THE POWER OF CHAOS

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1 INTRODUCTION

Nonlinear physics presents us with a perplexing variety of complicated fractal objects and strange sets. Notable examples include strange attractors for chaotic dynamical systems, regions of high vorticity in fully developed turbulence and fractal growth processes. The word "chaos" has in this context taken on a narrow technical meaning. When a physicist says that a certain system exhibits "chaos", he means that the system obeys deterministic laws of evolution, but that the outcome is highly sensitive to small uncertainties in the specification of the initial state. In a chaotic system any open ball of initial conditions, no matter how small, will in finite time spread over the extent of the entire asymptotically accessible phase space. Once this is grasped, the focus of theory shifts from attempting precise prediction (which is impossible) to description of the geometry of the space of possible outcomes, and evaluation of averages over this space. By now most of us appreciate the fact that the phase space of a generic nonlinear dynamical system is an infinitely interwoven mixture of islands of stability and regions of chaos. Possible trajectories are qualitatively of three distinct types: they are either asymptotically unstable (positive Lyapunov exponent), asymptotically marginal (vanishing Lyapunov) or asymptotically stable (negative Lyapunov). Here we shall concentrate on the unstable orbits which build up chaos. Furthermore, we shall concentrate on the quantum mechanics of systems which are classically chaotic.

Today this physics is perhaps not as "applicable" as the chaos theory–inspired potential improvements in weather prediction, but I suspect that Bohr's electrons as planetary orbits around atomic nucleus did not appear particularly "applicable" in their day either. What we shall describe here is very much in the spirit of early quantum mechanics, and were physicists of the period as familiar with classical chaos as we are today, this physics would have been developed in 1920's. The main idea is this: in the Bohr–de Broglie visualization of quantization, one places a wave instead of a particle on a Keplerian orbit around the hydrogen nucleus. The quantization condition is that only those orbits for which such wave is a standing wave are allowed; from this follows Balmer spectrum, colors in the world around us,

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and more sophisticated quantum theory of Schrödinger and others. Today we are very aware of the fact that elliptic orbits are a peculiarity of the Kepler problem, and that chaos is the rule; so can we generalize the Bohr quantization to chaotic systems? The answer was provided by Gutzwiller [1]-[2]: a chaotic system can be quantized by placing a wave on each of the infinity of unstable periodic orbits. We shall develop here the theory of such averages over infinities of unstable orbits [3] - but before launching into the requisite mathematics, we want to emphasize that the periodic orbits in quantum mechanics are not just a theorist toy; the theory itself arose from studies of eminently applicable semi-conductors, and the unstable periodic orbits have been measured in experiments [7] on the very paradigm of Bohr's atom, the hydrogen atom, this time in external field. Not that far in future, the classical mechanics textbooks will have a chapter on the Keplerian motions followed by a chapter on chaos; and the quantum mechanics textbooks will have a chapter on the Bohr atom followed by a chapter on a classically chaotic atom.

In retrospect many triumphs of both classical and quantum physics seem a stroke of luck: a few integrable problems, such as the harmonic oscillator and the Kepler problem, though "nongeneric", have gotten us very far. The success has lulled us into a habit of expecting simple solutions to simple equations - an expectation shattered for many by the recently acquired ability to numerically scan the phase space of non-integrable dynamical systems. The initial impression might be that all our analytic tools have failed us, and that the chaotic systems are amenable only to numerical statistical investigations. However, as we will show here, we already possess a perturbation theory of the deterministic chaos of predictive quality comparable to that of the traditional perturbation expansions for nearly integrable systems. This theory is based on the observation that the motion in dynamical systems of few degrees of freedom is often organized around a few *fundamental* cycles. These short cycles capture the skeletal topology of the motion in the sense that any long orbit can approximately be pieced together from the fundamental cycles. Computations with such systems require techniques reminiscent of statistical mechanics; however, the actual calculations are crisply deterministic throughout. The strategy will be to expand averages over chaotic phase space regions in terms of short unstable periodic orbits, with the small expansion parameter being the non-uniformity of the flow (here referred to as "curvature") across neighborhoods of periodic points. One of the surprises is that the quantum mechanics of classically chaotic systems is very much like the classical mechanics of chaotic systems; one needs nearly the same zeta functions and cycle expansions, with the same group theory factorizations and dependence on the topology of the classical flow. To get some feeling for how and why unstable cycles come about, we start by playing a game of pinball.

2 PINBALL CHAOS

A physicist's pinball [8, 9, 10, 11, 12] game is a pinball reduced to its bare essentials: three disks in a plane:

Figure 1. The 3-disk pinball with the disk radius : center separation ratio a:R = 1:2.5. (a) the three disks, with $\overline{12}$, $\overline{123}$ and $\overline{121232313}$ cycles indicated. (b) the fundamental domain, *i.e.* a wedge consisting of a section of a disk, two segments of symmetry axes acting as straight mirror walls, and an escape gap. The above cycles restricted to the fundamental domain are now the two fixed points $\overline{0}$ and $\overline{1}$ and the $\overline{100}$ cycle.

Point-like pinballs are shot at the disks from random starting positions and angles; they spend some time bouncing between the disks and then escape. For a physicist a good game consists in predicting the asymptotic lifetime (or the escape rate) of a pinball to many significant digits. The unstable cycles as a skeleton of chaos are almost visible here: a good strategy for keeping the ball bouncing as long as possible is to aim it as close as possible to a periodic trajectory. Short periodic trajectories are easily drawn and enumerated - some examples are given in fig. 1 - but it is rather hard to extract the systematics of the orbits from their physical space trajectories.

A clearer picture of the dynamics is obtained by constructing a phase space Poincaré section. We start [13] by exploiting the sixfold symmetry of the disks, and restrict the pinball to bouncing in the fundamental domain (fig. 1b). The whole pinball plane can be retiled by 6 copies of the fundamental domain, but the details are inessential for present considerations. We define our Poincaré section by marking x_i, the position of the ith bounce off the bottom wall (fig. 1b), and $\sin\phi_i$, the momentum component parallel to the bottom wall. $(x, \sin\phi)$ coordinates are a convenient choice, because they are phase-space volume preserving [14].

