TRACE FORMULAS IN CLASSICAL DYNAMICAL SYSTEMS

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INTRODUCTION

Dynamics is posed in terms of local equations, but the ergodic averages require global information. How can we use a local description of a flow to learn something about the global behavior? We shall now relate global averages to the eigenvalues of appropriate evolution operators. The eigenvalues are given by the zeros of corresponding determinants, and one way to evaluate determinants is to expand them in terms of traces, using the matrix identity $\log\det = \mathrm{tr}\log$. Traces of evolution operators can be evaluated as integrals over Dirac delta functions, and in this way the spectra of evolution operators become related to periodic orbits. If there is one idea that one should learn about chaotic dynamics, it is this: there is a fundamental local $\leftrightarrow$ global duality which says that

the spectrum of eigenvalues is dual to the spectrum of periodic orbits

For dynamics on the circle, this is called Fourier analysis; for dynamics on well-tiled manifolds, Selberg traces and zetas; and for generic nonlinear dynamical systems the duality is embodied in the formulas that we will now introduce: trace formulas, zeta functions and spectral determinants. These objects are to dynamics what partition functions are to statistical mechanics.

We start by discussing the necessity of studying the averages of observables in chaotic dynamics, and then cast the formulas for averages in a multiplicative form that motivates the introduction of evolution operators and the further formal developments to come.

Dynamical averaging

In chaotic dynamics detailed prediction is impossible, as any finitely specified initial condition, no matter how precise, will fill out the entire accessible phase space. Hence for chaotic dynamics one cannot follow individual trajectories for a long time; what is attainable is a description of the geometry of the set of possible outcomes, and evaluation of the long time averages. Examples of such averages are transport coefficients for chaotic dynamical flows, such as escape rate, mean drift and diffusion rate; power spectra; and a host of mathematical constructs such as generalized dimensions, entropies and Lyapunov exponents. Here we outline how such averages are evaluated within the framework of the periodic orbit theory. The key idea is to replace the expectation values of observables by the expectation values of generating functionals. This associates an evolution operator with a given observable, and relates the expectation value of the observable to the leading eigenvalue of the evolution operator.

Time averaging

Let a be any "observable", a function that associates to each point in phase space a number, a vector, or a tensor. The observable reports on a property of the dynamical system. The velocity field $a_i(x) = v_i(x)$ is an example of a vector observable; the length of this vector, or perhaps a temperature measured in an experiment at instant τ are examples of scalar observables. We define the *integrated observable* A^t as the time integral of the observable a evaluated along the trajectory of the initial point x_0,

$$A^t(x_0) = \int_0^t d\tau\, a(f^\tau(x_0))\,. \tag{1}$$

If the dynamics is given by an iterated mapping and the time is discrete, the integrated observable is given by

$$A^t(x_0) = \sum_{k=0}^{t-1} a(f^k(x_0)) \tag{2}$$

(we suppress possible vectorial indices for the time being). For example, if the observable is the velocity, $a_i(x) = v_i(x)$, its time integral $A_i^t(x_0)$ is the trajectory $A_i^t(x_0) = x_i(t)$. Another familiar example, for Hamiltonian flows, is the action associated with a trajectory passing through a phase space point x_0 (this function is the key to the quasiclassical quantization):

$$A^t(x_0) = \int_0^t d\tau\, \dot{\mathbf{q}}(\tau)\cdot\mathbf{p}(\tau)\,, \qquad x = [p,q]\,, \quad x_0 = [p(0),q(0)]\,.$$

The *time average* of the observable along the trajectory is defined by

$$\overline{a(x_0)} = \lim_{t\to\infty}\frac{1}{t}A^t(x_0)\,. \tag{3}$$

If a does not behave too wildly as a function of time — for example, $a_i(x) = x_i$ is bounded for bounded dynamical systems — $A^t(x_0)$ is expected to grow not faster than t, and the limit (3) might exist.

The time average depends on the trajectory, but not on the initial point. Starting at a later phase space point $f^T(x_0)$

$$\begin{aligned}\overline{a(f^T(x_0))} &= \lim_{t\to\infty}\frac{1}{t}\int_T^t d\tau\, a(f^\tau(x_0)) \\ &= \lim_{t\to\infty}\frac{1}{t}\left(\int_t^T d\tau\, a(f^\tau(x_0)) + \int_0^t d\tau\, a(f^\tau(x_0))\right) = \overline{a(x_0)}\end{aligned}$$

one obtains a finite contribution to $A^t(x_0)$ which vanishes in the limit $t\to\infty$.

