - simplification of the spectrum,
  - only *generic multi-scattering*.
- Ray-dynamical short-wave approximation of multi-scattering matrix via *periodic orbits*.

satisfying $\left(\boldsymbol{\Delta}^*_{\mathbf{r}} + \rho\omega^2\, \mathbb{1}\right)\mathbf{G}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$

where $\boldsymbol{\Delta}^*_{\mathbf{r}}\mathbf{G}(\mathbf{r},\mathbf{r}') \equiv \mu\nabla^2_{\mathbf{r}}\mathbf{G}(\mathbf{r},\mathbf{r}') + (\lambda+\mu)\boldsymbol{\nabla}_{\mathbf{r}}[\boldsymbol{\nabla}_{\mathbf{r}}\cdot\mathbf{G}(\mathbf{r},\mathbf{r}')]$

1. Frequency dependent and two dimensional GF:

$$\mathbf{G}(\mathbf{r},\mathbf{0}) = \frac{1}{4i\mu}\left\{H_0^{(+)}(k_T r)\,\mathbb{1} + \frac{1}{k_T{}^2}\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\left(H_0^{(+)}(k_T r) - H_0^{(+)}(k_L r)\right)\right\}$$

2. Expressed in terms of scattering states:

$$\mathbf{G}(\mathbf{r},\mathbf{r}') = \frac{1}{4i\rho\omega^2}\sum_{l=-\infty}^{\infty}\sum_{\pi\in\{L,T\}}\mathbf{u}_{l\pi}^{(\wedge)\sim}(\mathbf{r}_<)\otimes\mathbf{u}_{l\pi}^{(+)}(\mathbf{r}_>) = \frac{1}{4i\rho\omega^2}\sum_{l=-\infty}^{\infty}\left[\mathbf{u}_l^{(\wedge)\sim}(\mathbf{r}_<)\right]\cdot\left[\mathbf{u}_l^{(+)}(\mathbf{r}_>)\right]^t$$

where the operation "$\sim$" replaces $e^{il\theta}$ with $e^{-il\theta}$ and

$$[\mathbf{u}_l^{(\wedge)\sim}(\mathbf{r})] \equiv \begin{pmatrix} (u_{lL}^{(\wedge)})_r & (u_{lT}^{(\wedge)})_r \\ (u_{lL}^{(\wedge)})_\theta & (u_{lT}^{(\wedge)})_\theta \end{pmatrix} e^{-il\theta} \quad = \begin{pmatrix} \frac{\partial}{\partial r}J_l(k_L r) & \frac{-il}{r}J_l(k_T r) \\ \frac{-il}{r}J_l(k_L r) & -\frac{\partial}{\partial r}J_l(k_T r) \end{pmatrix} e^{-il\theta}$$

in polar coordinates.

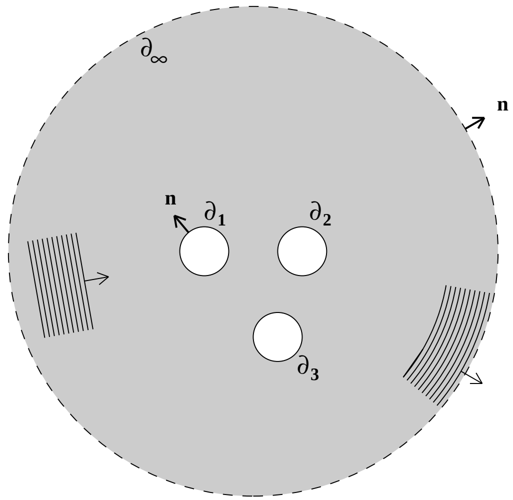
Boundary equations

From Betti's identity (analog of Green's theorem)

$$\int_\Omega \left\{ \mathbf{u}\cdot\boldsymbol{\Delta}^*\mathbf{v} - \mathbf{v}\cdot\boldsymbol{\Delta}^*\mathbf{u} \right\} d^2x = \int_{\partial\Omega} \left\{ \mathbf{u}\cdot\mathsf{t}[\mathbf{v}] - \mathbf{v}\cdot\mathsf{t}[\mathbf{u}] \right\} ds$$

we derive with the help of

1. $\left(\boldsymbol{\Delta}^*{}_{\mathbf{r}} + \rho\omega^2\,\mathbb{1}\right)\mathbf{G}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\,\mathbb{1}$,
2. $\mathsf{t}[\mathbf{u}(\mathbf{r})]\big|_{\mathbf{r}\in\partial\Omega_{\text{cavity}}} = \mathbf{0}$



the integral equations:

$$\underbrace{c(\mathbf{R})}_{0\ \text{or}\ 1}\mathbf{u}(\mathbf{R}) = \int_{\partial\Omega} ds \left\{ \mathbf{u}(\mathbf{s})\cdot\mathbf{t}\Big[\mathbf{G}(\mathbf{s},\mathbf{R})\Big] \underbrace{-\mathbf{t}\Big[\mathbf{u}(\mathbf{s})\Big]\cdot\mathbf{G}(\mathbf{R},\mathbf{s})}_{\text{if } \mathbf{s}\in\partial_\infty} \right\},$$

where $c(\mathbf{R})$ is *0* if $\mathbf{R}$ is inside the cavity
or *1* if $\mathbf{R}$ is *outside* in the elastic medium Ω.

Solving the equations

Expand the displacement field $\mathbf{u}(\mathbf{r})$ in Fourier modes

1. *at infinity in the global system:*

$$\mathbf{u} = \sum_{m,\pi;l,\sigma} \tfrac{1}{2} b^{(\infty,g)}_{m\pi} \left\{ \delta_{ml}\delta_{\pi\sigma} \left[\mathbf{\Psi}^{(-)}_{l}\right]^{t}_{\sigma;\cdot} + \mathbf{S}_{m\pi,l\sigma} \left[\mathbf{\Psi}^{(+)}_{l}\right]^{t}_{\sigma;\cdot} \right\} e^{il\theta^{(g)}}$$

   where $b^{(\infty,g)}_{m\pi} =: b_{m\pi} := b_{\mathbf{m}}$ are the Fourier coefficients (in angular momentum and polarization) at ∂_∞ (measured in the the global coordinate system).

   Note that the set $\{b_{m\pi}\}$ depends on the choice and details of the initial stimulation of the system.

2. *at the boundary of disk j in its local system:*

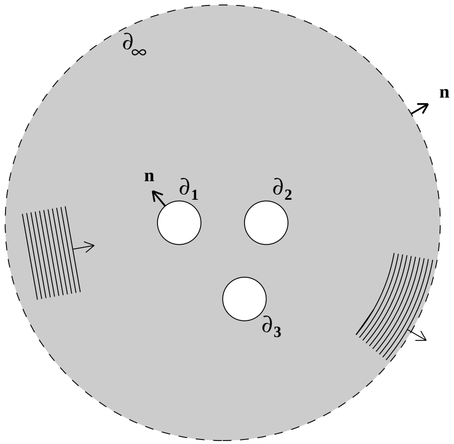
$$\mathbf{u} = \sum_{m'} \mathbf{B}^{(j)}_{m'}\left(\{b_{m\pi}\}\right) e^{im'\theta^{(j)}}$$

$$= \sum_{m} \left\{ \left(B^{(j)}_{m'}\right)_r \hat{\mathbf{r}}^{(j)} + \left(B^{(j)}_{m'}\right)_\theta \hat{\mathbf{\Theta}}^{(j)} \right\} e^{im'\theta^{(j)}}$$

   where the vectors $\mathbf{B}^{(j)}_{m'}\left(\{b_{m\pi}\}\right)$ are the Fourier coefficients (only in angular momentum!) of the resulting field at disk j (measured in its local coordinate system).

   Define $\widetilde{B}^{(j)}_{\mathbf{m}'}(b) := \widetilde{B}^{(j)}_{m'i}\left(\{b_{m\pi}\}\right) := \left(B^{(j)}_{m'}\right)_i$ with $i \in \{r,\theta\}$.

Infinite dimensional matrix equations



1. $\mathbf{R} \in$ cavity j, infinitely close to, but *not* on its boundary ∂_j:

   $\exists\ n$ homogeneous (integral) equations (labelled by the cavity index j) between the surface ∂_∞ at infinity and the surfaces of all n cavities (labelled by index j')

$$\underbrace{b \cdot \mathbf{C}^j}_{\partial\infty \to \partial j} = \sum_{j'} \underbrace{\widetilde{B}^{j'}(b) \cdot \mathbf{M}^{j'j}}_{\partial j' \to \partial j}$$

2. $\mathbf{R}$ at infinity (inside the elastic medium Ω):

   We deduce the $\mathbf{S}$-matrix from the field at infinity that is constrained by the boundaries at all n cavities in terms of the inhomogeneous integral equation

$$\underbrace{b \cdot \mathbf{S}}_{\text{inhom.}} = \underbrace{b \cdot \mathbf{1}}_{\partial\infty \to \partial\infty'} - i \sum_{j'} \underbrace{\widetilde{B}^{j'}(b) \cdot \mathbf{D}^{j'}}_{\partial j' \to \partial\infty'} .$$

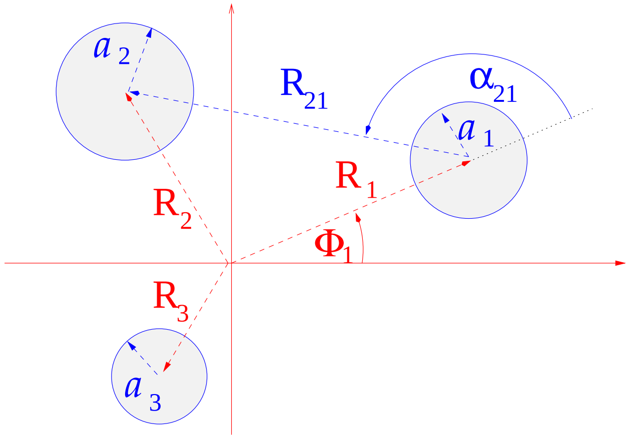
3. Note that the resulting eq.

$$b \cdot \mathbf{S} = b \cdot 1 - i\, b \cdot \sum_{j'} \sum_{j} \mathbf{C}^j \cdot \left(\mathbf{M}^{-1}\right)^{jj'} \cdot \mathbf{D}^{j'}$$

   has to hold for all possible choices of the initial stimulus b at $\partial\infty$ which therefore can be removed.

S-matrix for n cavities

$$\mathbf{S}(\omega) = \mathbf{1} - i \underbrace{\sum_{j=1}^{n}\sum_{j'=1}^{n} \mathbf{C}^{j}(\omega)\mathbf{M}^{-1\,jj'}(\omega)\mathbf{D}^{j'}(\omega)}_{\mathbf{T}\text{-matrix}}$$



$$\mathbf{D}^{j'}_{\mathbf{l}'\,\mathbf{m}'}(\omega) = -\frac{\pi a_{j'}}{\rho\omega^2}\ \mathbf{t}[\psi^{(\wedge)\sim}_{\mathbf{l}'}(a_{j'})] \cdot T^{\wedge j'\,G}_{\mathbf{l}'\,\mathbf{m}'}(R_j, \Phi_{j'})$$

(couples initial scattering channel $\mathbf{m}' = (m', \pi')$ to internal wave at cavity j'.)

$$\mathbf{M}^{j\,j'}_{\mathbf{l}\,\mathbf{l}'}(\omega) = \delta^{jj'}\delta_{\mathbf{l}\mathbf{l}'} + (1-\delta^{jj'})\frac{a_j}{a_{j'}}\left\{\mathbf{t}[\psi^{(\wedge)\sim}_{\mathbf{l}}(a_j)]\right\} \cdot T^{+j\,j'}_{\mathbf{l}\,\mathbf{l}'}(R_{jj'}, \alpha_{jj'}) \cdot \left\{\mathbf{t}[\psi^{(+)\sim}_{\mathbf{l}'}(a_{j'})]\right\}^{-1}$$

(inverse *multi*-scattering matrix: evolves internal wave

from channel $\mathbf{l}' = (l', \sigma')$ at cavity j' to the channel $\mathbf{l} = (l, \sigma)$ at cavity j.

The transfer matrix $\mathbf{M}^{jj'} - \delta^{jj'} \equiv \mathbf{A}^{jj'}$ is trace-class.)

$$\mathbf{C}^{\ j}_{\mathbf{m}\,\mathbf{l}}(\omega) = \rho\,\omega^2\ T^{\wedge G\,j}_{\mathbf{m}\,\mathbf{l}}(R_j, \Phi_j) \cdot \left\{\mathbf{t}[\psi^{(+)\sim}_{\mathbf{l}}(a_j)]\right\}^{-1} \frac{2i}{\pi a_j}$$

(couples internal wave at cavity j to out-going channel $\mathbf{m} = (m, \pi)$.)

$$T^{+jj'}_{l\sigma, l'\sigma'}(\omega) = \delta_{\sigma\sigma'}(-1)^{l'} H^{(1)}_{l-l'}(kR_{jj'})e^{il\alpha_{j'j} - il'\alpha_{jj'}}$$ with $T^{\wedge Gj} \cdot T^{\wedge jG} = \mathbb{1}$ and $T^{+jG} \cdot T^{\wedge Gj'} = T^{+jj'}$

(Translation matrix from coordinate system based at cavity j to coordinate system at cavity j'.)