We next mark the initial conditions $\Omega_{.\epsilon_1}$ which do not escape in one bounce. There are two strips of survivors, as the trajectories originating from one disk can hit either of the other two disks, or escape. We label the two strips with $\epsilon = 0, 1$. There are four strips $\Omega_{.\epsilon_1\epsilon_2}$ that survive four bounces, and so forth. Another way to look at the survivors after two bounces is to plot $\Omega_{\epsilon_1.\epsilon_2}$, the intersection of $\Omega_{.\epsilon_2}$ with the strips $\Omega_{\epsilon_1.}$ obtained by time reversal ($\sin\phi \rightarrow -\sin\phi$). Provided that the disks are sufficiently well separated, what emerges is a complete binary Cantor set with the usual Smale horseshoe [15] foliation and symbolic dynamics.

Figure 2. The Poincaré section of the phase space for the fundamental domain pinball, fig. 1b. Indicated are the fixed points $\overline{0}$, $\overline{1}$ and the 2-cycle $\overline{01}$,$\overline{10}$, together with strips which survive 1, 2, ... bounces. Iteration corresponds to bit shift; for example, region $\ldots 01.01 \ldots$ maps into $\ldots 010.1 \ldots$.

After n iterations the survivors are divided into 2^n distinct neighborhoods: the ith neighborhood consists of all points with itinerary $i = \epsilon_1\epsilon_2\epsilon_3 \ldots \epsilon_n$, $\epsilon_i = \{0, 1\}$. Each such patch contains a periodic point $\overline{\epsilon_1\epsilon_2\epsilon_3 \ldots \epsilon_n}$ with the basic block infinitely repeated. Periodic points are skeletal in the sense that as we look further and further, they stay put forever, while the finite covers shrink onto them. The periodic points are dense on the asymptotic repeller, and their number increases exponentially with cycle length. As we shall see, this exponential proliferation of cycles is not as dangerous as it might seem.

Before continuing with the pinball as an illustration of the origin and structure of unstable cycles, we turn briefly to the role of cycles in more general settings.

3 CYCLES AS THE SKELETON OF CHAOS

Consider a general three-dimensional flow sketched in fig. 3. To be interesting, the flow should be recurrent; otherwise it is a transient state that we cannot observe for long times. If the flow is recurrent, we can cut it by a Poincaré section; if it is a map of a compact disk domain onto itself, it must have at least one fixed point. Now consider the ways in which the flow can deform the neighborhood of a fixed point. There are essentially two possibilities: the neighborhood can return wrapped around the fixed point (the fixed point is stable or elliptic - see fig. 3a), or squeezed, stretched and folded (the fixed point is unstable or hyperbolic - see fig. 3b).

In the traditional approach we use integrable motions of fig. 3a as zeroth-order approximations to physical systems, and account for weak non-linearities perturbatively. We tend to think of a dynamical system as a smooth system whose evolution can be followed by integrating a set of differential equations. When this is actually followed through to asymptotic times, we discover that the strongly non-linear systems show an amazingly rich structure which is not at all apparent in their formulation in terms of differential equations. However, hidden in this apparent chaos is a rigid skeleton, a tree of *cycles* (periodic orbits) of increasing lengths and self-similar structure. The new insight is that the zeroth-order approximations to harshly chaotic dynamics should be very different from those for the nearly integrable systems: a good starting approximation here is the stretching and kneading of a baker's map of fig. 3b, rather than the winding of a harmonic oscillator of fig. 3a.

Figure 3. A recurrent flow around (a) an elliptic fixed point, (b) a hyperbolic fixed point.

For low dimensional deterministic dynamical systems the cycles (periodic orbits) provide a possibly optimal invariant description of a dynamical system, with the following virtues:

1. cycle symbol sequences are *topological* invariants: they give the spatial layout of a strange set

2. cycle eigenvalues are *metric* invariants: they give the scale of each piece of a strange set

3. cycles are *dense* on the asymptotic non-wandering set

4. cycles are ordered *hierarchically*: short cycles give good approximations to a strange set, longer cycles only refinements. Errors due to neglecting long cycles can be bounded, and typically fall off exponentially with the cutoff cycle length

5. cycles are *structurally robust*: eigenvalues of short cycles vary slowly with smooth parameter changes

6. asymptotic averages (such as generalized dimensions, escape rates, quantum mechanical eigenstates and other "thermodynamic" averages) can be computed from short cycles by means of *cycle expansions*

Points 1, 2: That the cycle topology and eigenvalues are invariant properties of dynamical systems follows from elementary considerations. If the same dynamics is given by a map f in one set of coordinates, and a map g in the next, then f and g (or any other good representation) are related by a reparametrization and a coordinate transformation $f = h^{-1} \circ g \circ h$. As both f and g are arbitrary representations of the dynamical system, the explicit form of the conjugacy h is of no interest, only the properties invariant under any transformation h are of general import. The most obvious invariant properties are topological; a fixed point must be a fixed point in any representation, a trajectory which exactly returns to the initial point (a cycle) must do so in any representation (fig. 4). Furthermore, a good representation should not mutilate the data; h must be a smooth transformation which maps nearby cycle points of f into nearby cycle points of g. This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metrically invariant. In particular, the cycle eigenvalues (eigenvalues of the Jacobians $df^{(n)}(x_k)/dx$ of periodic orbits $f^{(n)}(x_k) = x_k$) are invariant.

Figure 4. A fixed point and a cycle remain a fixed point and a cycle in any representation of a dynamical systems. Here (a) and (c) phase space is built from x and its derivatives. (b) and (d) might be the same trajectories in the time-delay coordinates.

Point 3:

The cycles are intuitively expected to be *dense* because on a connected chaotic set a typical trajectory is expected to behave ergodically, and pass infinitely many times arbitrarily close to any point on the set, including the initial point of the trajectory itself. Generically one expects to be able to gently move the initial point in such a way that that the trajectory returns precisely to the initial point (fig. 5). This is by no means guaranteed to work, and it must be checked for the particular system at hand. A variety of ergodic (but insufficiently mixing) counter-examples can be constructed - the most familiar being a quasiperiodic motion on a torus.

Point 5: An important virtue of cycles is their *structural robustness.* Many quantities customarily associated with dynamical systems depend on the notion of "structural stability" [15], *i.e.* robustness of strange sets to small parameter variations. This is certainly not a property of generic dynamical systems, such as the Hénon map [16] $(x, y) \to (1 - ax^2 + y, bx)$. For example, while numerical studies indicate that for $a = 1.4$, $b = 0.3$ the attractor is "strange", parameter variation as minute as changing a to $a = 1.39945219$ destroys this attractor and replaces is with a stable cycle of length 13. Still, the short unstable cycles of length less than 13 are structurally robust in the sense that they are only slightly distorted by such parameter changes, and averages computed using them as a skeleton are insensitive to small deformations of the strange set. In contrast, lack of structural stability wreaks havoc with long time averages such as Lyapunov exponents, for which there is no guarantee that they converge in any finite numerical computation.