The integrated observable $A^t(x_0)$ and the time average $\overline{a(x_0)}$ take a particularly simple form when evaluated on a periodic orbit. Define

$$A_p = \int_0^{T_p} d\tau\, a(f^\tau(x_0))\,, \qquad x_0 \in p\,, \tag{4}$$

where p is a prime cycle, and T_p is its period. A_p is a loop integral along the cycle, so it is an intrinsic property of the cycle, independent of the starting point $x_0 \in p$. (If the observable a is not a scalar but a vector or matrix we might have to be more precise in defining an average which is independent of the starting point around the cycle). If the trajectory retraces itself r times, we just obtain A_p repeated r times. Evaluation of the asymptotic time average (3) requires therefore only a single traversal of the cycle:

$$\overline{a_p} = \frac{1}{T_p} A_p\,. \tag{5}$$

However, $\overline{a(x_0)}$ is in general certainly a very wild function of x_0; for a nice hyperbolic system it takes the same value $\langle a \rangle$ for almost all initial x_0, but a different value (5) on any periodic orbit, that is, on a dense set of points. For example, for an open system such as the Sinai gas (an infinite two-dimensional periodic array of scattering disks) the phase space is dense with initial points that correspond to periodic runaway trajectories. The mean distance squared traversed by any such trajectory grows as $x(t)^2 \sim t^2$, and its contribution to the diffusion rate $D \approx x(t)^2/t$, (3) evaluated with $a(x) = x(t)^2$, diverges. Seemingly there is a paradox; even though intuition says the typical motion should be diffusive, we have an infinity of ballistic trajectories.

For chaotic dynamical systems, this paradox is resolved by robust averaging, that is, averaging also over the initial x, and worrying about the measure of the "pathological" trajectories.

Space averaging

The *space average* of a quantity a that may depend on the point x of phase space $\mathcal{M}$ and on the time t is given by the $(d+1)$-dimensional integral over the $(d+1)$ coordinates of the dynamical system:

$$\begin{aligned} \langle a(t) \rangle &= \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx\; a(x(t)) \\ |\mathcal{M}| &= \int_{\mathcal{M}} dx \;= \text{volume of } \mathcal{M}\,. \end{aligned} \tag{6}$$

The space $\mathcal{M}$ is assumed to have finite dimension and volume. For open systems like the 3-disk pinball we chose $\mathcal{M}$ such that it encloses the non–wandering set, and we discard all runaway trajectories. We define the *expectation value* $\langle a \rangle$ of an observable a to be the asymptotic time and space average over the "phase space" $\mathcal{M}$

$$\langle a \rangle = \lim_{t\to\infty} \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx\, \frac{1}{t} \int_0^t d\tau\; a(f^\tau(x))\,. \tag{7}$$

We use the same $\langle \cdots \rangle$ notation as for the space average (6), and distinguish the two by the presence of the time variable in the argument: if the quantity $\langle a(t) \rangle$ being averaged depends on time, then it is a space average, if it does not, it is the expectation value $\langle a \rangle$.

The expectation value is a space average of time averages. Each $x \in \mathcal{M}$ is used as a starting point of a time average. The averaging over space has the great advantage that it smears over the starting points which were problematic for the time average (like the periodic points). While easy to define, the expectation value $\langle a \rangle$ turns out not to be particularly tractable in practice. Such averages are more conveniently studied by introducing an auxiliary variable β, and investigating instead of $\langle a \rangle$ the space averages of form

$$\left\langle e^{\beta \cdot A^t} \right\rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx \, e^{\beta \cdot A^t(x)} . \tag{8}$$

In the present context β is an auxiliary variable of no particular physical meaning. In most applications β is a scalar, but if the observable is a $(d+1)$-dimensional vector $a_i(x) \in \mathbb{R}^{d+1}$, then β is a conjugate vector $\beta \in \mathbb{R}^{d+1}$; if the observable is a $(d+1) \times (d+1)$ tensor, β is also a rank-2 tensor, and so on. We will here mostly limit ourselves to scalar values of β.

If the limit $\overline{a(x_0)}$ for the time average (3) exists for "almost all" initial x_0 and the system is ergodic and mixing, we expect the time average along almost all trajectories to tend to the same value $\overline{a}$, and the integrated observable A^t to tend to $t\overline{a}$. The space average (8) is an integral over exponentials, which therefore also grows exponentially with time. So as $t \to \infty$ we would expect the space average of $\langle \exp(\beta \cdot A^t) \rangle$ itself to grow exponentially with time

$$\left\langle e^{\beta \cdot A^t} \right\rangle \sim e^{t s(\beta)} .$$

Hence we expect the rate of growth of the space average to go to a (parameter β dependent) limit

$$s(\beta) = \lim_{t \to \infty} \frac{1}{t} \ln \left\langle e^{\beta \cdot A^t} \right\rangle . \tag{9}$$

Now we have the first argument for why it is smarter to compute $\langle \exp(\beta \cdot A^t) \rangle$ rather than $\langle a \rangle$: the expectation value of the observable (7) and the moments of the integrated observable (1) can be computed by evaluating the derivatives of $s(\beta)$

$$\begin{aligned} \left.\frac{\partial s}{\partial \beta}\right|_{\beta=0} &= \lim_{t \to \infty} \frac{1}{t} \left\langle A^t \right\rangle = \langle a \rangle , \\ \left.\frac{\partial^2 s}{\partial \beta^2}\right|_{\beta=0} &= \lim_{t \to \infty} \frac{1}{t} \left(\left\langle A^t A^t \right\rangle - \left\langle A^t \right\rangle \left\langle A^t \right\rangle \right) \\ &= \lim_{t \to \infty} \frac{1}{t} \left\langle (A^t - t \langle a \rangle)^2 \right\rangle , \end{aligned} \tag{10}$$

and so forth. If we can compute the function $s(\beta)$, we have the desired expectation value without having to estimate any infinite time limits from finite time data.