Factorization of the spectral determinant

in incoherent and coherent parts:

$$\det \mathbf{S}^{(n)} = \frac{\mathrm{Det}\,[\mathbf{M} - i\mathbf{D}\mathbf{C}]}{\mathrm{Det}\,[\mathbf{M}]} = \left\{ \prod_{j=1}^{n} \det \mathbf{S}_j^{(1)}(\omega) \right\} \frac{\mathrm{Det}\,[\mathbf{M}(\omega^*)^\dagger]}{\mathrm{Det}\,[\mathbf{M}(\omega)]}$$

Same structure as in QM (see AW, Physics Reports **309**, 1999).

Product over one-disk scattering determinants:

- separable and existence guaranteed;
- no dependence on geometry of the scattering problem $\Rightarrow$ incoherent;
- excluded "volume" contribution deducible from this term

Ratio of complex-conjugated *genuine multi-scattering determinants*:

- existence linked to *trace-class* property of transfer matrix
- The genuine scattering resonances are the zeroes of $\mathrm{Det}\,\big[\mathbf{M}(\omega)\big]$.
- The one-cavity resonances cancel against the poles of $\mathrm{Det}\,\big[\mathbf{M}(\omega)\big]$.
- in QM: $\mathrm{Det}\,\mathbf{M}$ is the analog of the Gutzwiller's semiclassical spectral zeta-function: connection via cumulant expansion.

Cumulant expansion of $\mathrm{Det}\,\mathbf{M}(k)$

Expand $\mathrm{Det}\,\mathbf{M}(k){:=}\mathrm{Det}[\mathbf{1}{+}\mathbf{A}(k)]$ as *cumulant sum*:

$$\mathrm{Det}\,\mathbf{M}(k){=}\exp\mathrm{Tr}\ln[\mathbf{1}+\mathbf{A}(k)] = 1+\mathrm{Tr}\,\mathbf{A}(k) - \frac{1}{2}\left[\mathrm{Tr}\,\mathbf{A}^2 - (\mathrm{Tr}\,\mathbf{A})^2\right] + \cdots$$

(absolutely converging series since $\mathbf{A} \in$ trace-class!)

I.e., perform Taylor-expansion in book-keeping variable z

$$\mathrm{Det}[\mathbf{1}+z\mathbf{A}] = \sum_{m=0}^{\infty} z^m Q_m[\mathbf{A}] \qquad \text{(entire in z !),}$$

where the cumulants $Q_m[\mathbf{A}]$ and traces $\mathrm{Tr}[\mathbf{A}^i]$ satisfy the *Plemelj-Smithies* recursion relations:

$$Q_m[\mathbf{A}] = \frac{1}{m}\sum_{i=1}^{m}(-1)^{i+1}Q_{m-i}[\mathbf{A}]\times\mathrm{Tr}[\mathbf{A}^i] \qquad (m\geq 1).$$

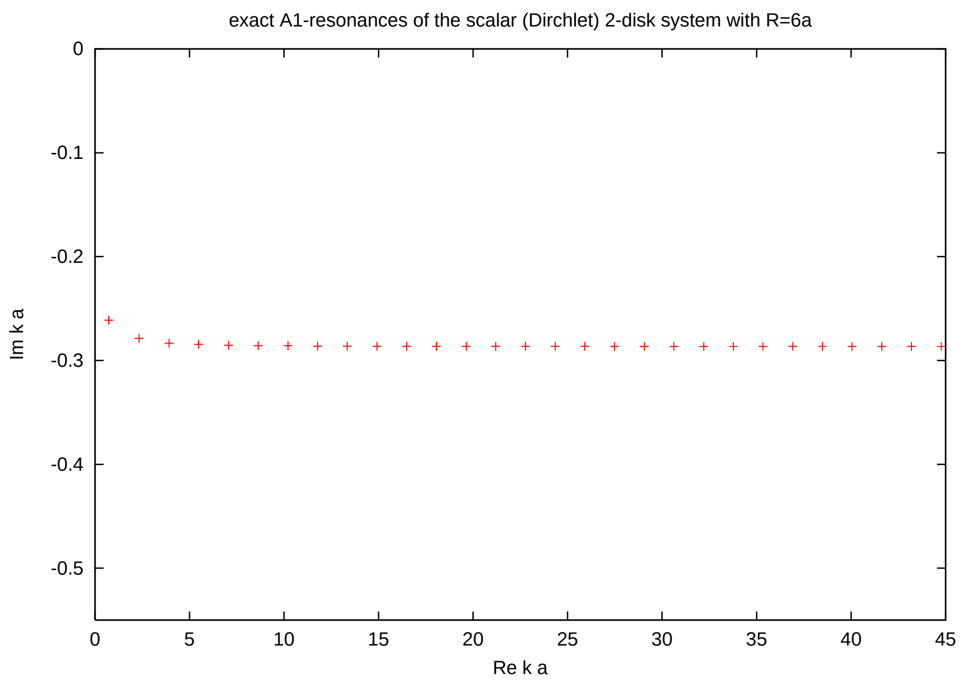
Cumulants are wave-dynamical analogs of the ray-dynamical *curvatures* of the *cycle expansion*.

Expand traces $\mathrm{Tr}[\mathbf{A}^m]$ into *quantum cycles/itineraries*

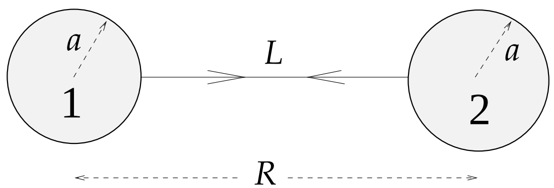
$$\mathrm{Tr}[\mathbf{A}^m] = \sum_{j_1=1}^{n}\cdots\sum_{j_m=1}^{n}\underbrace{\mathbf{A}^{j_1j_2}\mathbf{A}^{j_2j_3}\ldots\mathbf{A}^{j_mj_1}}_{\text{itinerary: } j_1j_2\cdots j_m}$$

Quantum cycles are the wave-dynamical analogs of the ray-dynamical *periodic orbits*.

Quantum mechanical resonances of scalar 2-disk system



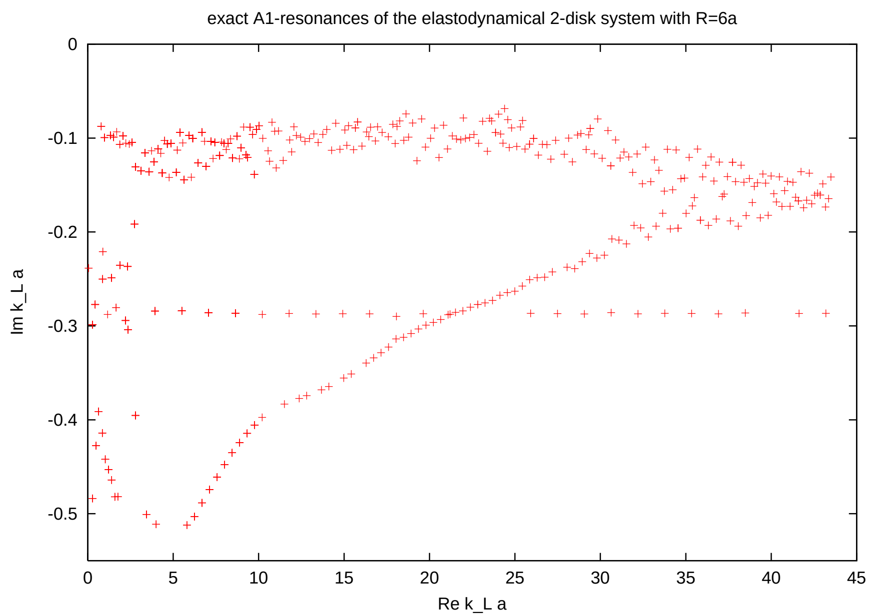
A1 representation and $R = 6a$

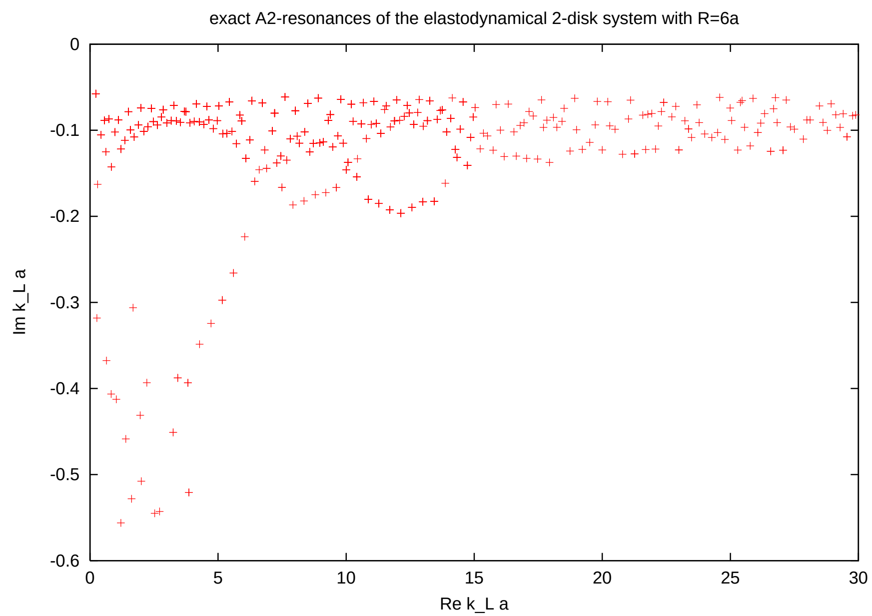


2-cavity resonances

Resonances of elastodynamical 2-disk system with traction-free boundary conditions

(for polyethelene and $R = 6a$)





A1-represention

A2-representation

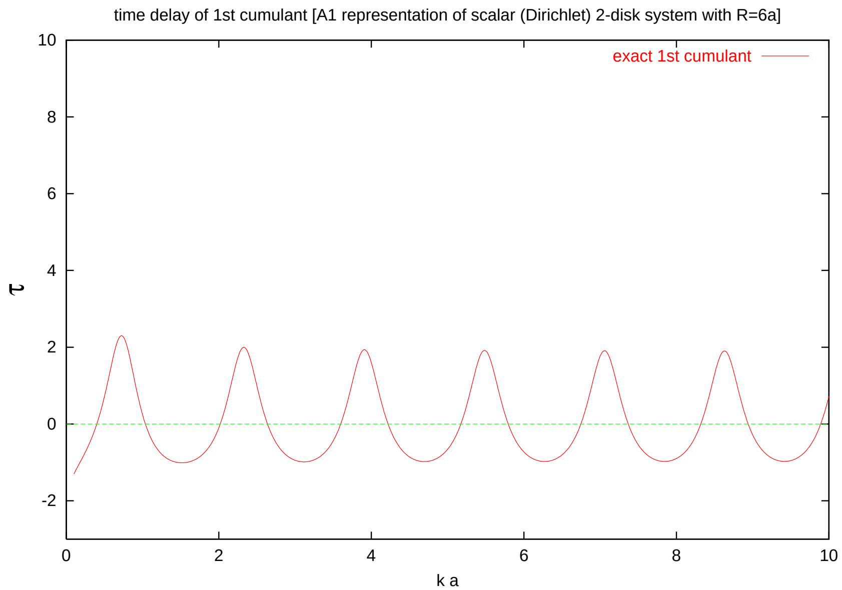
Time delay τ in 2-disk system of QM

Remember:

$$\det \mathbf{S}^{(n)} = \left\{ \prod_{j=1}^{n} \det \mathbf{S}_j^{(1)}(\omega) \right\} \frac{\mathrm{Det}\,\mathbf{M}(\omega)^\dagger}{\mathrm{Det}\,\mathbf{M}(\omega)}$$