Figure 5. (a) A close recurrence of an unstable trajectory (b) can be exploited to locate a nearby cycle.

The theoretical advance that we will concentrate on here is **point 4:** we now know how to control the errors due to neglecting longer cycles. As we shall explain in sect. 5, even though the number of invariants is infinite (unlike, for example, the number of Casimir invariants for a compact Lie group) the dynamics can be well approximated to any finite accuracy by a small finite set of invariants. The origin of this convergence is geometrical, as we shall now show by returning to our game of pinball.

4 PINBALL ESCAPE RATE

Consider fig. 2 again. In each bounce the initial conditions get thinned out, yielding twice as many thin strips as at the previous bounce. The phase-space volume is preserved by the flow, so they are contracted along the stable eigendirections, and ejected along the unstable eigendirections; the total fraction of survivors after n

bounces is proportional to

$$\Gamma_n = \sum_i^{(n)} l_i. \tag{1}$$

where i is a binary label of the ith strip, and l_i is the width of the ith strip. One expects the sum (1) to fall off exponentially with n, and tend to a limit

$$\Gamma_n = e^{-n/T_n} \to e^{-n/T} = e^{-n\gamma}. \tag{2}$$

T is the *asymptotic lifetime* of a random initial pinball; $\gamma = 1/T$ is the pinball *escape rate* [17]. We shall now show that this asymptotic escape rate can be extracted from a highly convergent *exact* expansion by reformulating the sum (1) in terms of unstable periodic orbits.

Each neighborhood i in fig. 2 contains a periodic point $\overline{i}$. The finer the intervals, the smaller is the variation in flow across them, and the strip width l_i is well approximated by the contraction around the periodic point, $l_i = a_i/|\Lambda_i|$. Here Λ_i is the expanding eigenvalue of the linearized map evaluated on the periodic point i, and a_i is a prefactor defined by

$$a_i = l_i|\Lambda_i|. \tag{3}$$

Now we make the only approximation in our derivation of the ζ function: for large n the prefactors $a_i \approx O(1)$ are overwhelmed by the exponential growth of Λ_i, so we neglect them. This is called the *hyperbolicity* assumption. a_i reflect the particular distribution of incoming pinballs; the asymptotic trajectories are strongly mixed by bouncing chaotically between the disks and we expect them to be insensitive to smooth variations in the initial distribution. If the hyperbolicity assumption is justified, we can replace l_i in (1) by $1/\Lambda_i$ and form a formal sum over all periodic orbits of all lengths:

$$\begin{aligned}\Omega(z) \;&=\; \sum_{n=1}^{\infty} z^n \sum_i^{(n)} |\Lambda_i|^{-1} = z/|\Lambda_{\overline{0}}| + z/|\Lambda_{\overline{1}}| + z^2/|\Lambda_{\overline{00}}| + z^2/|\Lambda_{\overline{01}}| \\ &\quad + z^2/|\Lambda_{\overline{10}}| + z^2/|\Lambda_{\overline{11}}| + z^3/|\Lambda_{\overline{000}}| + z^3/|\Lambda_{\overline{001}}| + \ldots \end{aligned} \tag{4}$$

As for large n the nth level sum (2) tends to the limit $e^{-n\gamma}$, the escape rate γ is determined by the smallest $z = e^{\gamma}$ for which (4) diverges:

$$\Omega(z) \approx \sum_{n=0}^{\infty} (ze^{-\gamma})^n = \frac{1}{1 - ze^{-\gamma}} \tag{5}$$

This observation is the motivation for introducing the sum (4). Rather than attempting to estimate the escape rate from the $n \to \infty$ limit of preasymptotic sums (1), we shall determine γ from the singularities of (4).

If a trajectory retraces itself r times, its expanding eigenvalue is Λ_p^r, where p is a *prime* cycle. A prime cycle is a single traversal of the orbit; its label is a non-repeating symbol string. There is only one prime cycle for each cyclic permutation class. For example, $p = \overline{0011} = \overline{1001} = \overline{1100} = \overline{0110}$ is prime, but $\overline{0101} = \overline{01}$ is not. The stability of a cycle is (by the chain rule) the same everywhere along the orbit, so each prime cycle of length n_p contributes n_p terms to the sum (4). Hence (4) can be rewritten as

$$\Omega(z) = \sum_p n_p \sum_{r=1}^{\infty} \left(z^{n_p}|\Lambda_p{}^{-1}|\right)^r = \sum_p \frac{n_p z^{n_p}|\Lambda_p^{-1}|}{1 - z^{n_p}|\Lambda_p^{-1}|}.$$

where the index p runs through all distinct *prime* cycles. The $n_p z^{n_p}$ factors in the sum suggest rewriting it as a logarithmic derivative $\Omega(z) = -z\frac{d}{dz}\ln\zeta(z)$. The resulting infinite product

$$1/\zeta(z) = \prod_p (1 - t_p), \quad t_p = z^{n_p}|\Lambda_p^{-1}|, \tag{6}$$

is an example of a dynamical ζ function [21]. The name is motivated by the (purely formal) similarity of the infinite product to the Euler product representation of the Riemann ζ function.

The above derivation of the ζ function formula for the escape rate has one shortcoming; it estimates the fraction of survivors as a function of the number of pinball bounces. However, the physically interesting quantity is the mean lifetime; giving the same weight to all paths with the same number of bounces overestimates the contributions of the long trajectories and underestimates the short trajectories (remember, the flight paths between disks are of different lengths). The correct weight [20, 22] is obtained by replacing the discrete "topological" time n_p by the actual cycle period T_p in (6):

$$t_p = z^{T_p \gamma}|\Lambda_p^{-1}|. \tag{7}$$

Perhaps more surprisingly, (6) also yields *quantum* resonances, with the quantum amplitude associated with a given cycle

$$t_p = \frac{1}{\sqrt{\Lambda_p}} e^{\frac{i}{\hbar}S_p(E) + i\pi m_p} \,, \tag{8}$$

essentially (in a somewhat vague sense) the square root of the classical weight (we refer the reader to refs. [10, 11, 12] for detailed discussions).

Expression (6) is the main result of this section; the problem of estimating the asymptotic escape rates from finite n sums such as (1) is now reduced to a study of the singularities of the ζ function (6). The escape rate is related by (5) to a divergence of $\Omega(z)$, and $\Omega(z)$ diverges whenever $1/\zeta(z)$ or $\zeta(z)$ has a zero. Glancing back, we see that the derivation is very general, and should work for any average over any strange set which satisfies two conditions: 1. the weight associated with a part of the set is multiplicative along the trajectory; 2. the set is organized in such a way that the nearby points in the symbolic dynamics have nearby weights.