Suppose we could evaluate $s(\beta)$ and its derivatives. What are such formulas good for? A typical application is to the problem of describing a particle scattering elastically off a two-dimensional triangular array of disks. If the disks are sufficiently large to block any infinite length free flights, the particle will diffuse chaotically, and the transport coefficient of interest is the diffusion constant given by $x(t)^2 \approx 4Dt$. In contrast to D estimated numerically from trajectories $x(t)$ for finite but large t, the above formulas yield the asymptotic D without any extrapolations to the $t \to \infty$ limit. For example, for $a_i = v_i$ and zero mean drift $\langle v_i \rangle = 0$, the diffusion constant is given by the curvature of $s(\beta)$ at $\beta = 0$,

$$D = \lim_{t \to \infty} \frac{1}{2dt} \left\langle x(t)^2 \right\rangle = \frac{1}{2d} \sum_{i=1}^{d} \left.\frac{\partial^2 s}{\partial \beta_i^2}\right|_{\beta=0} , \tag{11}$$

so if we can evaluate derivatives of $s(\beta)$, we can compute transport coefficients that characterize deterministic diffusion. The periodic orbit theory yields an explicit closed form expression for D.

Averaging in open systems

If the $\mathcal{M}$ is a compact region or set of regions to which the dynamics is confined for all times, (7) is a sensible definition of the expectation value. However, if we have a repeller or an attractor for which trajectories can exit the region without ever returning we might be in trouble. For example, a pinball in our 3-disk pinball might fly off to Mars, contributing to the expectation value in a not very useful way.

For an attractor whose basin is not included in $\mathcal{M}$, and for sufficiently large t a trajectory originating in $\mathcal{M}$ might never renter it again,

$$\int_{\mathcal{M}} dy\, \delta(y - f^t(x_0)) = 0 \qquad \text{for } t > t_{exit}\,.$$

For a repeller $f^t(x_0)$ will eventually leave the region of integration unless we choose x_0 to be on the repeller, so the identity

$$\int_{\mathcal{M}} dy\, \delta(y - f^t(x_0)) = 1 \qquad \text{iff } x_0 \in \text{ non–wandering set} \tag{12}$$

might be correct only on a fractal set $\mathcal{M}$ of zero Lebesgue measure. Clearly, we need to modify the definition of the expectation value so it refers only to the dynamics on the set of trajectories which are confined for all times, that is the non–wandering set.

Designate by $\mathcal{M}$ a phase space region that encloses all interesting initial points, say the 3-disk Poincaré section constructed from the disk boundaries and all possible incidence angles. The volume of the phase space containing all trajectories which start out within the phase space region $\mathcal{M}$ and recur within that region at the time t

$$|\mathcal{M}(t)| = \int_{\mathcal{M}} dx dy\, \delta\big(y - f^t(x)\big) \;\sim\; |\mathcal{M}| e^{-\gamma t}\,. \tag{13}$$

is expected to decay exponentially, with the *escape rate* γ. The integral over x takes care of all possible initial pinballs; the integral over y checks whether they are still within $\mathcal{M}$ by the time t. If the dynamics is bounded, and $\mathcal{M}$ envelops the entire accessible phase space, $|\mathcal{M}(t)|=$ const. for all t. However, if trajectories exit $\mathcal{M}$ the recurrence volume $|\mathcal{M}(t)|$ decreases with time. For example, any trajectory that falls off a pinball table is gone for good.

For open systems we replace the space average (8) by

$$\left\langle e^{\beta\cdot A^t}\right\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dx\, e^{\beta\cdot A^t(x)+\gamma t}\,. \tag{14}$$

The interpretation is that by compensating for the loss of trajectories with factor $e^{\gamma t}$ we have a correctly normalized $\beta = 0$ average $\langle 1\rangle = 1$.

We now turn to the problem of evaluating $\left\langle e^{\beta\cdot A^t}\right\rangle$.

Evolution operators

The above simple shift of focus, from studying $\langle a\rangle$ to studying $\langle \exp(\beta\cdot A^t)\rangle$ is the key to all what follows. Make the dependence on the flow explicit by rewriting this quantity as

$$\left\langle e^{\beta\cdot A^t}\right\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dx \int_{\mathcal{M}} dy\, \delta\big(y - f^t(x)\big)\, e^{\beta\cdot A^t(x)}\,. \tag{15}$$

Here $\delta(y - f^t(x))$ is the Dirac delta function: for a deterministic flow an initial point x maps into a unique point y at time t. Formally, all we have done above is to insert the identity

$$1 = \int_{\mathcal{M}} dy\, \delta\big(y - f^t(x)\big) \; , \tag{16}$$

that is, we are averaging over trajectories that remain in $\mathcal{M}$ for all times. However, having made this substitution we have replaced the study of individual trajectories $f^t(x)$ by the study of the evolution of the totality of initial conditions. Instead of temporal averages (a long trajectory which explores the phase space ergodically) we shall always work with finite time, topologically partitioned space averages. We now introduce the *evolution operator* that acts on bounded scalar functions $h(x)$ of the phase space as

$$\mathcal{L}^t \circ h(y) = \int_{\mathcal{M}} dx\, \delta\big(y - f^t(x)\big)\, e^{\beta \cdot A^t(x)} h(x)\,, \tag{17}$$

and rewrite the expectation value of the generating function (15) as

$$\left\langle e^{\beta \cdot A^t} \right\rangle = \left\langle \mathcal{L}^t \circ \iota \right\rangle \; ,$$

where $\iota(x)$ is the constant function that always returns 1. For chaotic dynamics any smooth intial distribution $\rho(x)$ would be just as good.