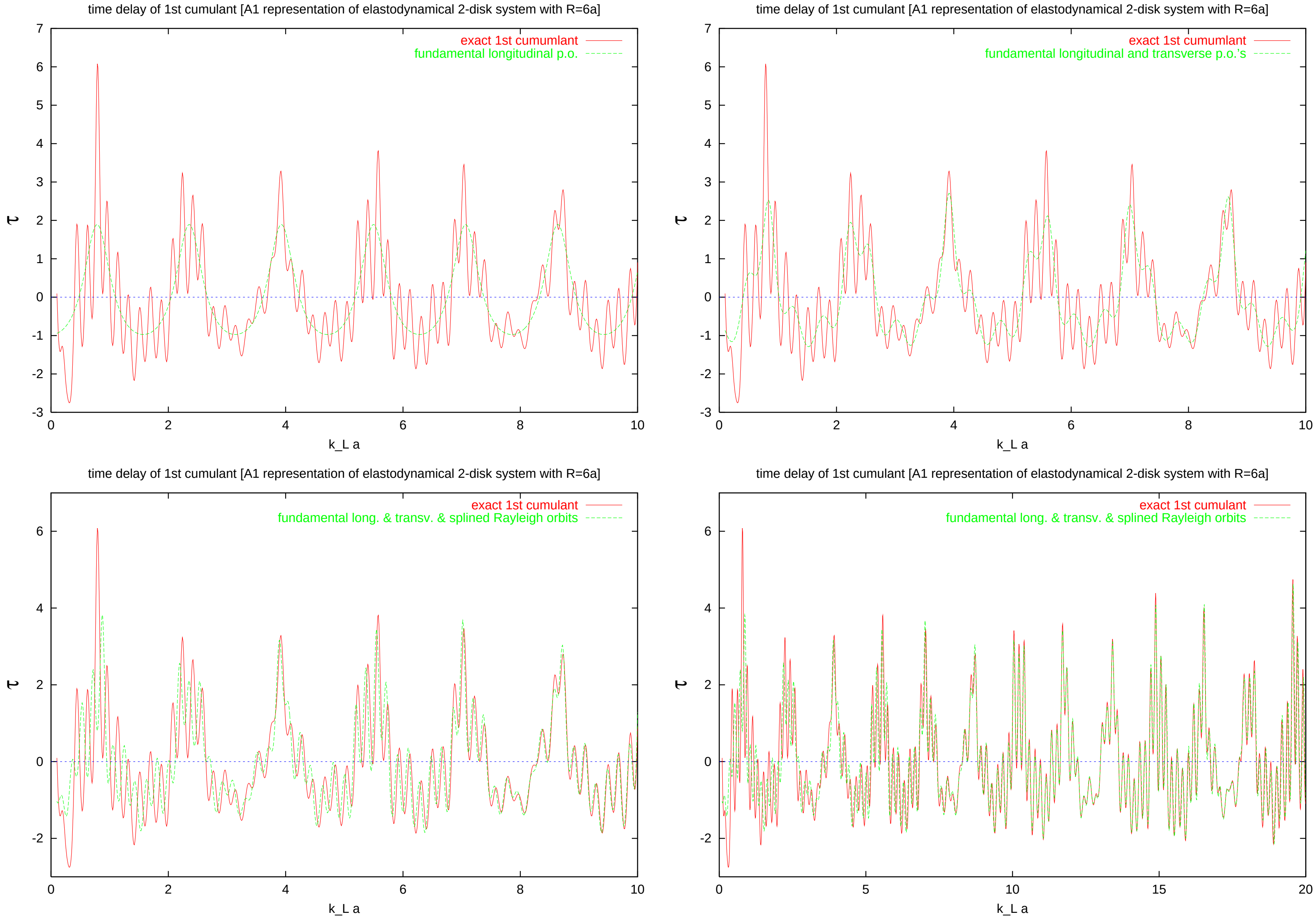
Definitions on the real ω axis:

$\eta_{\mathrm{tot}}(\omega)$	$= i^{-1} \ln \det \mathbf{S}^{(\mathbf{n})}$	total phase shift
$\eta_{\mathrm{clust}}(\omega)$	$= i^{-1} \ln \frac{\mathrm{Det}\,\mathbf{M}(\omega)^\dagger}{\mathrm{Det}\,\mathbf{M}(\omega)} = i \ln \frac{\mathrm{Det}\,\mathbf{M}(\omega)}{\lvert \mathrm{Det}\,\mathbf{M}(\omega) \rvert}$	cluster phase shift
$\tau(\omega)$	$= \frac{d}{d\omega} \eta_{\mathrm{clust}}(\omega)$	Wigner's time delay



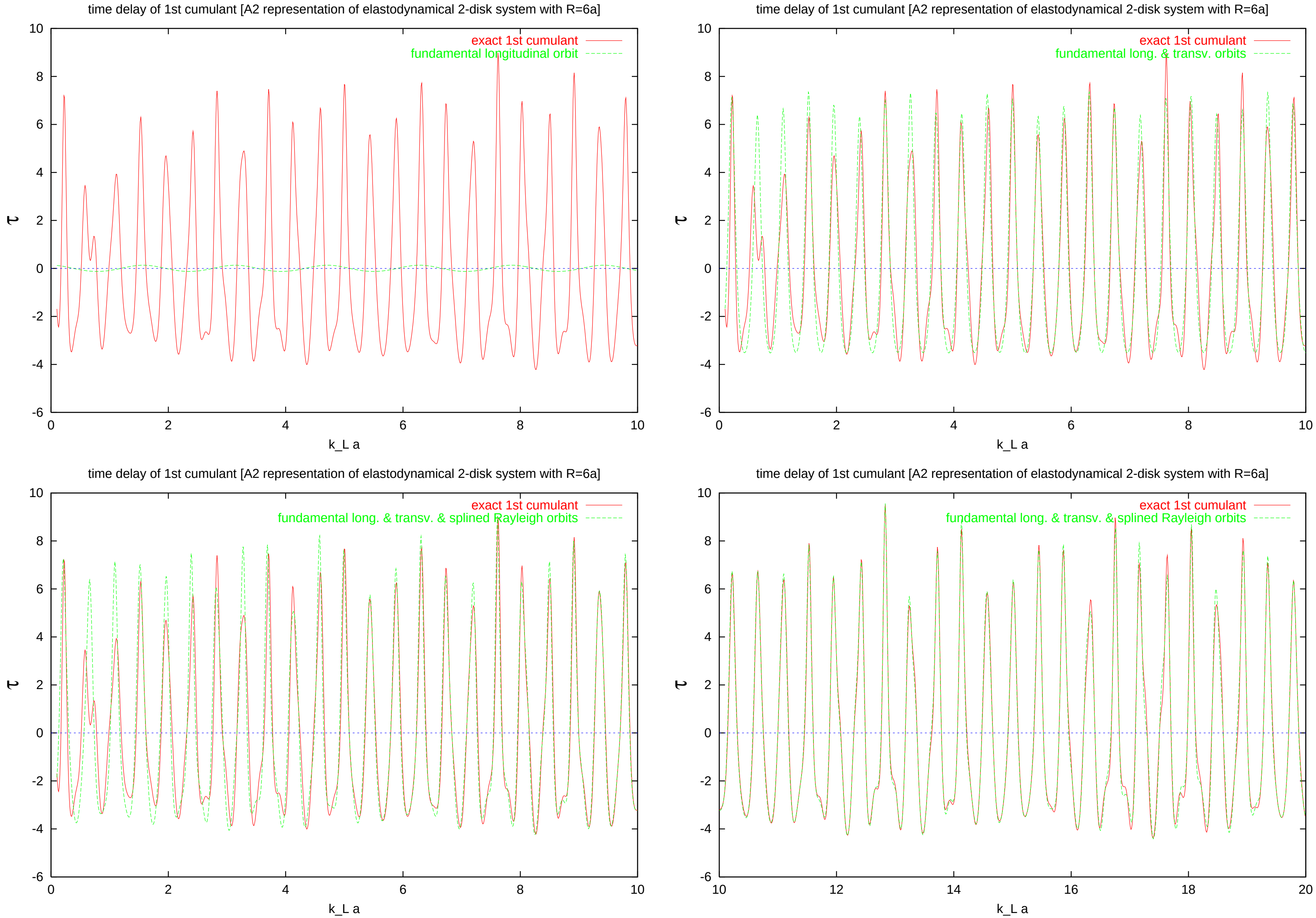
(in 1st cumulant approximation $\mathrm{Det}\,\mathbf{M} = 1 + \mathrm{Tr}(A)$, A1 representation, $R = 6a$)

time delay τ in 2-cavity system of elastodynamics



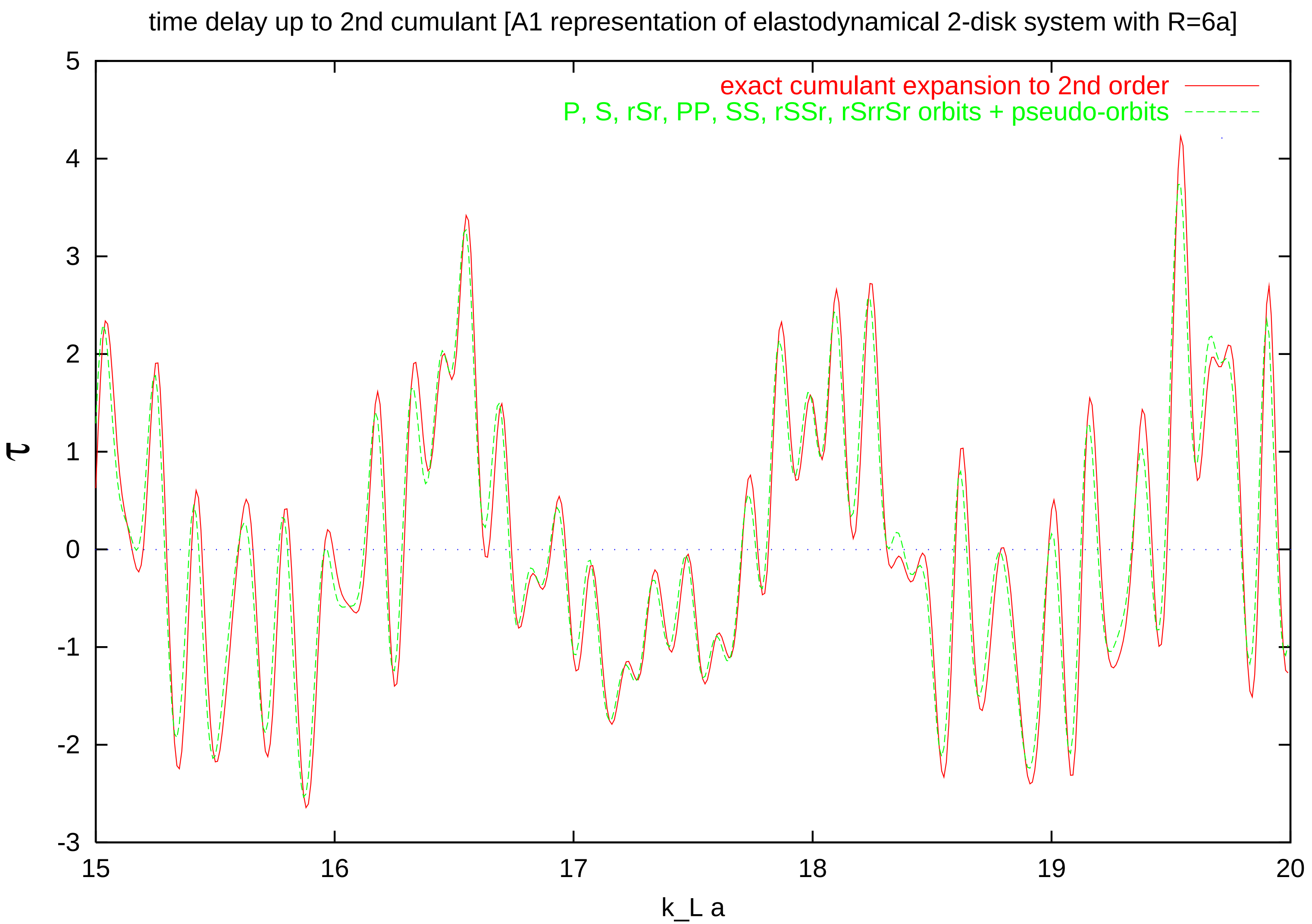
(in 1st cumulant approximation $\mathrm{Det}\,\mathbf{M} = 1 + \mathrm{Tr}(A)$, A1 representation, $R = 6a$)

time delay τ in 2-cavity system of elastodynamics



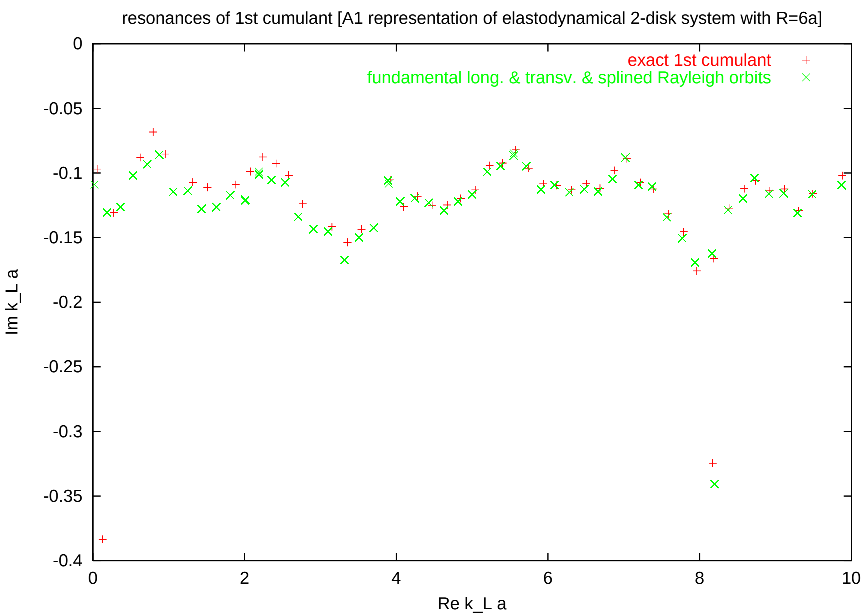
(in 1st cumulant approximation $\mathrm{Det}\,\mathbf{M} = 1 + \mathrm{Tr}(A)$, A2 representation, $R = 6a$)