We conclude this section by noting that if one is interested in correlation spectra [20] rather than just the leading resonances, the correct function to study is not (6), but a Selberg type product discussed elsewhere [23, 24]. For present purposes $1/\zeta$ is good enough, and much more transparent illustration of shadowing of long cycles by short cycles discussed below.

5 CYCLE EXPANSIONS AND CURVATURES

How are formulas such as (6) used? We start by computing the lengths and eigenvalues of shortest cycles. In our pinball example this can be done by elementary geometrical optics; in general potentials or maps this requires some numerical integrations and Newton's method searches for periodic solutions. The result is a table like this [12] (the disk radius : center separation ratio is a:R = 1:6):

p	. T_p	. Λ_p
o	4.0	9.89897948557
i	4.26794919243	-11.7714551964
io	8.31652948517	-124.094801992
oio	12.3217466162	-1240.54255704
ioi	12.580807741	1449.54507485
oioo	16.3222764744	-12295.706862
ioii	16.8490718592	-17079.0190089
iooi	16.5852429061	14459.97595
ooioo	20.3223300257	-121733.838705
...	.	.

The next step is the key step in our approach [25, 26]: we observe that the expansion of the Euler product (6)

$$1/\zeta = 1 - \sum_f t_f - \sum_n c_n, \tag{9}$$

allows a regrouping of terms into dominant *fundamental* contributions t_f and decreasing *curvature* corrections c_n. For example, if the strange set is labelled by binary symbol sequences, as is the pinball of fig. 3, then the Euler product (6) is given by

$$\begin{aligned} 1/\zeta &= (1-t_0)(1-t_1)(1-t_{10})(1-t_{100})(1-t_{101})(1-t_{1000}) \\ &\quad (1-t_{1001})(1-t_{1011})(1-t_{10000})(1-t_{10001}) \end{aligned}$$

$$(1 - t_{10010})(1 - t_{10011})(1 - t_{10101})(1 - t_{10111})\ldots \quad (10)$$

The curvature expansion is obtained by multiplying out the Euler product and grouping together the terms of the same total symbol string length:

$$\begin{aligned} 1/\zeta &= 1 - t_0 - t_1 - [t_{10} - t_1 t_0] - [(t_{100} - t_{10}t_0) - (t_{101} - t_{10}t_1)] \\ &\quad -[(t_{1000} - t_0 t_{100}) + (t_{1110} - t_1 t_{110}) \\ &\quad +(t_{1001} - t_1 t_{001} - t_{101}t_0 + t_{10}t_0 t_1)] - \ldots \end{aligned} \quad (11)$$

The fundamental cycles t_0, t_1 have no shorter approximants; they are the "building blocks" of the dynamics in the sense that all longer orbits can be approximately pieced together from them. We call the sum of all terms of same total length n (grouped in brackets above) the nth curvature correction c_n, for geometrical reasons we shall soon try to explain. It is also often possible to pair off individual longer cycles and their shorter approximants (grouped in parenthesis above). If all orbits are weighted equally ($t_p = z^{n_p}$), such combinations cancel exactly; if orbits of similar symbolic dynamics have similar weights, the weights in such combinations will almost cancel.

Given the curvature expansion (11), the calculation is straightforward. We substitute the eigenvalues and lengths of prime cycles (for the example at hand, up to 5 pinball bounces - total of 14 cycles) into the curvature expansion (11), and obtain a polynomial approximation to $1/\zeta$. We then vary the exponent γ in (7) and determine the escape rate γ by finding the smallest γ for which (11) vanishes. The zeros are easily determined by standard numerical methods (such as the Newton algorithm), with accuracy as good as 7 significant digits for the pinball example considered here.

The convergence can be illustrated by listing γ computed from truncations of (11) to different maximal cycle lengths. (disk radius to disk-disk center separation ratio is a:R = 1:6). The first column gives the maximal cycle length used, the second the estimate of the classical escape rate from the full 3-disk cycle expansion, the third from the fundamental domain expansion [13].

n	full 3 disks	fund. dom.
1		0.407693
2	0.43578	0.410280
3	0.40491	0.410336
4	0.40945	0.410338
5	0.41037	0.410338
6	0.41034	

For comparison, a numerical simulation of ref. [10] yields $\gamma = .410\ldots$, and the $n = 2, 3$ approximations of ref. [10] yield 0.3102, 0.4508 respectively.

If one has some experience with numerical estimates of dimensions, one realizes that the convergence here is very impressive; only three input numbers (the two fixed points $\overline{0}$, $\overline{1}$ and the 2-cycle $\overline{10}$) already yield the escape rate to 4 significant digits! We have omitted an infinity of unstable cycles; so why does approximating the dynamics by a finite number of cycle eigenvalues work so well?

Figure 6. Approximation to (a) a smooth dynamics by (b) the skeleton of periodic points, together with their linearized neighborhoods.

A typical curvature expansion term in (11) is a *difference* of a long cycle $\{ab\}$ minus its shadowing approximation by shorter cycles $\{a\}$ and $\{b\}$:

$$t_{ab} - t_a t_b = t_{ab}(1 - t_a t_b/t_{ab}) = t_{ab}(1 - |\frac{\Lambda_{ab}}{\Lambda_a \Lambda_b}| e^{(T_a + T_b - T_{ab})\gamma}) \qquad (12)$$

To understand why this should be small compared to t_{ab}, try to visualize the description of a chaotic dynamical system in terms of cycles as a tessellation of the dynamical system, with smooth flow approximated by its periodic orbit skeleton, each "face" centered on a periodic point, and the scale of the "face" determined by the linearization of the flow around the periodic point (see fig. 6).

The orbits that follow the same symbolic dynamics, such as $\{ab\}$ and a "pseudo orbit" $\{a\}\{b\}$, lie physically close; longer and longer orbits resolve the dynamics with finer and finer resolution in the phase space. If the weights associated with the orbits are multiplicative along the flow (for example, products of derivatives) and the flow is smooth, the term in parenthesis in (12) falls off *exponentially* with the cycle length, and therefore the curvature expansions are expected to be highly convergent.

We have here used the curvature expansions to evaluate pinball escape rates, but the technique is much more general. It is applicable [26] to a broad class of "thermodynamical" averages such as those used in the extraction of generalized dimensions, as well as to the quantum periodic orbits sums [11, 12] to which we now turn.