We shall refer to the kernel of the evolution operator in (17) as $\mathcal{L}^t(y,x)$ where

$$\mathcal{L}^t(y,x) \;=\; \delta\big(y - f^t(x)\big)\; e^{\beta \cdot A^t(x)}\,. \tag{18}$$

The evolution operator is different for different observables, as its definition depends on the choice of the integrated observable A in the exponential. Its role, as $t \to \infty$, is to generate the expectation value of a, but before showing that we verify some of its basic properties.

By its definition, the integral over the observable a is additive along the trajectory

$$\begin{aligned} A^{t_1+t_2}(x_0) &= \int_0^{t_1} d\tau\, a(x(\tau)) \;+\; \int_{t_1}^{t_1+t_2} d\tau\, a(x(\tau)) \\ &= A^{t_1}(x_0) \;+\; A^{t_2}(f^{t_1}(x_0))\,. \end{aligned}$$

If $A^t(x)$ is additive along the trajectory, the evolution operator generates a semigroup; $\mathcal{L}^{t_2} \circ \mathcal{L}^{t_1} = \mathcal{L}^{t_1+t_2}$. The order of the composition of operators can be a little confusing. What is meant is that

$$\mathcal{L}^{t_2} \circ \mathcal{L}^{t_1} \circ h(x) = \int_{\mathcal{M}} dy\, \delta(x - f^{t_2}(y)) e^{\beta \cdot A^{t_2}(y)} (\mathcal{L}^{t_1} \circ h)(y) = \mathcal{L}^{t_1+t_2} h(x)\,. \tag{19}$$

This semigroup property is the main reason why (15) is preferable to (7) as a starting point for evaluation of dynamical averages; their value in the asymptotic $t \to \infty$ limit can be recovered by means of evolution operators.

The topological entropy, the asymptotic rate of growth of the number of topologically distinct trajectories, is given by the leading eigenvalue of the transition matrix. It is our goal now to generalize this observation and relate general asymptotic averages over chaotic dynamics to eigenvalues of evolution operators. As we shall now show, in the case of piecewise-linear approximations to dynamics these operators reduce to finite matrices, but for generic smooth flows, they are infinite-dimensional linear operators, and finding smart ways of computing their eigenvalues requires some thought.

Transfer operators

We start by showing that for a piecewise-linear map with a finite Markov partition, the evolution operator is a finite dimensional transfer operator, a straightforward generalization of the topological transition matrices.

Consider as an example of expanding 1-d map $f(x)$, with $|f_s'(x)| > 1$ and monotone on two non-overlapping intervals, a piecewise-linear 2–branch repeller with slopes Λ_0 and Λ_1:

$$f(x) = \begin{cases} \Lambda_0 x & \text{if } x \in \mathcal{M}_0 = [0, 1/\Lambda_0] \\ \Lambda_1(1-x) & \text{if } x \in \mathcal{M}_1 = [1 - 1/\Lambda_1, 1] \end{cases} . \tag{20}$$

As both $f(\mathcal{M}_0)$ and $f(\mathcal{M}_1)$ map onto the entire unit interval $\mathcal{M}$, the partition into $\{\mathcal{M}_0, \mathcal{M}_1\}$ is a finite Markov partition with the unrestricted 2-letter alphabet $s_n \in \{0, 1\}$, with the transition matrix

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

The transition matrix T counts the allowed transitions, so $T_{dm} = 0$ or 1; however, it is just as easy to keep track of matrix elements T_{dm} that take other ranges of values. Due to the piecewise linearity, the simplest of evolution operators, the Perron-Frobenius operator acts on a piecewise constant function $\rho(x) = (\rho_0, \rho_1)$ as a $[2\times 2]$ transfer matrix with matrix elements

$$\begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} \to \mathcal{L} \circ \rho = \begin{pmatrix} \frac{1}{|\Lambda_0|} & \frac{1}{|\Lambda_1|} \\ \frac{1}{|\Lambda_0|} & \frac{1}{|\Lambda_1|} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} . \tag{21}$$

More generally, T_{dm} is the "penalty" paid for choosing a particular transition, and T is a finite memory *Markov* or *transfer* matrix. Piecewise-linear approximations to dynamical systems yield finite-dimensional transfer operators, provided that the symbolic dynamics is a subshift of finite type.

If we increase the number of steps remembered, the transition matrix grows big quickly, as N-ary dynamics with n-step memory requires an $[N^n \times N^n]$ matrix. As the matrix is very sparse, we shall find it convenient to use the Markov graph representations for T.