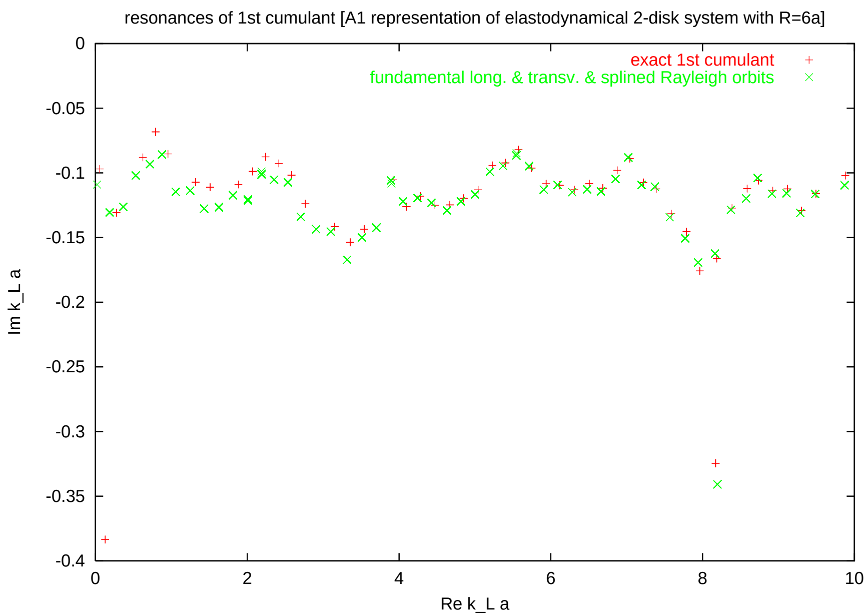
time delay τ in 2-cavity system of elastodynamics



(in 2nd cumulant approximation, A1 representation, $R = 6a$)

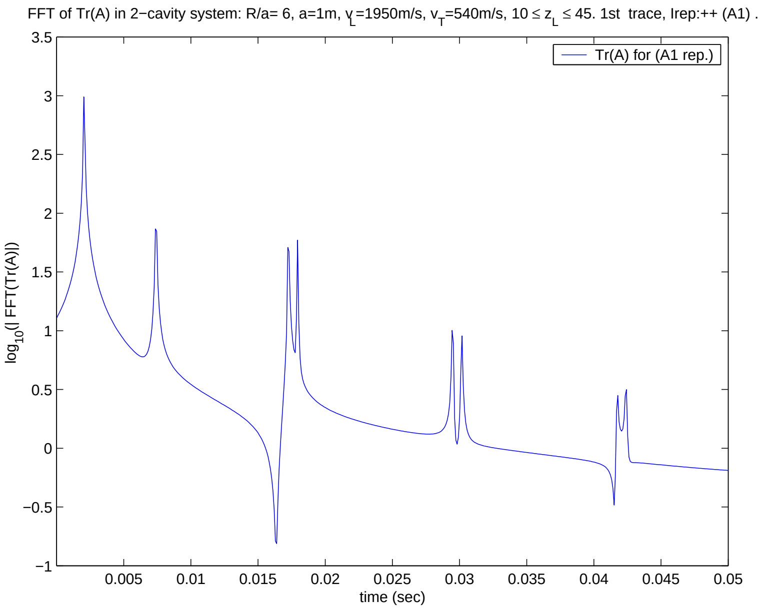
resonances of 2-cavity system of elastodynamics

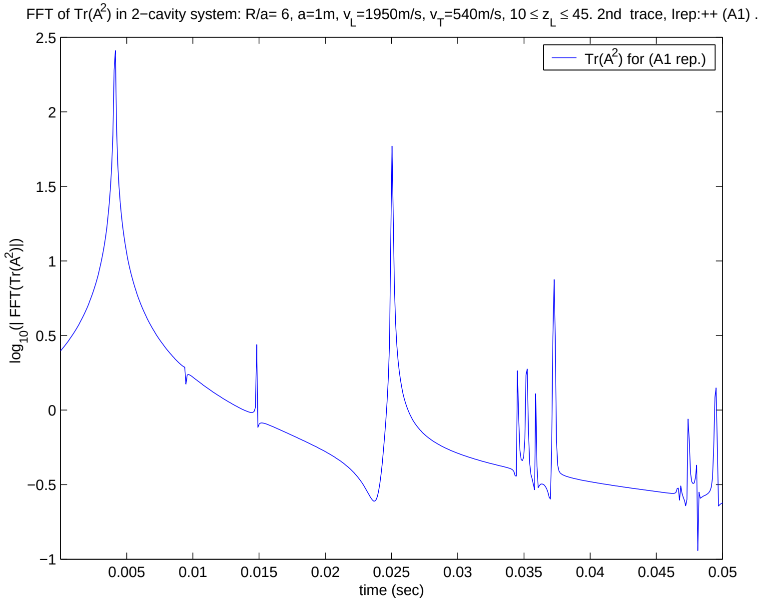




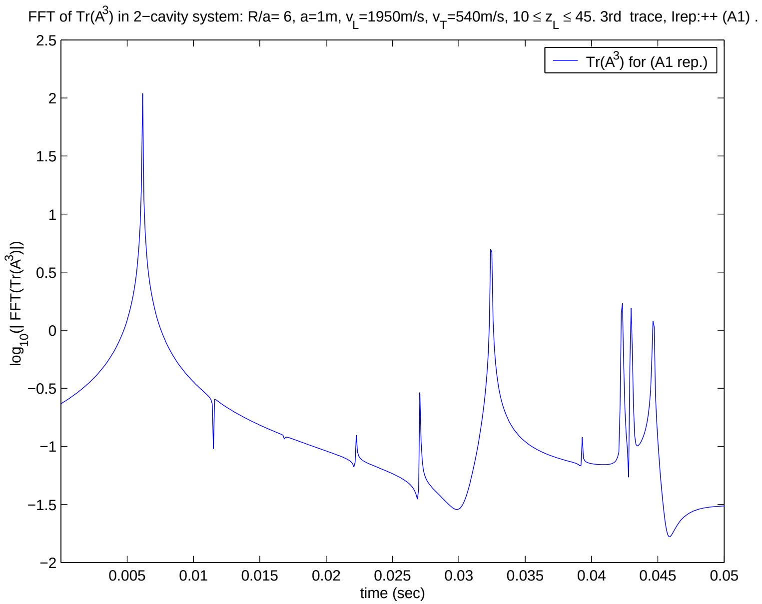
(in 1st cumulant approximation $\mathrm{Det}\,\mathbf{M} = 1 + \mathrm{Tr}(A)$, A1 representation, $R = 6a$)

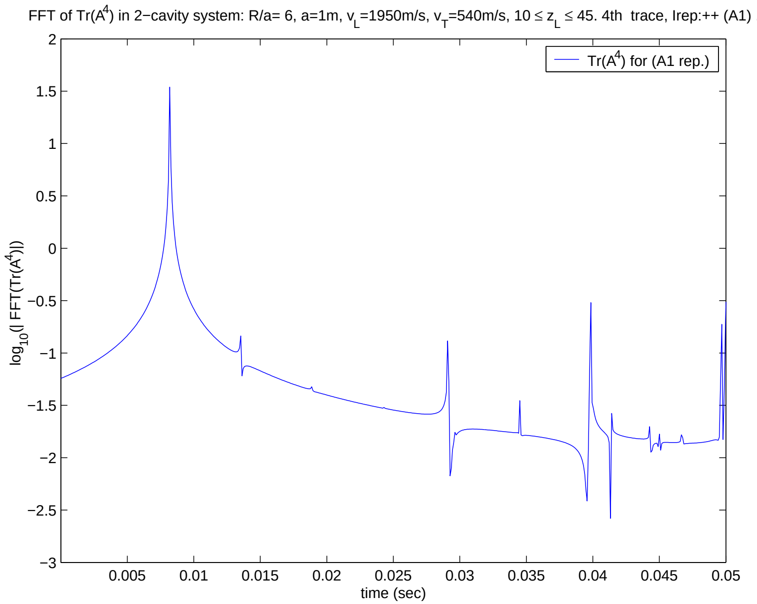
FFT of $\mathrm{Tr}\mathbf{A}$ to $\mathrm{Tr}\mathbf{A}^4$ in 2-cavity system of elastodynamics





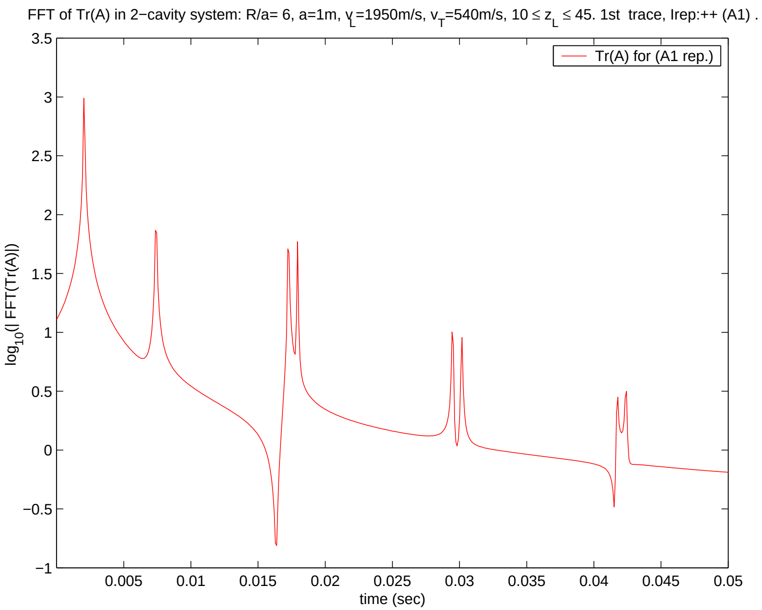
(A1 rep.)

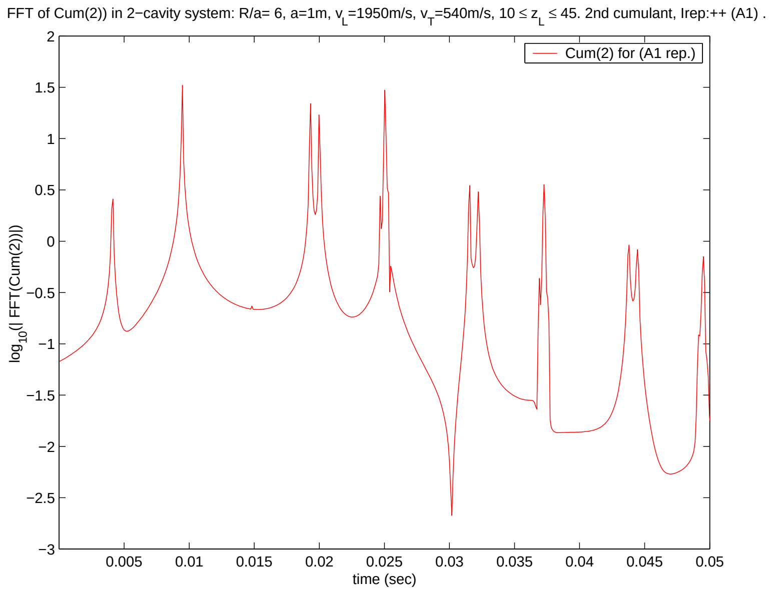




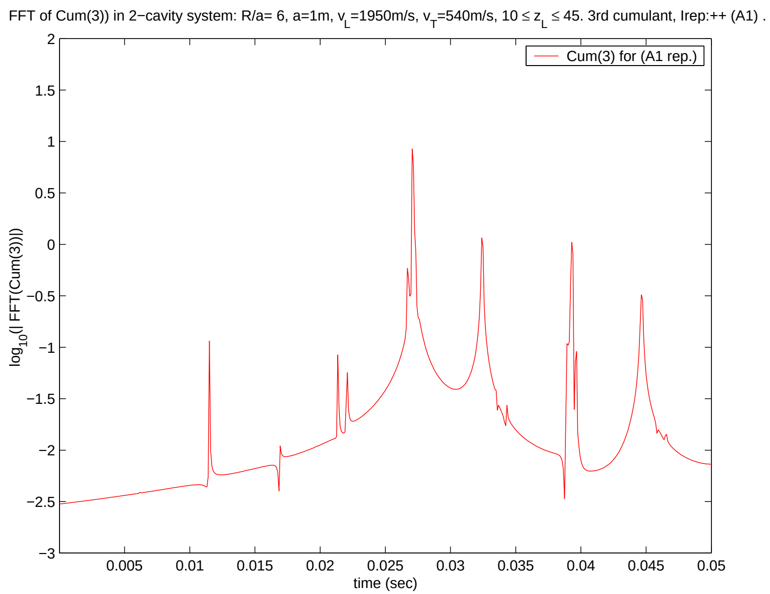
(A1 rep.)