6 QUANTUM CYCLE EXPANSIONS

In this section we illustrate the structure of quantum resonances by considering few simple cycle expansions [12].

The simplest conceivable system with exponentially growing errors has only one periodic orbit (for example, there is only one unstable orbit bouncing between two disks in a plane), and the cycle expansion (11) is simply

$$1/\zeta(k) = 1 - \frac{e^{iLk+i\pi m}}{\sqrt{\Lambda}} , \tag{13}$$

where Λ is the expanding eigenvalue of the cycle Jacobian, L is the cycle length, m is the Maslov index (here $m = 2$ for the 2 disk reflections), and the free motion action is $S(k) = \int \mathbf{p}.d\mathbf{q} = \hbar k L$. The resonances $1/\zeta(k_n) = 0$ are given by

$$k_n = \frac{\pi}{L}(m + 2n) - i\lambda/2, \quad n \in Z , \tag{14}$$

where $\lambda = \ln|\Lambda|/T$ is the cycle Lyapunov exponent (for constant velocity $v = 1$ motion the cycle period T equals the cycle length L). The real part of the wave–number k goes through a resonance for each n, the number of nodes in a standing wave, and the imaginary part indicates that the resonance is unstable, with the decay rate proportional to the cycle Lyapunov exponent.

The $1/\zeta(k)$ function is only the leading part of the Gutzwiller QM periodic orbit expansion; the full expression is given by the Selberg-type zeta function $Z(k) = \prod 1/\zeta_r(k)$, which for a single fixed point takes a particularly simple form

$$Z(k) = \prod_{r=0}^{\infty} (1 - \frac{e^{iLk+i\pi m}}{\sqrt{\Lambda}\Lambda^r}) , \tag{15}$$

with spectrum

$$k_{r,n} = \frac{\pi}{L}(m + 2n + rs) - i\lambda(1/2 + r), \quad n \in Z , \quad r = 0, 1, 2, \dots , \tag{16}$$

where $s = 0$ for positive Λ, and $s = 1$ for negative Λ. Hence the full Gutzwiller spectrum forms a lattice in the complex k plane, spaced horizontally by π/L and vertically by λ. In practice, only the resonances $\mathrm{Im}(k) = -\lambda/2$ closest to the real axis are important. The more distant resonances which give exponentially small contributions, are, as we shall see, of a considerable theoretical interest. The exact Green's functions for disks in a plane are known [27, 28, 29] and this could be a good playground for studying effects of the non-leading terms in semiclassical expansions. In the present context it suffices to make a mental note of the fact that the simplest conceivable problem already has an infinity of resonances in the negative $\mathrm{Im}(k)$ direction. It is also instructive to plot $|Z(k)|$ or the phase of $Z(K)$ in the complex k plane. For $\mathrm{Im}(k) > \lambda/2$, the $e^{iLk}/\Lambda^{1/2+r}$ term in (15) is always smaller than 1, $Z(k)$ is of order 1 throughout the upper complex k half-plane, and there are no resonances there. However, for large negative $\mathrm{Im}(k)$, the second terms in (15) factors grow exponentially in magnitude, and the nice regular lattice of zeros (16) hides a wildly varying $Z(k)$. Also note that while the cycle expansions of $1/\zeta(k)$ can yield numerically very accurate leading resonance spectra [11, 12], in contrast to $Z(k)$ they cannot be extended to the non-leading resonances [30, 31]; the breakdown of analyticity is accompanied with generation of spurious zeros along the border of analyticity.

The above example illustrates qualitatively the QM Gutzwiller spectrum; one expects infinite families of zeros, with the convergence of the theory controlled by their imaginary parts. The imaginary parts converge for reasons we have already discussed in context of cycle expansions on classical strange sets (imaginary k axis corresponds to the real γ axis in (7)); complex phases only add oscillations which do not increase the magnitude of terms in cycle expansions and thus do not destroy their convergence.

Though unstable, the 2-disk system is not chaotic: it has only one, rather than exponentially proliferating number (positive entropy) of unstable orbits. The effect of entropy can be illustrated by another simple model, a dynamical system with complete binary symbolic dynamics and uniform Lyapunov exponent for all orbits. As we have seen above, if the symbolic dynamics is a subshift of finite type, a cycle expansion (6) for a chaotic system (infinity of unstable cycles) consists of fundamental cycles and curvature corrections. If the dynamics is binary and expansion uniform, the curvature corrections vanish, and the factors in (15) become simply:

$$\begin{aligned} 1/\zeta_r(k) &= 1 - 2t \\ t &= e^{iLk}/\sqrt{\Lambda}\Lambda^r \ . \end{aligned} \tag{17}$$

While there is most likely no quantum system with such simple cycle structure (in classical dynamics a tent map is an example [26] of such system), we need this model only to illustrate the qualitative effect of positive entropy on quantum spectra.

The factor 2 in (17) reflects the fact that in this model the number of topologically distinct unstable trajectories of topological length n grows as 2^n; the spectrum has the same lattice structure as for the single unstable cycle, but all the resonances are now shifted *upwards* in the $\mathrm{Im}(k)$ direction by the entropy $h = \ln 2/T$:

$$k_{r,n} = \frac{\pi}{L}(m + 2n + rs) - i\left(\lambda(\frac{1}{2} + r) - h\right) \ . \tag{18}$$

Positive entropy *lowers* the decay rate; the physical reason is that for a chaotic system the local instability of the system (Lyapunov exponent) is being compensated by the back–scatter into the system (the number of orbits that have not escaped by time t, *ie.* the entropy). In classical bound systems probability conservation amounts to relation [32] $\lambda = h$, but the QM situation is subtler: factor $1/2$ in $\lambda/2 - h$ in (18) reflects the fact that the QM probability = (amplitude)^2. The probability (or unitarity) conservation constraints in quantum chaotic context are still poorly understood. While the main effect of the positive entropy is to bring the leading resonance closer to the Re(k) axis, the whole lattice of non-leading resonances in the lower complex k half–plane *remains*, and the spacing between the non-leading resonances is still given by the Lyapunov exponent.

We emphasize this, because even if the bound state leading resonances moved exactly onto the real axis, it is hard to imagine a mechanism by which contributions of individual cycles of wildly differing phases and magnitudes would conspire to eliminate all the non-leading resonances in the complex plane. On the other hand, existence of such resonances is a direct violation of quantum mechanical unitarity which guarantees that the bound state energy eigenvalues (for example, the hydrogen spectrum) are real. So even if cycle expansions yield the energy eigenvalues with good accuracy, we expect that the semi-classical saddle-point approximation that was the basis for the Gutzwiller cycle formulas will always lead to spurious subexponential eigenvalues in the complex energy plane.