Trace formulas

Our extraction of the spectrum of $\mathcal{L}$ commences with the evaluation of the trace. To compute an expectation value we have to integrate over all the values of the kernel $\mathcal{L}^t(x, y)$. If $\mathcal{L}^t$ were a matrix we would be computing a weighted sum of its eigenvalues which is dominated by the leading eigenvalue as $t \to \infty$. As the trace of $\mathcal{L}^t$ is also dominated by the leading eigenvalue as $t \to \infty$, we might just as well look at the trace

$$\operatorname{tr} \mathcal{L}^t = \int dx\, \mathcal{L}^t(x, x) = \int dx\, \delta\big(x - f^t(x)\big)\, e^{\beta \cdot A^t(x)} . \tag{22}$$

Assume that $\mathcal{L}$ has a spectrum of discrete eigenvalues $s_0, s_1, s_2, \cdots$ ordered so that $\mathrm{Re}\, s_\alpha \le \mathrm{Re}\, s_{\alpha+1}$. We ignore for the time being the question of what function space the eigenfunctions belong to, as we shall compute the eigenvalues directly, without constructing the corresponding eigenfunctions.

By definition, the trace is the sum over eigenvalues

$$\mathrm{tr}\,\mathcal{L}^t = \sum_{\alpha=0}^{\infty} e^{s_\alpha t}\,, \tag{23}$$

so if we can evaluate the above delta-function integral, we have a formula for eigenvalue sums (for the time being we choose not to worry about convergence of such sums). Integrating Dirac delta functions is easy: $\int_{\mathcal{M}} dx\, \delta(x) = 1$ if $0 \in \mathcal{M}$, zero otherwise. Integral over a one-dimensional Dirac delta function picks up the Jacobian of its argument evaluated at all of its zeros:

$$\int dx\, \delta(h(x)) = \sum_{x\in \mathrm{Zero}\,[h]} \frac{1}{|h(x)'|}\,, \tag{24}$$

and in d dimensions the denominator is replaced by

$$\int dx\, \delta(h(x)) = \sum_{x\in \mathrm{Zero}\,[h]} \frac{1}{\left|\det \frac{\partial h(x)}{\partial x}\right|}\,. \tag{25}$$

An evolution operator propagates a density of inital conditions $h(x)$ to time t:

$$\mathcal{L}^t \circ h(x) = \int dy\, \delta(x - f^t(y))h(y) = \sum_{y=f^{-t}(x)} \frac{h(y)}{|f^t(y)'|}\,.$$

As the case of discrete time mappings is somewhat simpler, we first derive the trace formula for maps, and then for flows. The final formula (39) covers both cases.

Hyperbolicity assumption

According to (25) the trace (22) picks up a contribution whenever $x - f^T(x) = 0$, that is whenever x belongs to a periodic orbit. The contribution of a prime cycle p of period T_p for a map f can be evaluated by restricting the integration to an infinitesimal neighborhood $\mathcal{M}_p$ around the cycle, with $x_p \in p$ a cycle point

$$\mathrm{tr}\,{}_p\mathcal{L}^{T_p} = \int_{\mathcal{M}_p} dx\, \delta\big(x - f^{T_p}(x)\big) = \frac{1}{\left|\det\big(\mathbf{1} - \mathbf{J}_p\big)\right|} = \prod_{i=1}^{d} \frac{1}{|1-\Lambda_{p,i}|}\,. \tag{26}$$

(Here we set the observable $e^{A_p} = 1$ for the time being). Periodic orbit Jacobian matrix $\mathbf{J}_p$ is also known as the monodromy matrix, and its eigenvalues $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ as the Floquet multipliers. The integral can be carried out only if $\mathbf{J}_p$ has no eigenvalue of unit magnitude. Sort the eigenvalues $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ of the p-cycle $[d\times d]$ transverse Jacobian matrix $\mathbf{J}_p$ into sets $\{e, m, c\}$:

$$\begin{aligned}
\text{expanding eigenvalues: } e &= \{\Lambda_{p,i} : |\Lambda_{p,i}| > 1\} \\
\text{marginal eigenvalues: } m &= \{\Lambda_{p,i} : |\Lambda_{p,i}| = 1\} \\
\text{contracting eigenvalues: } c &= \{\Lambda_{p,i} : |\Lambda_{p,i}| < 1\}
\end{aligned}$$

We assume that no eigenvalue is marginal (as we shall show in (37), the longitudinal $\Lambda_{p,d+1} = 1$ eigenvalue for flows can be eliminated by restricting the consideration to the transverse Jacobian matrix $\mathbf{J}_p$), and factorize the trace (26) into a product over the expanding and the contracting eigenvalues

$$\left|\det\big(\mathbf{1} - \mathbf{J}_p\big)\right|^{-1} = \frac{1}{|\Lambda_p|} \prod_e \frac{1}{1 - 1/\Lambda_{p,e}} \prod_c \frac{1}{1-\Lambda_{p,c}}\,, \tag{27}$$

where $\Lambda_p = \prod_e \Lambda_{p,e}$ is the product of expanding eigenvalues. In the above both $\Lambda_{p,c}$ and $1/\Lambda_{p,e}$ are smaller than 1 in absolute value, and as long as they are real we are allowed to drop the absolute value brackets $|\cdots|$.