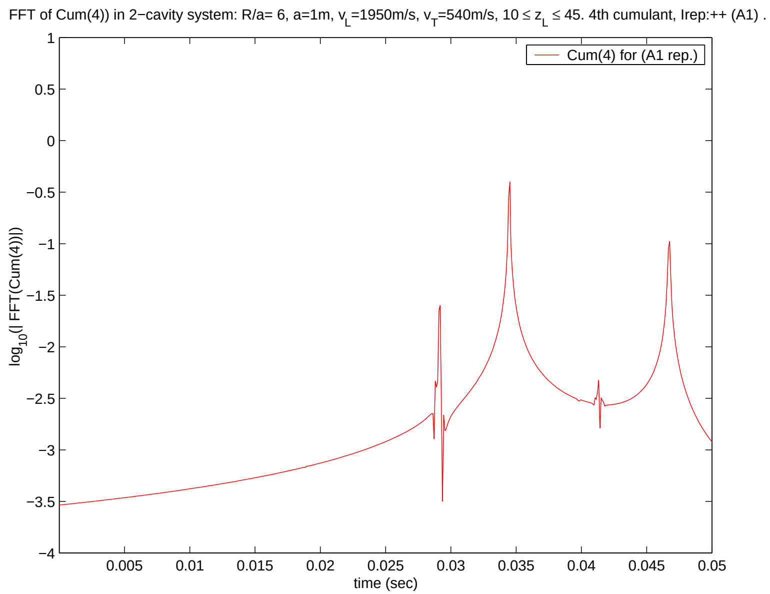
FFT of 1st to 4th cumulant in 2-cavity system of elastodynamics





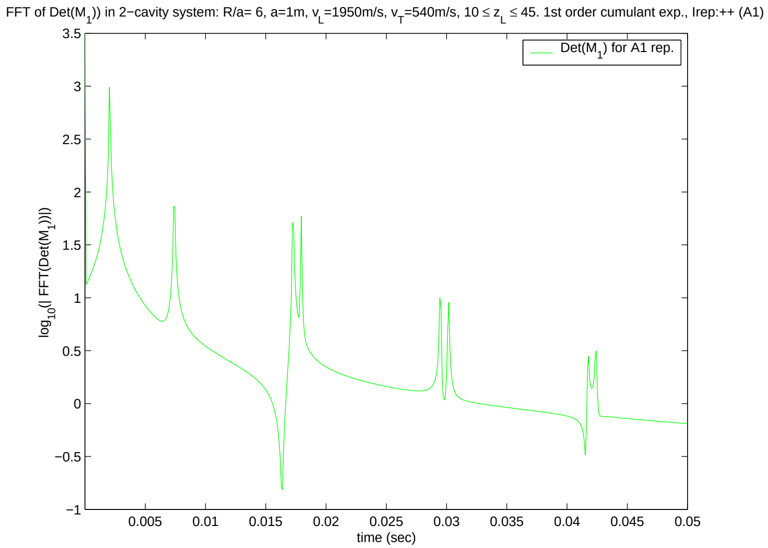
(A1 rep.)

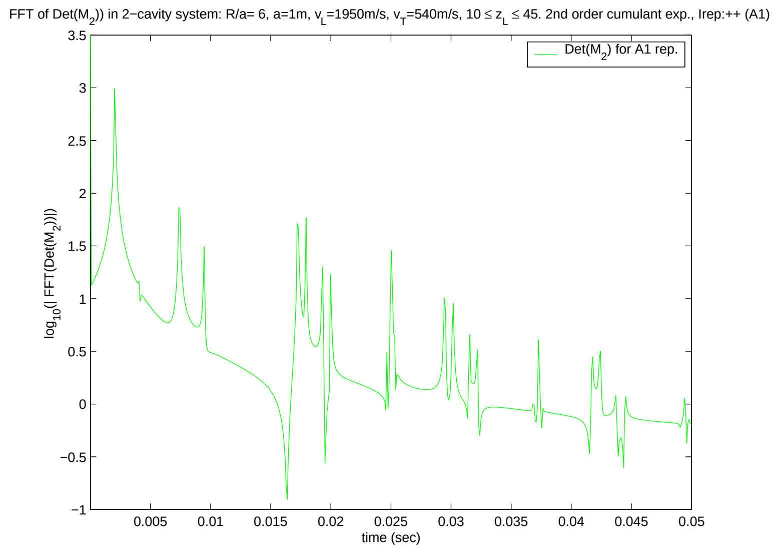




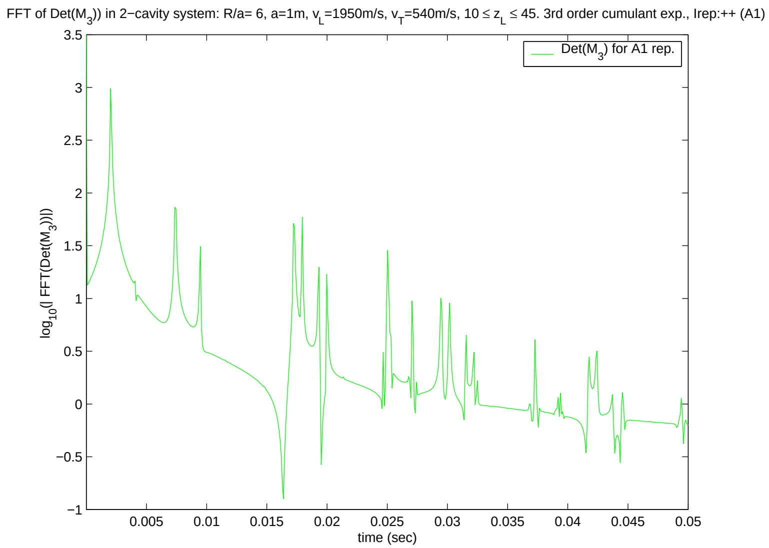
(A1 rep.)

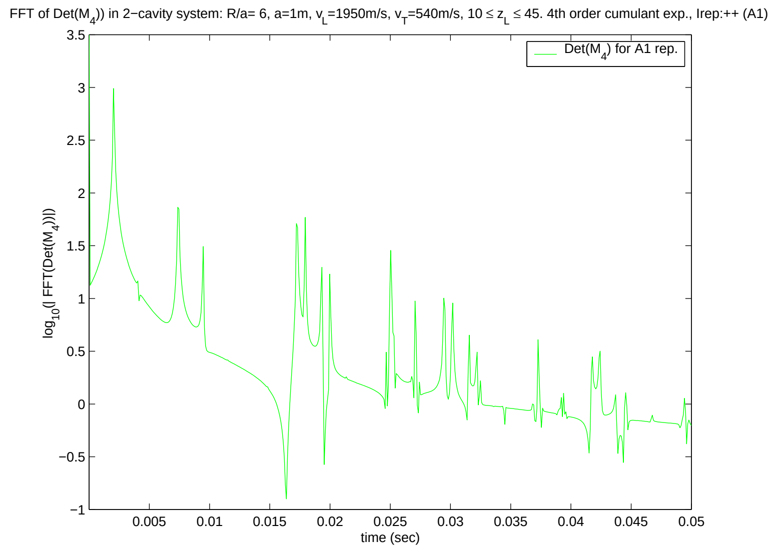
FFT of cumulant expansion (1st to 4th order) in 2-cavity





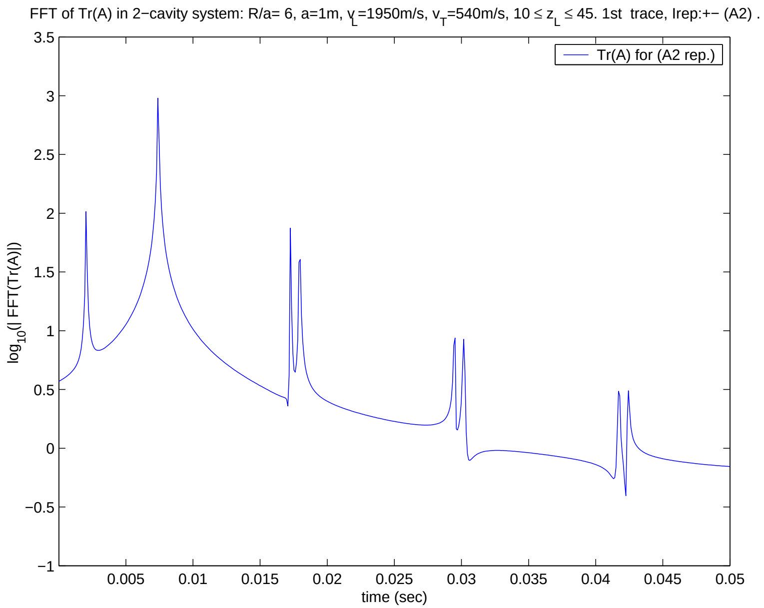
(A1 rep.)

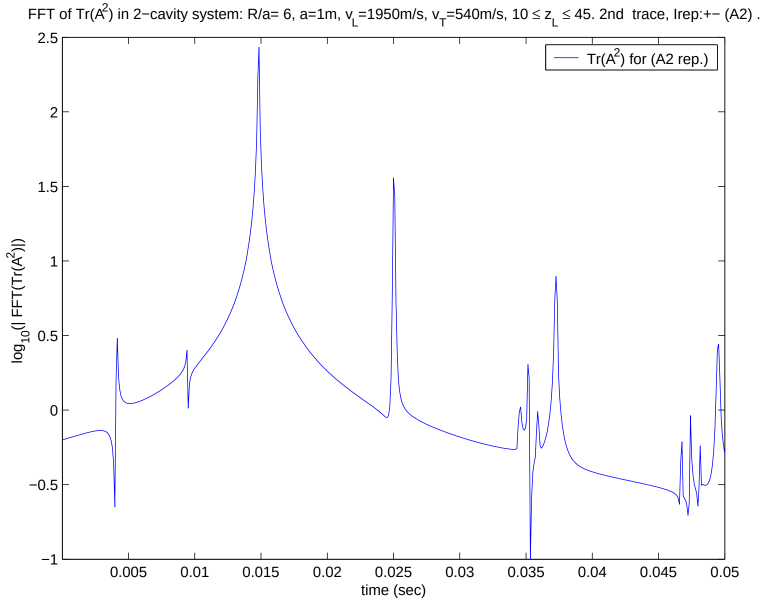




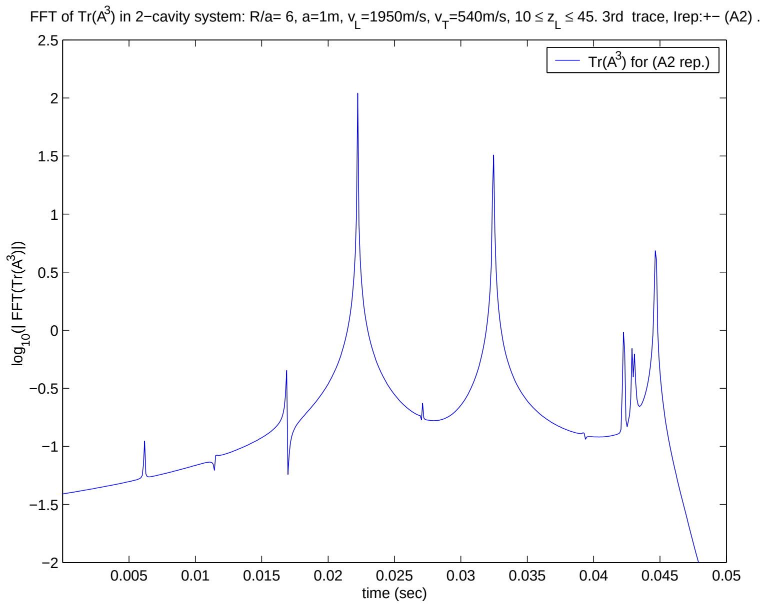
(A1 rep.)

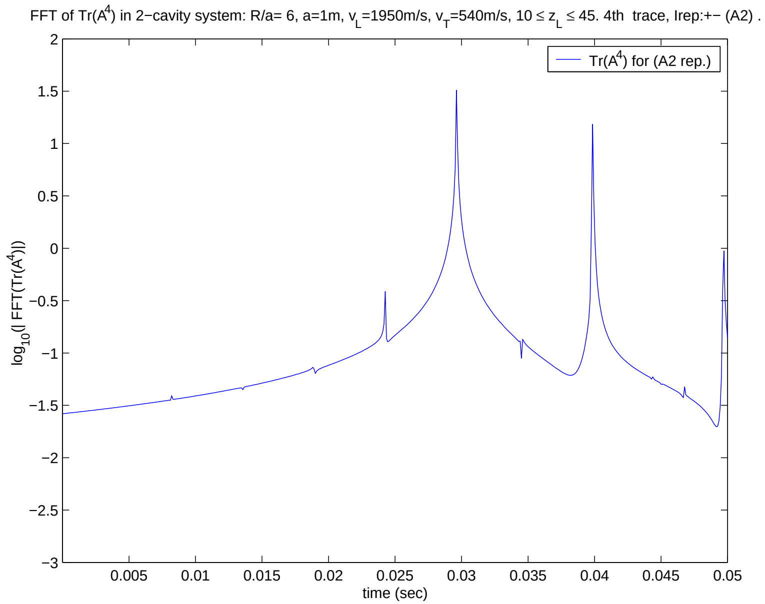
FFT of $\mathrm{Tr}\mathbf{A}$ to $\mathrm{Tr}\mathbf{A}^4$ in 2-cavity system of elastodynamics





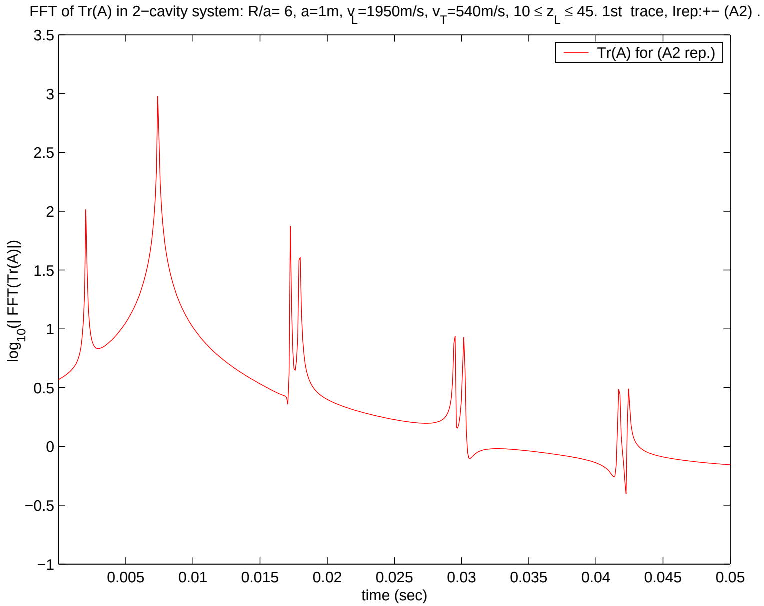
(A2 rep.)