It therefore comes as something of a shock to learn that for the genuine Selberg zeta [33, 34], interpretable as the QM spectrum of a Laplacian on a space of negative constant curvature, this unlikely conspiracy takes place, and the cycle expansion in this context is *exact*.

The above examples are simplistic, but they do capture the essential features of quantum resonance spectra for problems with sufficiently simple symbolic dynamics; one gets qualitatively right estimates of resonance locations and separations, beat oscillations in spectra [12], *etc.*. We illustrate this by a brief discussion of the anisotropic Kepler problem.

7 ANISOTROPIC KEPLER PROBLEM

In this section we apply the cycle expansions technique to the anisotropic Kepler problem, given by the Hamiltonian

$$E = -\frac{1}{2} = \frac{u^2}{2\mu} + \frac{v^2}{2\nu} - \frac{1}{(x^2+y^2)^{1/2}} \ . \qquad (19)$$

This model has been extensively studied by Gutzwiller; we refer the reader to his excellent monograph [2] for an overview of theoretical and experimental work related to the problem.

While in the isotropic Kepler problem an orbit returns precisely to the initial position and velocity after one Kepler period, in the anisotropic Kepler problem a generic orbit misses the initial point and continues wandering chaotically though the phase space - numerical evidence [2] even suggests that chaotic trajectories take the full measure of the available phase space.

The anisotropic Kepler problem arises in solid state physics, where electrons associated with donor impurities in silicon and germanium semiconductor crystals have anisotropic (and experimentally measurable) effective masses. For example [35], for silicon the anisotropy mass ratio is $\mu/\nu = 4.810$, and for germanium $\mu/\nu = 19.48$. The quantum mechanical spectra are known to high accuracy, both theoretically [36] and experimentally [37]. Furthermore, Gutzwiller [38] has obtained by Ising model techniques rather good estimates for dozens of the lowest eigenvalues, using 50 cycles of topological length 10. Hence it is possible to test the spectra obtained by the cycle expansion methods against both the exact spectra, and those obtained by trace formulas.

The anisotropic Kepler problem is also uniquely suited to testing the validity of quantum cycle expansions for a deeper theoretical reason: Gutzwiller has provided strong evidence that for anisotropies as large as the physically interesting ones all cycles (computed so far) are unstable and in 1-1 correspondence with a simple symbolic dynamics. For a Hamiltonian system such claim is a very surprising; we know of no other example of a bound Hamiltonian system with simple symbolic dynamics. But, as explained above, for simple symbolic dynamics the cycles can be combined into "shadowed" curvature combinations, and cycle expansions are expected to converge well.

We have carried out a preliminary check of cycle expansions for the anisotropic Kepler problem, and the results are promising. The cycle expansion evaluation of spectra for the anisotropic Kepler problem differs from the Gutzwiller's periodic orbit calculation [38] in several important aspects:

1. The zeta functions that we expand are already asymptotic, arranged in such a way that spectrum is dominated by short cycles, with long cycles contributing exponentially small curvature corrections. This is expected to lead to exponentially better convergence than that obtainable from the Gutzwiller trace sums, which require extrapolation to asymptotics from finite cycle lengths.

2. the backbone of our calculation are collision orbits; they were not utilized in previous calculations.

3. the fact that series of cycles accumulate to limits with finite stability and action might cause cycle expansions to diverge. However, our preliminary results indicate that the Maslov phases induce cancellations between successive terms in the cycle expansion, and that the reminders fall off exponentially.

4. symmetries are used to restrict the dynamics to a fundamental domain, factorize the spectra, and simplify the symbolic dynamics.

Inserting a total of 18 cycles gleaned from tables of ref. [39] into the cycle expansions (11), we obtain the spectrum of fig. 7. Cycle expansion of the Selberg-type Zeta function [40, 24] yields (within the accuracy of the present calculation) the same leading eigenvalues.

These results are preliminary in the sense that we have not rechecked the Maslov and symmetry factors used, and that a new calculation of cycles dictated by the fundamental domain symbolic dynamics is on the way [41] . If all works well, the accuracy of semi-classical expansions based periodic orbit theory should be improvable by orders of magnitude beyond the original results of ref. [38], and place us into position to attack more seriously the problems of unitarity, the WKB corrections and so on.

Figure 7. The symmetric A_1 subspace $\log|1/\zeta_{A_1}(k)|$ as function of the complex wave–number k for the anisotropic Kepler problem with anisotropy $\mu/\nu = 5$, estimated from the cycle expansion (11) truncated to the fundamental domain cycles up to topological length 6. If $n < \log|1/\zeta(k)| \leq n+1$, the complex k point is marked black for n odd, white for n even. The zeros of $|1/\zeta(k)|$ are the centers of the rapidly varying elliptic regions. The quantum-mechanical eigenvalues from ref. [36] lie on the real k axis at values indicated by vertical lines. Note that the $\mathrm{Re}(k)$ and $\mathrm{Im}(k)$ scales differ by a factor of about 50, and that the leading resonances lie on the real axis within the numerical convergence of the cycle expansions (estimated to about 5%).

Briefly, cycle expansions for the anisotropic Kepler problem are set up as follows [41]:

Scale conventions. The homogeneity of the potential (19) can be used to to scale out the energy dependence and bring action to form analogous to the quantum phase in the pinball problem (13)

$$\frac{1}{\hbar}S_p(E) = kT_p(E) \ , \tag{20}$$

where the wave–number k is the square root of the energy expressed in Rydberg units

$$k = \sqrt{E_R/E}\ , \quad E_R = -\frac{m_0 e^4}{2\kappa^2\hbar^2}\ , \tag{21}$$

and e = electric charge. For silicon $m_0 = (m_1 m_2)^{1/2} = 0.4177$ electron mass, and the dielectric constant $\kappa = 11.4$. The quantum cycle weight (8) is given by

$$t_p = \frac{1}{\sqrt{\Lambda_p}} e^{ikT_p + i\pi m_p}\ , \tag{22}$$

with m_p the Maslov index of the p cycle.

Symbolic dynamics. Effective use of cycle expansions requires identification of the shortest, fundamental cycles, and control of the symbolic dynamics for long cycles.