The *hyperbolicity assumption* requires that the stabilities of all cycles included in the trace sums be exponentially bounded away from unity:

$$\begin{array}{ll} |\Lambda_{p,i}| > e^{\lambda_e T_p} & \text{for any } p, \text{ any expanding eigenvalue } i \in e \\ |\Lambda_{p,i}| < e^{-\lambda_c T_p} & \text{for any } p, \text{ any contracting eigenvalue } i \in c\,, \end{array} \tag{28}$$

where $\lambda_e, \lambda_c > 0$ are strictly positive bounds on the expanding, contracting cycle Lyapunov exponents. If a dynamical system satisfies the hyperbolicity assumption (for example, the well separated 3-disk system clearly does), the $\mathcal{L}^t$ spectrum will be relatively easy to control. If the expansion/contraction is slower than exponential, let us say $|\Lambda_{p,i}| \sim {T_p}^2$, the system may exhibit "phase transitions", and the analysis is much harder.

A trace formula for maps

If the evolution is given by a discrete time mapping, and all periodic points have stability eigenvalues $|\Lambda_{p,i}| \neq 1$ strictly bounded away from unity, the trace $\mathcal{L}^n$ is given by the sum over all periodic points i of period n:

$$\operatorname{tr} \mathcal{L}^n = \int dx\, \mathcal{L}^n(x,x) = \sum_{x_i \in \mathrm{Fix} f^n} \frac{e^{\beta \cdot A^n(x_i)}}{|\det(\mathbf{1} - \mathbf{J}^n(x_i))|}\,. \tag{29}$$

Here $\mathrm{Fix}\, f^n = \{x : f^n(x) = x\}$ is the set of all periodic points of period n. The weight follows from the properties of the Dirac delta function (25) by taking the determinant of $\partial_i(x_j - f^n(x)_j)$. If a trajectory retraces itself r times, its Jacobian matrix is $\mathbf{J}_p^r$, where $\mathbf{J}_p$ is the $[d \times d]$ Jacobian matrix evaluated along a single traversal of the prime cycle p. As we saw in (4), the integrated observable A^n also has a similar property: If a prime cycle p trajectory retraces itself r times, $n = rn_p$, we obtain A_p repeated r times, $A^n(x_i) = rA_p$, $x_i \in p$. A prime cycle is a single traversal of the orbit, and its label is a non-repeating symbol string. There is only one prime cycle for each cyclic permutation class. For example, the four cycle points $\overline{0011} = \overline{1001} = \overline{1100} = \overline{0110}$ belong to the same prime cycle $p = 0011$ of length 4. As both the stability of a cycle and the weight A_p are the same everywhere along the orbit, each prime cycle of length n_p contributes n_p terms to the sum, one for each cycle point. Hence (29) can be rewritten as a sum over all prime cycles and their repeats

$$\operatorname{tr} \mathcal{L}^n = \sum_p n_p \sum_{r=1}^{\infty} \frac{e^{r\beta \cdot A_p}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} \delta_{n,n_p r}\,, \tag{30}$$

with the Kronecker delta $\delta_{n,n_p r}$ projecting out the periodic contributions of total period n. This constraint is awkward, and will be more awkward still for the continuous time flows, where it will yield a series of Dirac delta spikes (38). Such sums are familiar from the density-of-states sums of statistical mechanics, where they are dealt with in the same way as we shall do here: we smoothen this distribution by taking a Laplace transform which rids us of the $\delta_{n,n_p r}$ constraint.

We define the trace formula for maps to be the Laplace transform of $\operatorname{tr} \mathcal{L}^n$ which, for discrete time mappings, is simply the generating function for the trace sums

$$\sum_{n=1}^{\infty} z^n \operatorname{tr} \mathcal{L}^n = \operatorname{tr} \frac{z\mathcal{L}}{1 - z\mathcal{L}} = \sum_p n_p \sum_{r=1}^{\infty} \frac{z^{n_p r} e^{r\beta \cdot A_p}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|}\,. \tag{31}$$

Expressing the trace as (23), the sum of the eigenvalues of $\mathcal{L}$, we obtain the *trace formula for maps*:

$$\sum_{\alpha=0}^{\infty} \frac{z e^{s_\alpha}}{1 - z e^{s_\alpha}} = \sum_p n_p \sum_{r=1}^{\infty} \frac{z^{n_p r}\, e^{r\beta \cdot A_p}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} \,. \tag{32}$$

This is our first example of the duality between the spectrum of eigenvalues and the spectrum of periodic orbits, announced in the introduction to this chapter.

A trace formula for transfer operators

For a piecewise-linear map with a finite Markov partition, we can explicitly evaluate the trace formula. We illustrate this by the piecewise linear repeller (20). By the piecewise linearity and the chain rule $\Lambda_p = \Lambda_0^{n_0}\Lambda_1^{n_1}$, where the cycle p contains n_0 symbols 0 and n_1 symbols 1, the trace (29) reduces to

$$\operatorname{tr} \mathcal{L}^n = \sum_{m=0}^{n} \binom{n}{m} \frac{1}{|1 - \Lambda_0^m \Lambda_1^{n-m}|} = \sum_{k=0}^{\infty} \left(\frac{1}{|\Lambda_0|\Lambda_0^k} + \frac{1}{|\Lambda_1|\Lambda_1^k} \right)^n \,, \tag{33}$$

as is also clear form the explicit form of the transfer matrix (21).