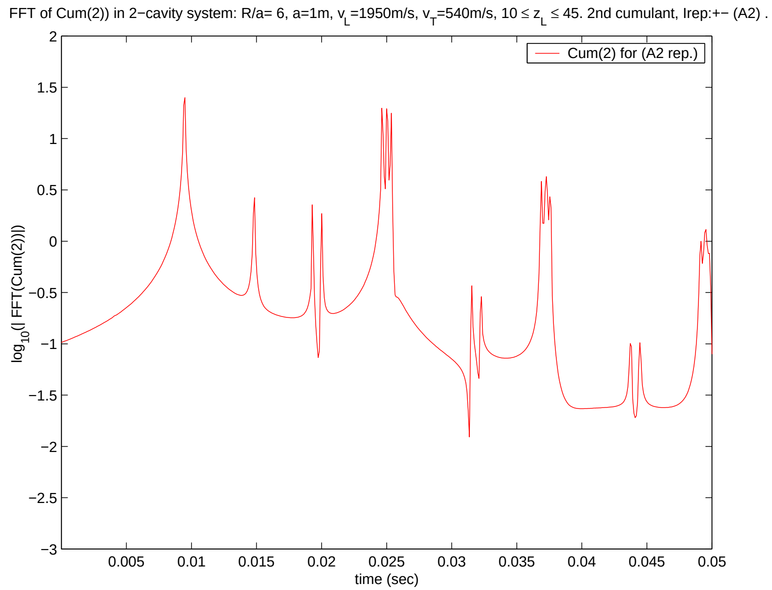




(A2 rep.)

FFT of 1st to 4th cumulant in 2-cavity system of elastodynamics





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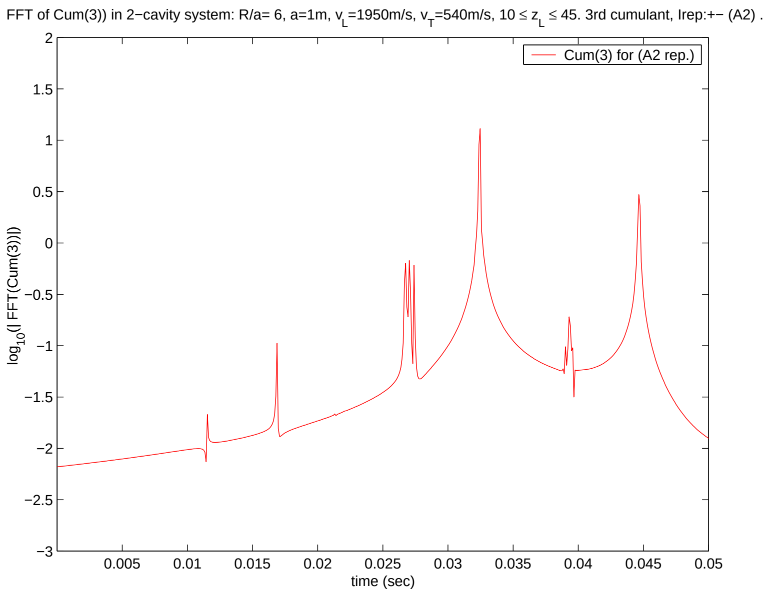
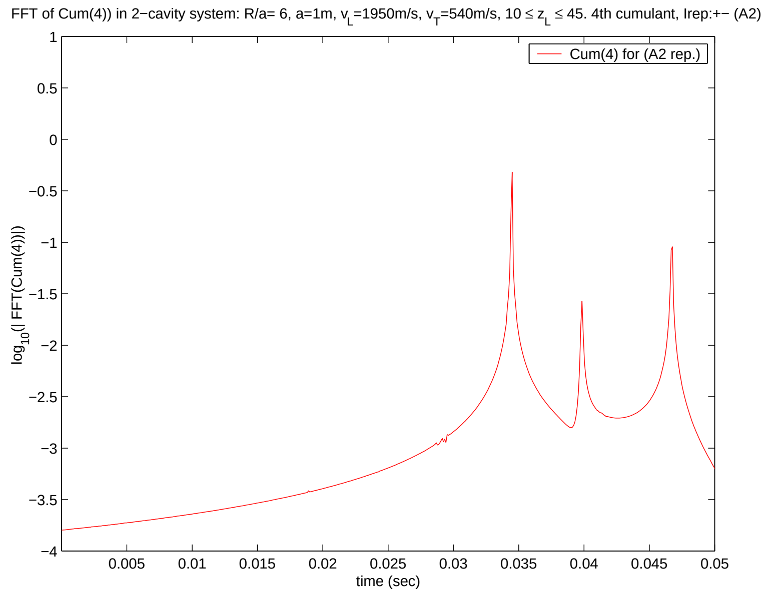
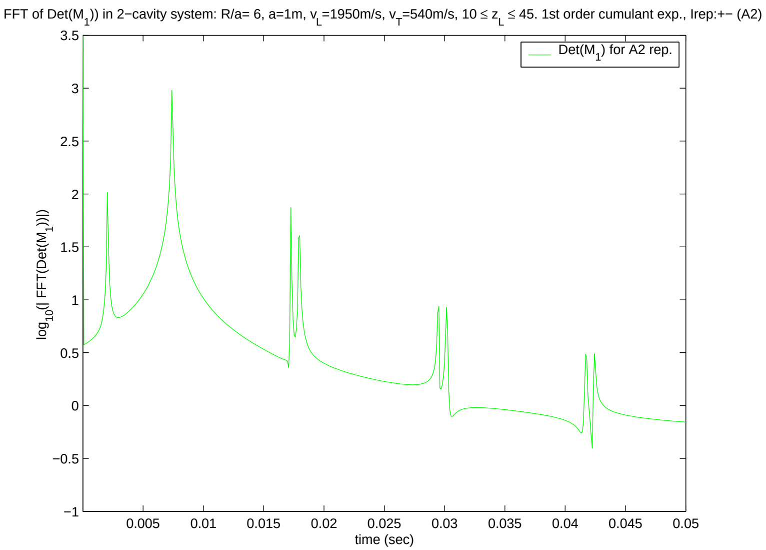
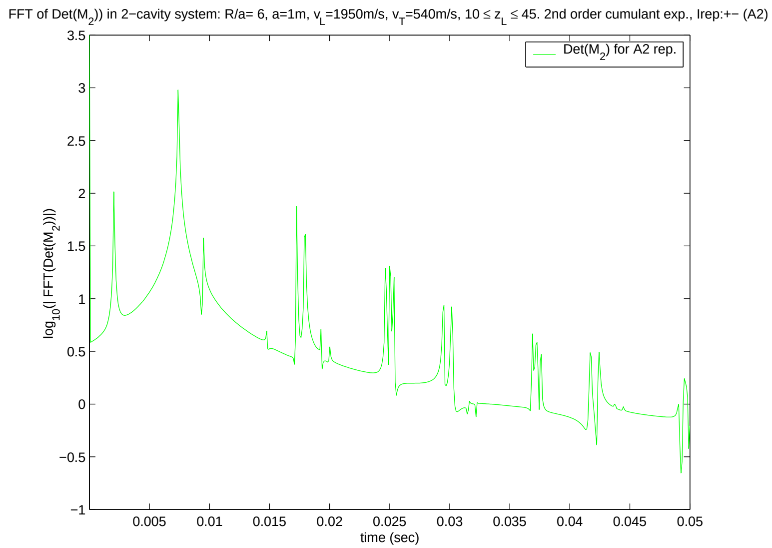




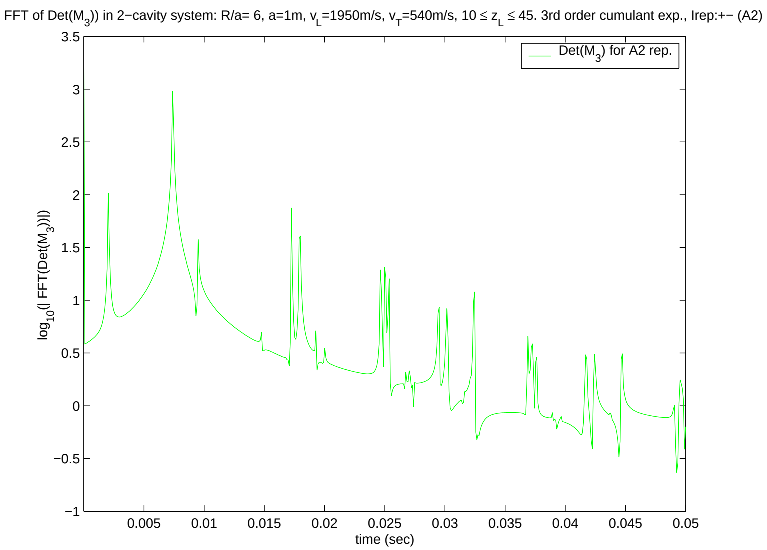
(A2 rep.)

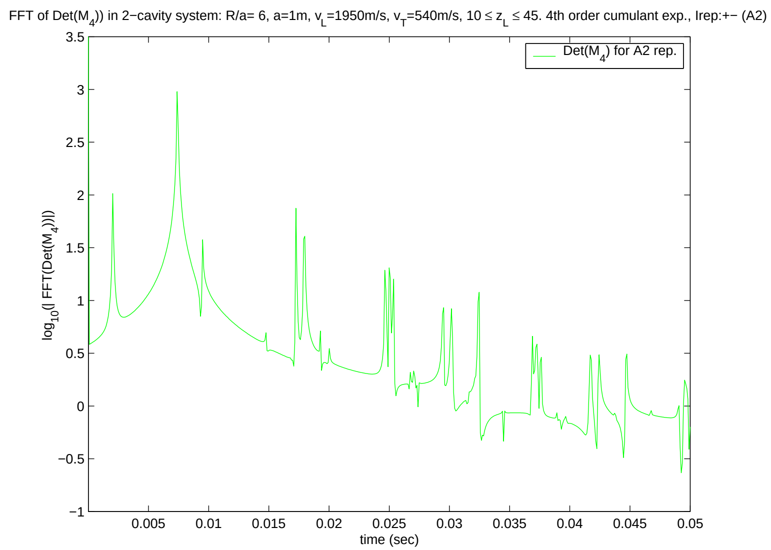
FFT of cumulant expansion (1st to 4th order) in 2-cavity system





(A2 rep.)



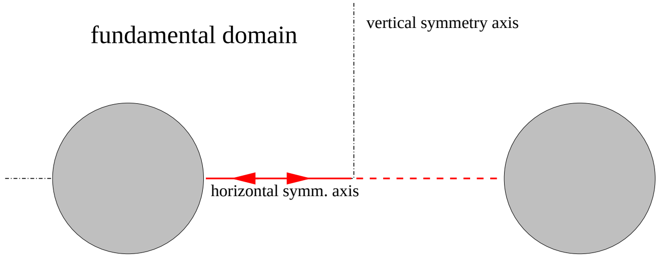


(A2 rep.)

Periodic orbits of two-cavity problem

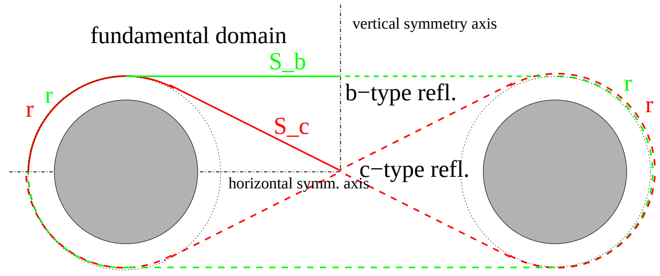
1. Topological length 1:

P & S orbits



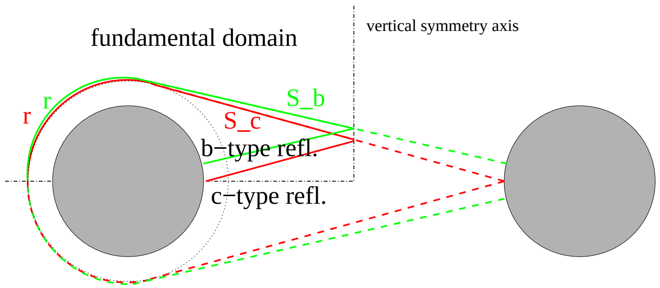
&

"dogbone" r S_b r & "butterfly" r S_cr orbit



2. Topological length 2:

r S_bS_br & r S_c S_cr orbit

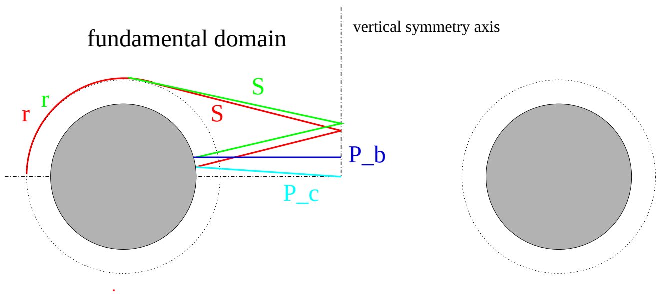


& **PP** & **SS** & $\mathbf{rS_brrS_br}$ & $\mathbf{rS_brrS_cr}$

& $\mathbf{rS_crrS_br}$ & $\mathbf{rS_crrS_cr}$

3. Topological length 3:

r S P_b S r & r S P_c S r



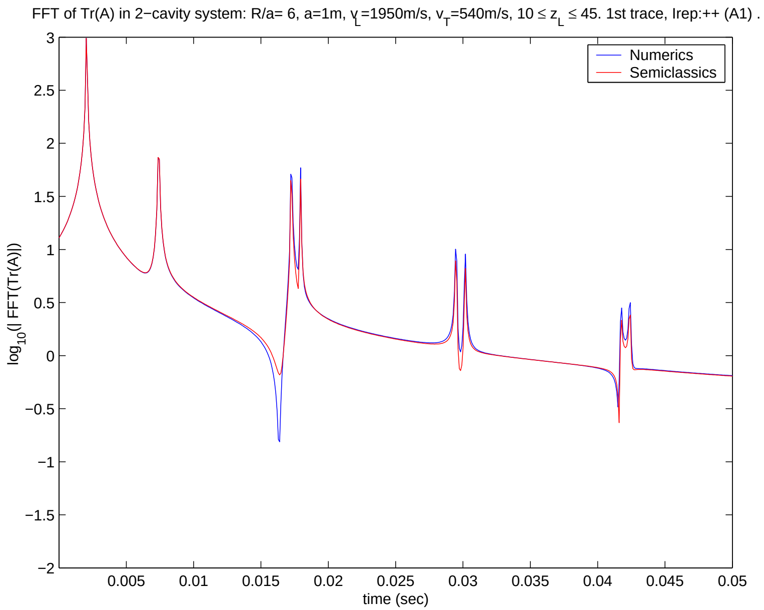
& PPP & SSS & $\mathbf{rS_brrS_brrS_br}$ & $\mathbf{rS_brrS_brrS_cr}$

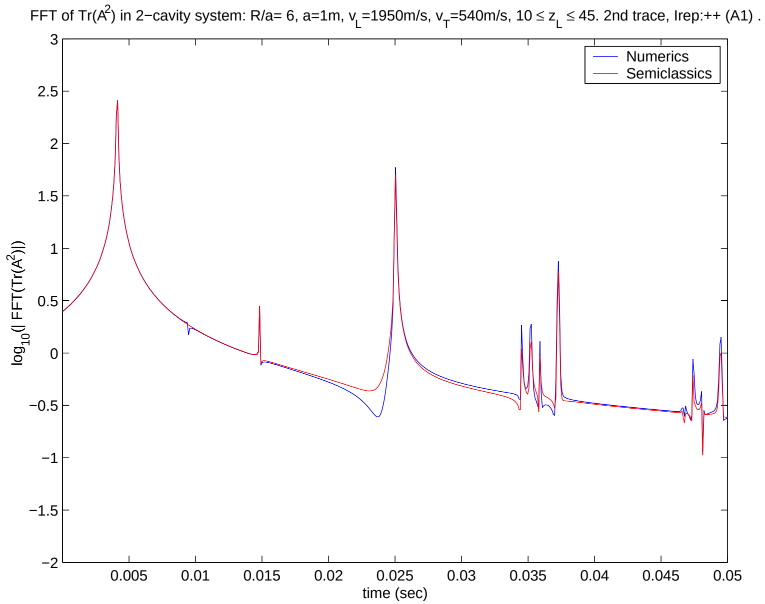
& $\mathbf{rS_brrS_crrS_cr}$ & $\mathbf{rS_crrS_crrS_cr}$ & $\mathbf{rS_bS_brrS_br}$

& $\mathbf{rS_bS_brrS_cr}$ & $\mathbf{rS_cS_crrS_br}$ & $\mathbf{rS_cS_crrS_br}$

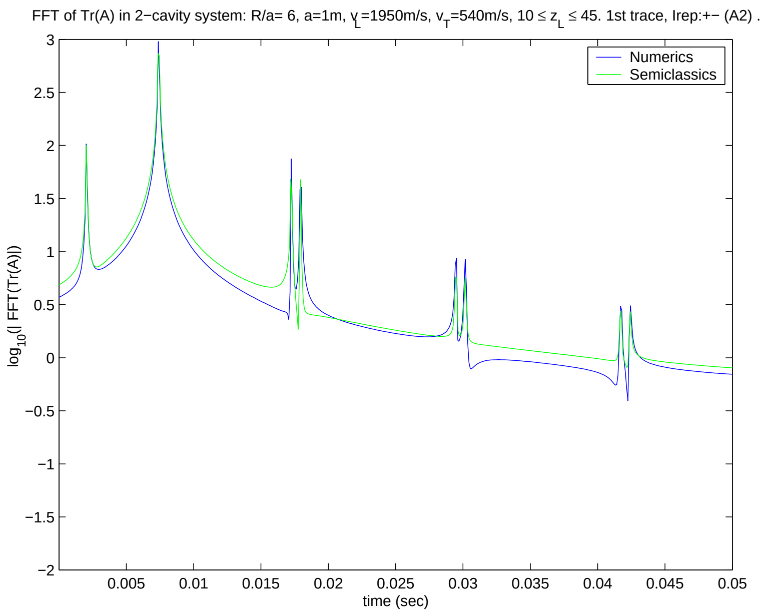
& $\mathbf{rSS_bSr}$ & $\mathbf{rSS_cSr}$ & cyclic permutations.

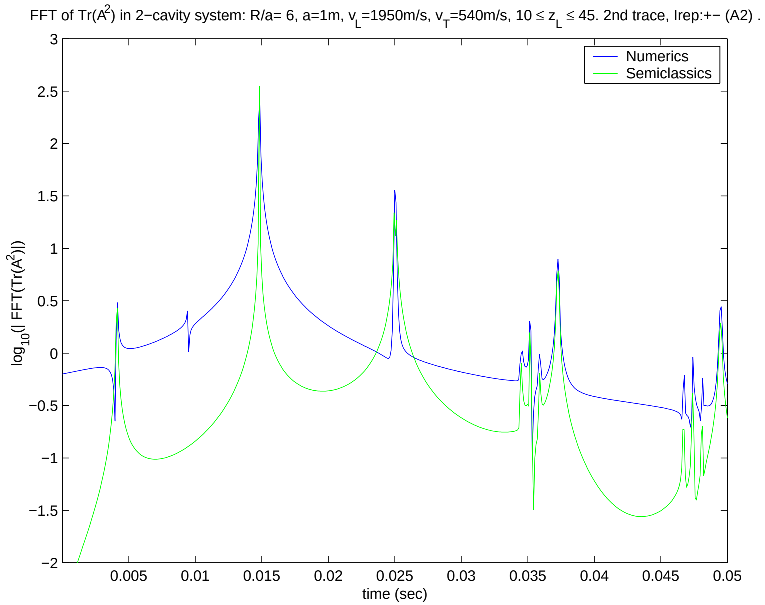
FFT of $\mathrm{Tr}\mathbf{A}$ and $\mathrm{Tr}\mathbf{A}^2$: wave vs. ray dynamics





(A1 rep.)

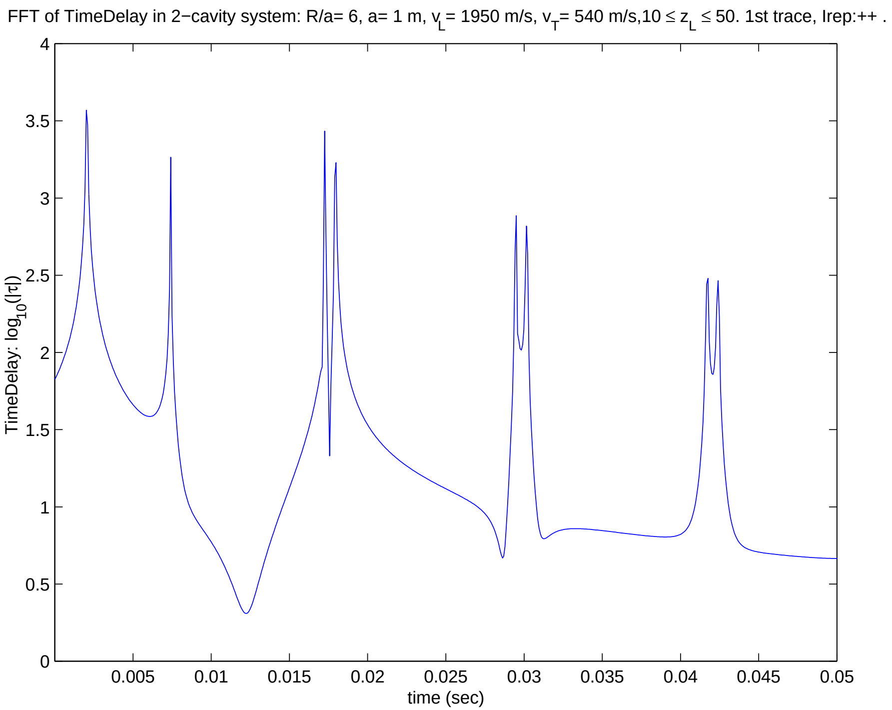




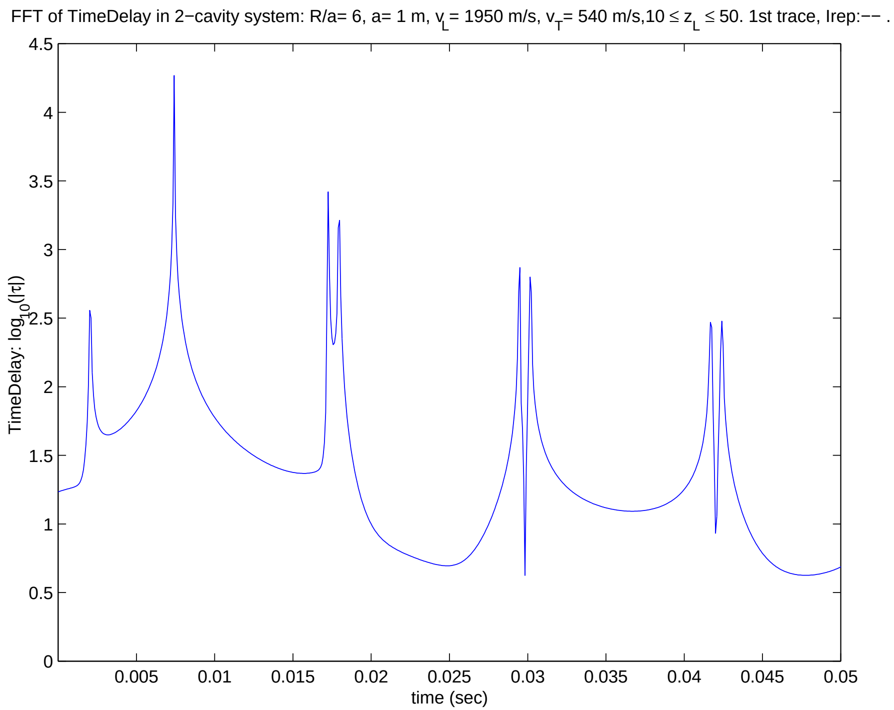
(A2 rep.)

FFT of $\frac{d}{dk}\mathrm{Tr}\,\mathbf{A}$ in 2-cavity system of elastodynamics

$R = 6a$ (polyethelene)



A1 rep.



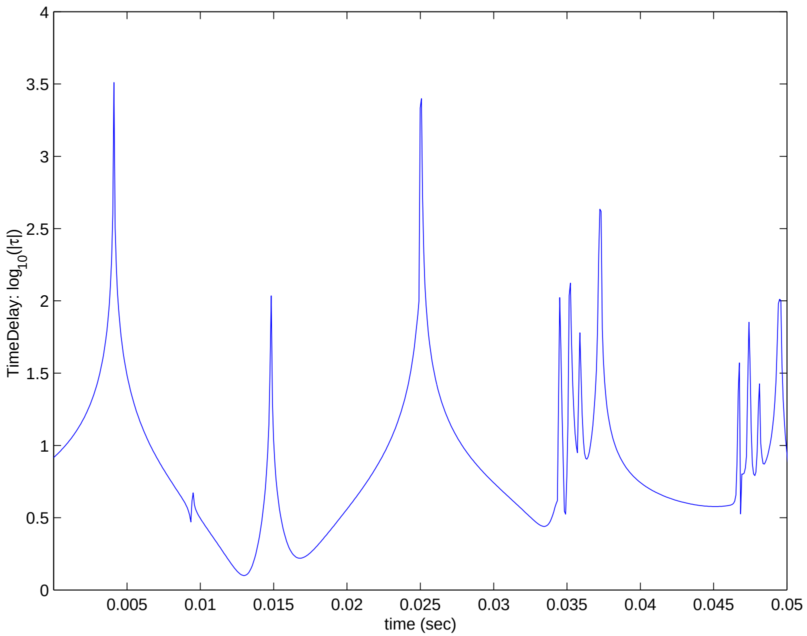
B1 rep.

courtesy of Niels Søndergaard

FFT of $\frac{d}{dk}\mathrm{Tr}\mathbf{A}^2$ in 2-cavity system of elastodynamics

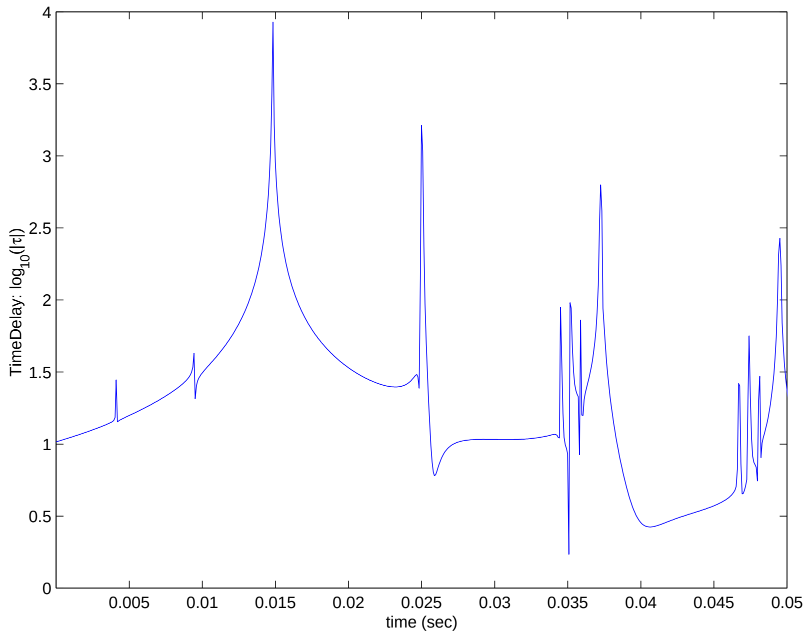
$R = 6a$ (polyethelene)

FFT of TimeDelay in 2−cavity system: R/a= 6, a= 1 m, v_L= 1950 m/s, v_T= 540 m/s,10 ≤ z_L ≤ 50. 2nd trace, Irep:++ .