Anisotropy $\mu > \nu$ causes the orbits that cross y axis to fall faster toward the x axis, forcing them to cross the x axis before the next y axis crossing. As the motion along the x axis is sluggish compared with the motion across it, the orbit can cross (for sufficiently large [42, 43] anisotropy $\mu > 9/8$) the x axis any number of times before the next y axis crossing. Let x_i be the initial x-axis crossing, and x_{i+1} the next crossing. Then any trajectory can be pieced together from following four types of topologically distinct segments [39], distinguished by whether the segment does or does not cross the y axis:

$$\begin{array}{ccl} 11 & \leftarrow & x_i > 0, x_{i+1} > 0 \\ 01 & \leftarrow & x_i > 0, x_{i+1} < 0 \\ 10 & \leftarrow & x_i < 0, x_{i+1} > 0 \\ 00 & \leftarrow & x_i < 0, x_{i+1} < 0 \end{array} \tag{23}$$

In terms of the labels $\{11, 10, 01, 00\}$, the dynamics is a complete 4-symbol dynamics. This symbolic dynamics, introduced by Gutzwiller and Devaney, has not been proven to be of covering type: it is known that for every symbol sequence there exists at least one orbit, but it is only conjectured [2] that this orbit is unique. The conjecture is based on investigations of the foliation of the phase space, together with much numerical evidence [44], but it runs very much against what we know about generic Hamiltonian systems. Indeed, elliptic islands have been found [45] for small anisotropies (not much above $\mu/\nu > 9/8$), and in another model that has for long time looked as chaotic, careful symbolic dynamics work has ferreted out stability islands [46]. Such islands, if they exist for $\mu/\nu \approx 5 - 20$, are certainly very small, and we shall proceed as though the anisotropic Kepler problem is fully chaotic.

Symmetrization. Our cycle expansions start with a rather technical, but extremely useful step for setting up actual calculations [13]: we restrict the dynamics to the fundamental domain. The symmetry of the 2-dimensional anisotropic Kepler problem is C_{2v}, the group of x, y axes reflections and rotations by 180^o. We take a quadrant in the x-y plane as the fundamental domain. A global trajectory is reduced to the fundamental domain by reflecting the trajectory back into the fundamental domain at every symmetry line crossing; we split the trajectory into segments between successive x axis crossings, and mark the segment by symbol "i" if it includes an y axis reflection, and "o" otherwise. The global symbolic dynamics $\ldots s_n s_{n+1} s_{n+2} \ldots$ (23) is reduced to the fundamental domain symbolic dynamics [13] by a simple rule; $a_n = o$ if $s_n = s_{n+1}$, and $a_n = 0$ otherwise. For example, $0111 \to iioo$, $0000011111 \to ioooo$.

This reduces the global unrestricted 4-symbol dynamics to the simplest imaginable symbol dynamics, the unrestricted binary dynamics (just as in the 3-disk problem discussed above).

Reduction to the fundamental domain also alters the cycle weights and eigenvalues by group–theoretic factors. In our preliminary calculation, the total effect of the Maslov phases and symmetry factors in the A_1 symmetric subspace is an overall $(-1)^{n_p}$ sign factor in (22), with $n_p =$ topological length of the fundamental domain cycle.

Fundamental cycles. The next step in construction of cycle expansions is identification of the fundamental cycles. In our formulation of the anisotropic Kepler problem, there are six shortest global orbits that play role of the fundamental cycles.

The first type of fundamental cycle is rather obvious. The Kepler problem has a pair of counter-rotating circular orbits; in the anisotropic Kepler problem there is a corresponding pair of counter-rotating "pseudo-circular" orbits. In the fundamental domain, the two global "pseudo-circular" orbits reduce to the i fixed point, an orbit starts on the x axis and is bounced back into itself by a reflection off the y axis. In the crudest attempt [47] at periodic orbit quantization one simply uses the action of these orbits to find "standing wave" configurations; for reasons explained in the previous section, this is a fair attempt at getting a rough estimate of the spectrum, though omission of the infinity of other orbits necessarily yields energy eigenvalues with negative imaginary parts.

The next class of fundamental cycles are the four collision orbits along the x and y axes. Though they have been ignored in the previous trace calculations [39], they play the central role in our cycle expansions for the anisotropic Kepler problem, helping us to control the asymptotics of periods and stabilities of long cycles. The anisotropy destroys all Keplerian ellipses with the exception of the these four collision orbits whose periods are given by the third Kelper law [48]:

$$T = 2\pi a^{3/2} m^{1/2} , \tag{24}$$

where m is the mass and a is the major semi-axis. With the $E = -1/2$ scale convention, the collision orbits reach the boundary of the x–y space at $r = 2$, so the major semi-axis is $a = 1$. For the anisotropy ratio $\mu/\nu = 5$ (with mass scale fixed by setting $\mu\nu = 1$) the collision orbit periods are

$$\begin{aligned} T_x &= 2\pi 5^{1/4} &&= 9.3955535\ldots \\ T_y &= 2\pi/5^{1/4} &&= 4.2018193\ldots \end{aligned} \tag{25}$$

The stabilities of the collision orbits are also known in analytic form; Yoshida [49] gives a pretty formula for the expanding eigenvalue

$$\Lambda_y = (|c| + \sqrt{1+c^2})^2\,, \quad c = \cos\left(\frac{\pi}{2}\sqrt{9-8\mu/\nu}\right)\,. \tag{26}$$

We have not checked this formula numerically, but if it is correct, it implies that the stability of the collision orbit along the y axis is finite, but so large (of order of 10^7 for silicon) that we can safely omit it for time being from cycle expansions.

Cycle accumulation sequences. The x-axis collision orbit is densely enveloped by infinite sequences of orbits with arbitrarily large numbers of successive x-axis crossings (long $\ldots ooooo \ldots$ subsequences). We identify this collision orbit with the $\overline{o}$ fixed point, of well defined Kepler period $T_o = T_x$. Numerical results [41] indicate that the periods of the sequence of orbits io, ioo, $io\ldots oo$ which twine around the $\overline{o}$ collision orbit converge to

$$T_{io^n} = T_x - \frac{b}{\rho^n} \tag{27}$$

For $\mu/\nu = 5$, $\rho = 1.76\ldots$. Similarly, sequences of cycles of type iio^n hug closer and closer to an x-collision orbit followed by a y-collision orbit, and their periods converge to $T_y + T_x = 13.59737$ with the same geometric factor. However, the stability exponents in sequences that hug the y axis grow very large with n, as large as for the y-axis collision orbit, and longer cycles can be safely omitted.