The Perron-Frobenius operator trace formula for the piecewise-linear map (20) follows from (31):

$$\operatorname{tr} \frac{z\mathcal{L}}{1 - z\mathcal{L}} = \frac{\frac{z}{|\Lambda_0 - 1|} + \frac{z}{|\Lambda_1 - 1|}}{1 - z\left(\frac{1}{|\Lambda_0 - 1|} + \frac{1}{|\Lambda_1 - 1|}\right)} \,. \tag{34}$$

A trace formula for flows

As any point along a cycle returns to itself exactly at each cycle period, the eigenvalue corresponding to the eigenvector along the flow necessarily equals unity for all periodic orbits. Hence for flows the trace integral $\operatorname{tr} \mathcal{L}^t$ requires a separate treatment for the longitudinal direction. To evaluate the contribution of a prime cycle p of period T_p, restrict the integration to an infinitesimally thin tube $\mathcal{M}_p$ enveloping the cycle and choose a local coordinate system with a longitudinal coordinate $dx_\parallel$ along the direction of the flow, and d transverse coordinates $x_\perp$

$$\operatorname{tr}_p \mathcal{L}^t = \int_{\mathcal{M}_p} dx_\perp dx_\parallel\, \delta\left(x_\perp - f_\perp^t(x)\right) \delta\left(x_\parallel - f_\parallel^t(x)\right) \,. \tag{35}$$

(Here we again set the observable $\exp(\beta \cdot A^t) = 1$ for the time being). Let $v(x)$ be the magnitude of the velocity at the point x along the flow. $v(x)$ is strictly positive, as otherwise the orbit would stagnate for infinite time at $v(x) = 0$ points, and get nowhere. Therefore we can trade in the longitudinal variable $x_\parallel$ for the flight time τ, with jacobian $dx_\parallel = v\, d\tau$. In terms of the time variable $f_\parallel^t(x)$ contributes for every T_p return to the same point along the trajectory, and the delta function along the flow becomes $\delta(\tau - (t+\tau) \bmod T_p)$, independent of τ. So the integral along the trajectory yields a contribution whenever the time t is a multiple of the cycle period T_p

$$\begin{aligned} \int_{\mathcal{M}_p} dx_\parallel\, \delta\left(x_\parallel - f_\parallel^t(x)\right) &= \sum_{r=1}^{\infty} \delta(t - rT_p) \int_p d\tau \\ &= T_p \sum_{r=1}^{\infty} \delta(t - rT_p) \,. \end{aligned} \tag{36}$$

Linearization of the periodic flow in a plane perpendicular to the orbit yields

$$\int_{\mathcal{M}_p} dx_\perp \delta\Big(x_\perp - f_\perp^{rT_p}(x)\Big) = \frac{1}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} \,, \tag{37}$$

where $\mathbf{J}_p$ is the p-cycle $[d\times d]$ *transverse* Jacobian matrix, and as in (28) we have to assume hyperbolicity, that is that the magnitudes of all transverse eigenvalues are bounded away from unity.

Substituting (36), (37) into (35), we obtain an expression for $\mathrm{tr}\,\mathcal{L}^t$ as a sum over all prime cycles p and their repetitions

$$\mathrm{tr}\,\mathcal{L}^t = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} \delta(t - rT_p) \,. \tag{38}$$

A Laplace transform smoothes the above sum over Dirac delta spikes and yields*

$$\int_{0_+}^{\infty} dt\, e^{-st}\, \mathrm{tr}\,\mathcal{L}^t = \mathrm{tr}\,\frac{1}{s-\mathcal{A}} = \sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} \,.$$

Here $\mathcal{A}$ is the infinitesimal evolution generator, $\mathcal{L}^t = e^{t\mathcal{A}}$, and $(s-\mathcal{A})^{-1}$ is its resolvent. The *classical trace formula* for flows

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} \tag{39}$$

is our second example of the duality between the (local) cycles and (global) eigenvalues†.

If T_p takes integer values, we can replace $e^{-s} \to z$ throughout. We see that the trace formula for maps (32) is a special case of the trace formula for flows. The relation between the continuous and discrete time cases can be summarized as follows:

$$\begin{aligned} T_p &\leftrightarrow n_p \\ e^{-s} &\leftrightarrow z \\ e^{t\mathcal{A}} &\leftrightarrow \mathcal{L}^n \end{aligned} \tag{40}$$

We could now proceed to estimate the location of the leading singularity of $\mathrm{tr}\,\mathcal{L}(s)$ by extrapolating finite cycle length truncations of (39) by methods such as Padé approximants. However, it pays to first perform a simple resummation which converts this divergence into a *zero* of a related function. We shall accomplish this in (49), after we complete our offering of trace formulas.

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*We should caution the reader that in taking the Laplace transform we have ignored a possible $t \to 0_+$ volume term, as we do not know how to regularize the delta function kernel in this limit. In the quantum (or heat kernel) case this limit gives rise to the Weyl mean density of states. A more careful treatment should regularize the divergent sum in (39) and might assign to such volume term some interesting role in the theory of classical resonance spectra.

†Continuous time flow traces weighted by the cycle periods were introduced by Bowen[1] who treated them as Poincaré section suspensions weighted by the "time ceiling" function. They were used by Perry and Pollicott[2]. The derivation presented here was designed[3] to parallel as closely as possible the derivation of the Gutzwiller semiclassical trace formula.