A1 rep.

FFT of TimeDelay in 2−cavity system: R/a= 6, a= 1 m, v_L= 1950 m/s, v_T= 540 m/s,Cutoff at 10 ≤ z_L ≤ 50. 2nd trace, Irep:−− .

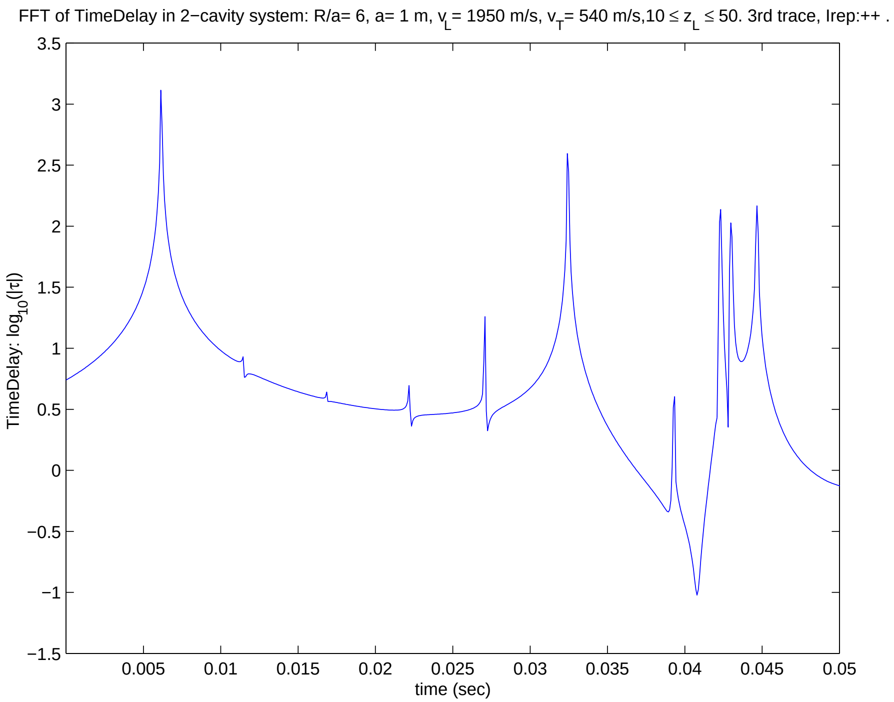


B1 rep.

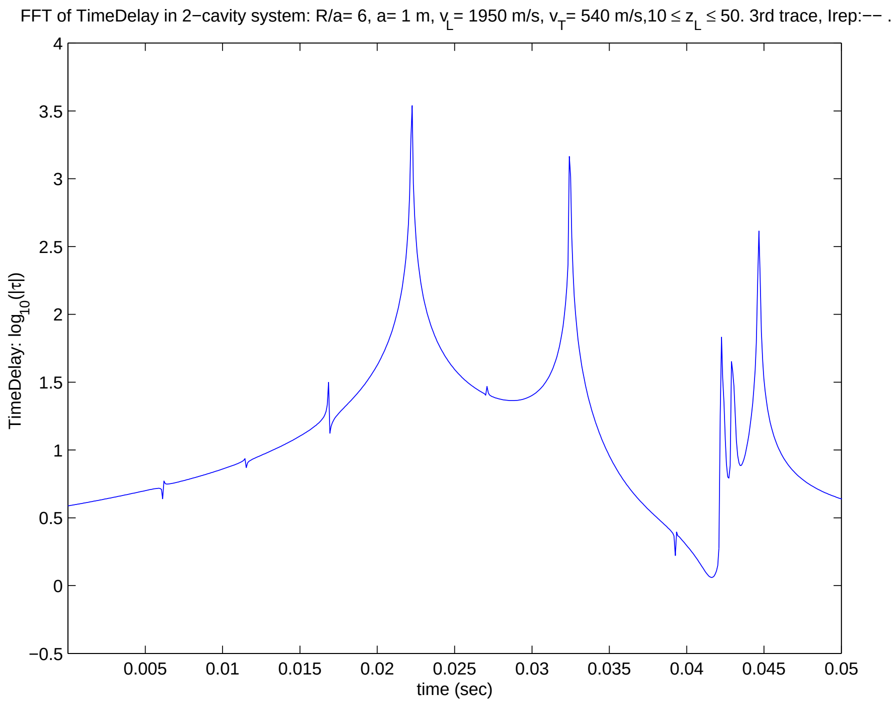
courtesy of Niels Søndergaard

FFT of $\frac{d}{dk}\mathrm{Tr}\mathbf{A}^3$ in 2-cavity system of elastodynamics

$R = 6a$ (polyethelene)



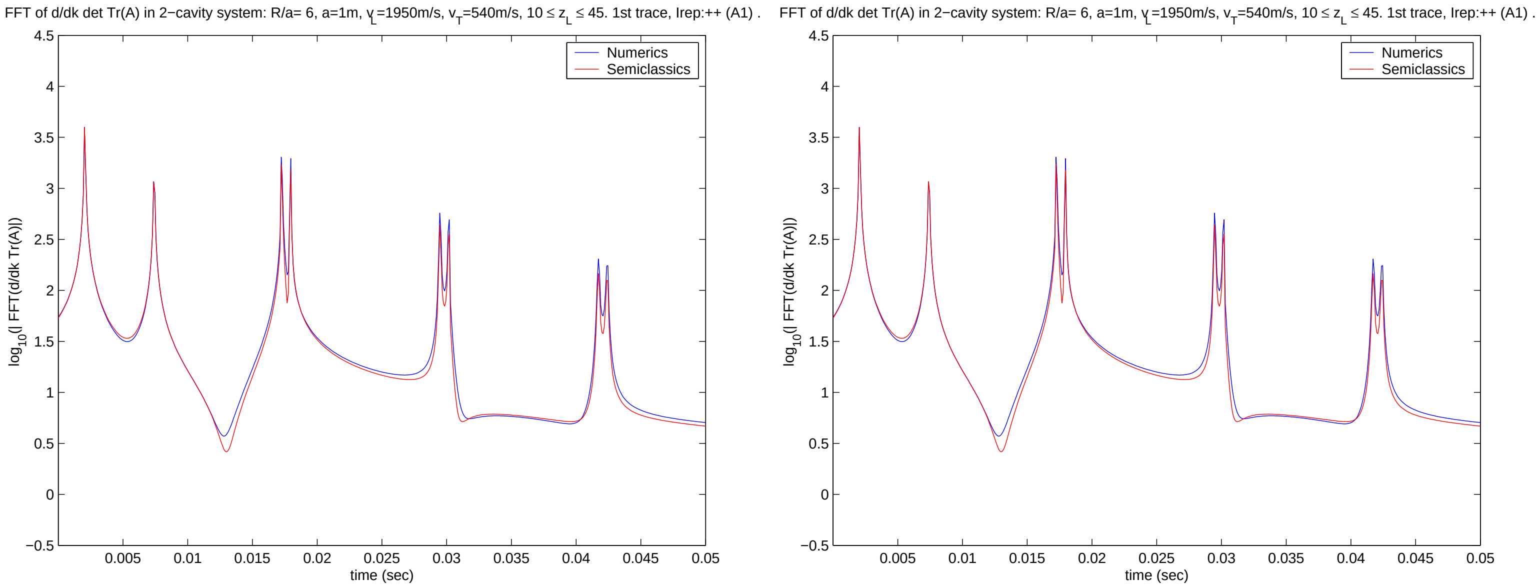
A1 rep.



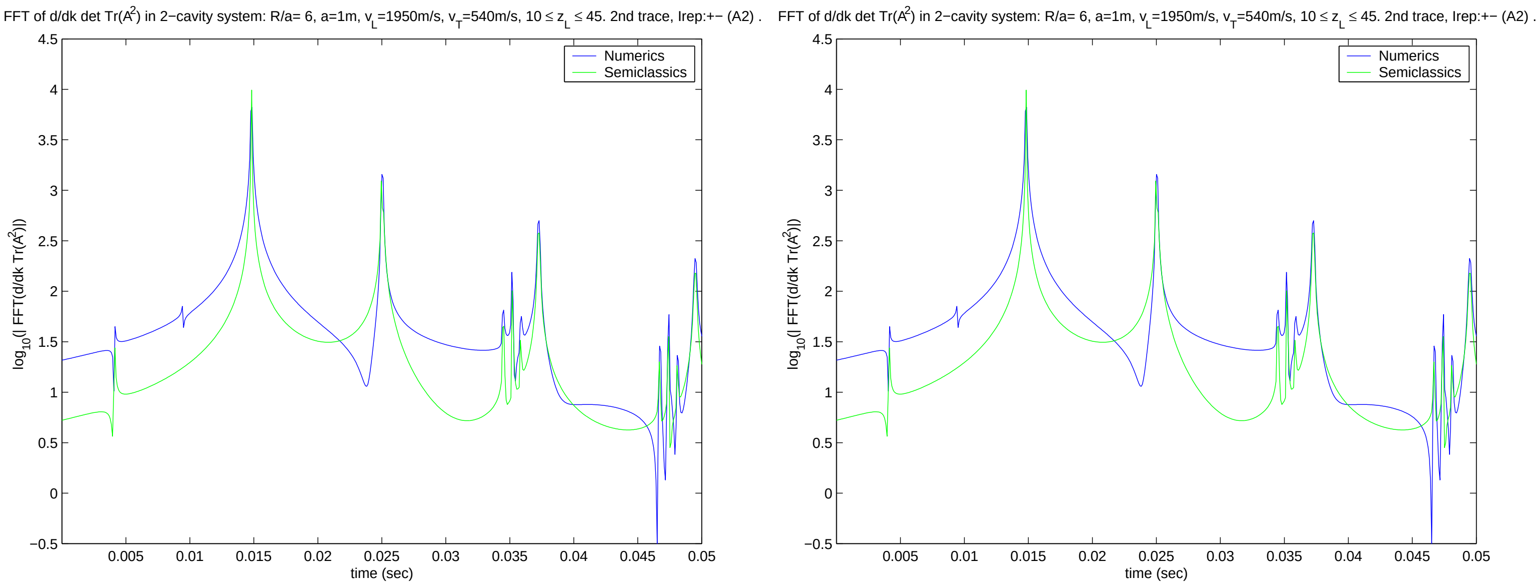
B1 rep.

courtesy of Niels Søndergaard

FFT of $\frac{d}{dk}\mathrm{Tr}\mathbf{A}$ & $\frac{d}{dk}\mathrm{Tr}\mathbf{A}^2$ in 2-cavity system of elastodynamics



(A1 rep.)



(A2 rep.)

Summary and Outlook

1. We have presented a scheme for finding the exact scattering resonances, cluster phase shifts, and Wigner time delays in a system of several cylindrical cavities in the case of plane strain isotropic elastodynamics with free boundary conditions.
2. The method is formally similar to previous derivations in quantum mechanics. Therefore we expect that also for elastodynamics there should exist a corresponding Gutzwiller-Voros determinant in strongly hyperbolic systems. Promising agreement has been found in the two-cavity system, at least to second order in the cumulant expansion.
3. The two-cavity scattering system clearly indicates that *surface orbits of Rayleigh type* are of importance for the *leading part of the spectrum*. They have no counterpart in a quantum scattering problem.
4. Because of the superposition with the 2-bounce geometrical orbit, the two-cavity scattering problem becomes chaotic, in contast to the quantum analog problem.
5. The Rayleigh orbits are of different type than the *creeping* orbits which are exponentially damped.
6. Also there will be orbits containing mode-conversions starting at third order in the cumulant expansion.
7. The three-cavity system will have mode-conversion already at second order in the cumulant expansion.