Numerical work [41] indicates that also the stabilities of the io^n cycles converge geometrically $\mu_{io^n} = \log|\Lambda_{io^n}| = \mu_x - c/\sigma^n$, though not nearly as smoothly as for the cycle periods. For $\mu/\nu = 5$, $\sigma \approx 2$, and $\mu_x = 2.13\ldots$.

Hence with increasing n the weights of the io^n cycles are well approximated by the collision orbit

$$t_{io^n} = t_x e^{-ikb/\rho^n - c/\sigma^n + i\pi m_{io^n}}\,. \tag{28}$$

As is clear from refs. [31, 50], if there are many orbits of comparable stability, they have to be summed before exponential convergence of the cycle expansion can work. For accumulating sequences the natural labels are not binary but n-ary, where $n-1$ is the number of consecutive occurances [43, 31, 50] of symbol o between two consecutive i's. The appropriate truncations of cycle expansions include all n fixed points,

$$\begin{aligned} 1/\zeta &= 1 - \hat{t}_1 - (\hat{t}_{12} - t_1\hat{t}_2) - (\hat{t}_{112} - t_1\hat{t}_{12}) \\ & \quad -(\hat{t}_{23} - t_2\hat{t}_3) - (\hat{t}_{122} - t_{12}\hat{t}_2) - (\hat{t}_{1112} - t_1\hat{t}_{112}) - \cdots \end{aligned} \tag{29}$$

where $\hat{t}_p$ stands for t_p together with its infinite tail sequence of cycles of form (see table 3 of ref. [31])

$$\hat{t}_1 = \sum_{a=1}^{\infty} t_a \ , \quad \hat{t}_{12} = \sum_{a=2}^{\infty} t_{1a} \ , \quad \hat{t}_{a_1 \cdots a_{k-1} a_k} = \sum_{a=a_k}^{\infty} t_{a_1 \cdots a_{k-1} a} \, . \tag{30}$$

An infinity of orbits of type io, ioo, $io\ldots oo$, all of the same stability and action, presages disaster; the cycle expansion might not converge. However, we are saved by the fact that for each extra bounce the Maslov phase induces a prefactor -1. The geometric convergence (27) of the actions and the stabilities leads to geometric convergence of the alternating infinite series $\sum t_{\ldots o^n}$. For example, the fixed point approximaton to $1/\zeta$ is given by

$$\zeta^{(1)}(k) = 1 - \sum_{n=1}^{\infty} t_n = 1 - \sum_{n=1}^{N-1} t_n - t_x(-1)^N \left(ik\frac{B}{\rho^N} + \frac{C}{\sigma^N} \right) \ . \tag{31}$$

Putting all the above pieces together with the numerical values of actions and stabilities for all cycles up to given length (here up to length 6) we obtain a cycle expansion for $1/\zeta(k)$ and extract the resonance spectrum plotted in fig. 7.

Everything about this spectrum is pretty much as expected from the theory of cycle expansions [26]. As long as the product $k\Delta_N < \pi$, where Δ_N is some mean error in phase factors due to order N truncation of the cycle expansion, we expect the zeros of truncated cycle expansions to be exponentially close to the $N \to \infty$ limits. This means that for a finite number of cycles only a finite number of lowest eigenvalues is determined, and that these eigenvalues converge exponentially to their limits. In this sense the cycle expansions are still inferior to the usual spectra calculations by matrix diagonalizations (an example in the classical context is given in ref. [52]; this is also an example where better understanding of the transfer operator also leads to a good estimate for *all* eigenvalues). As far as we can tell, all leading eigenvalues lie very close to the real axis (from either side), but not exactly it. By estimates of sect. 6 the non-leading resonances should have $\mathrm{Im}(k)$ of order $-\lambda$. For the silicon anisotropy $\lambda \approx .2$, in qualitative agreement with the typical non-leading resonance distance to the real axis in fig. 7. For $1/\zeta(k)$ the non-leading resonances are outside the region of convergence, and the non-leading zeros in this figure are spurious effects of the finite convergence radius on the truncated cycle expansions. This is partially fixed up by using $Z(k)$ cycle expansions, which for hyperbolic systems are entire functions in the k complex plane. Still, in all fairness, the non-leading resonances are uncomfortably close to the real axis, and further refinements of the theory are needed. In order to improve the spectrum we need either to go beyond the semiclassical leading terms, or find a way of imposing unitarity. In this connection we note that the expansions advocated in ref. [51] are actually the same as the ones we have been using [23, 52, 53]. We find the conjecture that Riemann-Siegel type formulas exist unlikely to hold in this context; the Riemann-Siegel formulas use the self-duality of the Riemann and other ζ functions, but there is no evidence of such symmetry for generic Hamiltonian flows.

8 SUMMARY

We have here illustrated the utility of cycles and their curvatures by the simple pinball example of fig. 1 and by a preliminary check of the spectrum of the anisotropic Kepler problem. The cycle expansions such as (9) outperform the pedestrian methods such as the nth level trace sum (1) because the higher terms in the cycle expansion (9) are deviations of longer primitive cycles from their approximations by shorter cycles $c_{10} = -t_{10} + t_1 t_0$, $c_{1001} = -t_{1001} + t_1 t_{001} + t_{101} t_0 - t_{10} t_0 t_1, \ldots$, which vanish exactly in polygonal approximations and are expected to fall off exponentially for smooth hyperbolic flows.

In our approach the lessons of classical periodic orbit theory play a crucial role insofar that they dictate expansions that would probably seem less than obvious if one starts with semiclassical expansions of quantum Green's functions: in particular, omission of the collision orbits would destroy shadowing, and make cycle expansions not any more convergent than the trace formulas used previously. Much of the spectrum can be qualitatively understood already in terms of the fundamental cycles. Comparison with the exact spectrum of [36] and Gutzwiller's best result[38] shows that with considerably less labor cycle expansions reproduce the spectrum to comparable accuracy; one basically constructs a polynomial of low order (5th, for example) in the topological cycle length and determines its zeros.

To summarize: a motion on a strange attractor can be approximated by shadowing the orbit by a sequence of nearby periodic orbits of finite length. This notion is here made precise by decomposing orbits into primitive cycles, and evaluating associated curvatures. A curvature measures the deviation of a longer cycle from its approximation by shorter cycles; the smoothness of the dynamical system implies exponential fall-off for (almost) all curvatures. The technical prerequisite for implementing this shadowing is a good understanding of the symbolic dynamics of the classical dynamical system. The resulting curvature expansions offer an efficient method for evaluating classical and quantum periodic orbit sums; accurate estimates can be obtained by using as input the actions and eigenvalues of as few as 2–14 prime cycles.

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