An asymptotic trace formula

In order to illuminate the above manipulations and relate them to something we already possess intuition about, we now derive a heuristic sum from the exact trace formula (32). The Laplace transforms (32) or (39) are designed to capture the time $\to \infty$ asymptotic behavior of the trace sums. By the hyperbolicity assumption (28) for $t = T_p r$ large the cycle weight approaches

$$\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right| \to |\Lambda_p|^r , \tag{41}$$

where Λ_p is the product of the expanding eigenvalues of $\mathbf{J}_p$. Define $\Gamma(z)$ as the approximate trace formula obtained by replacing the cycle weights $\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|$ by $|\Lambda_p|^r$ in (32). Equivalently, think of this as a replacement of the evolution operator (18) by a transfer operator. For concreteness consider a dynamical system whose symbolic dynamics is complete binary, for example the 3-disk system. In this case distinct periodic points that contribute to the nth periodic points sum (30) are labelled by all admissible itineraries composed of sequences of letters $s_i \in \{0, 1\}$:

$$\begin{aligned}
\Gamma(z) &= \sum_{n=1}^{\infty} z^n \Gamma_n = \sum_{n=1}^{\infty} z^n \sum_{x_i \in \mathrm{Fix} f^n} \frac{e^{\beta \cdot A^n(x_i)}}{|\Lambda_i|} \\
&= z \left\{ \frac{e^{\beta\cdot A_0}}{|\Lambda_0|} + \frac{e^{\beta\cdot A_1}}{|\Lambda_1|} \right\} + z^2 \left\{ \frac{e^{2\beta\cdot A_0}}{|\Lambda_0|^2} + \frac{e^{\beta\cdot A_{01}}}{|\Lambda_{01}|} + \frac{e^{\beta\cdot A_{10}}}{|\Lambda_{10}|} + \frac{e^{2\beta\cdot A_1}}{|\Lambda_1|^2} \right\} \\
&\quad + z^3 \left\{ \frac{e^{3\beta\cdot A_0}}{|\Lambda_0|^3} + \frac{e^{\beta\cdot A_{001}}}{|\Lambda_{001}|} + \frac{e^{\beta\cdot A_{010}}}{|\Lambda_{010}|} + \frac{e^{\beta\cdot A_{100}}}{|\Lambda_{100}|} + \dots \right\}
\end{aligned}$$

Both the cycle averages A_i and the stabilities Λ_i are the same for all points $x_i \in p$ in a cycle p. Summing over repeats of all prime cycles we obtain[‡]

$$\Gamma(z) = \sum_p \frac{n_p t_p}{1 - t_p} , \qquad t_p = z^{n_p} e^{\beta \cdot A_p} / |\Lambda_p| . \tag{42}$$

If the weights $e^{\beta \cdot A^n(x)}$ are multiplicative along the flow, and the flow is hyperbolic, for given β the magnitude of each $|\exp(\beta \cdot A^n(x_i))/\Lambda_i|$ term is bounded by some constant M^n, the total number grows as 2^n (or e^{hn} in general), and the sum is convergent for sufficiently small $|z| < 1/2M$. For large n the n-th level sum (29) tends to the leading $\mathcal{L}^n$ eigenvalue e^{ns_0}. Hence s_0 is determined by the smallest $z = e^{-s_0}$ for which (32) diverges,

$$\Gamma(z) \approx \sum_{n=1}^{\infty} \left(z e^{s_0}\right)^n = \frac{z e^{s_0}}{1 - z e^{s_0}} . \tag{43}$$

If one is interested only in the leading eigenvalue of $\mathcal{L}$, it suffices to consider the approximate trace $\Gamma(z)$. We will use this fact below to motivate the introduction of dynamical zeta functions (53).

Spectral determinants

The problem with trace formulas (32), (39) and (42) is that they diverge at $z = e^{-s_0}$, respectively $s = s_0$, that is, precisely where one would like to use them. While

[‡]Note that we could not perform such sum over r in the exact trace formula (32) as $\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right| \neq \left|\det\left(\mathbf{1} - \mathbf{J}_p\right)\right|^r$; the correct way to resum the exact trace formulas is given in (51).

this does not prevent numerical estimation of some "thermodynamic" averages for iterated mappings, in the case of the Gutzwiller trace formula this leads to a perplexing observation that crude estimates of the radius of convergence seem to put the entire physical spectrum out of reach. We shall now partially cure this problem by going from trace formulas to determinants. Determinants tend to have larger analyticity domains because if $\operatorname{tr}\mathcal{L}/(1-z\mathcal{L}) = \frac{d}{dz}\ln\det(1-z\mathcal{L})$ diverges at a particular value of z, then $\det(1-z\mathcal{L})$ might have an isolated zero there, and a zero of a function is easier to determine than its radius of convergence.

We establish the relation between determinants and traces first for maps and then for flows. The inverse of the eigenvalues of a linear operator are given by the zeros of the determinant

$$\det(1-z\mathcal{L}) = \prod_{\alpha}(1-z\lambda_\alpha)\,. \tag{44}$$

For finite matrices this is the characteristic determinant; for operators this is the Hadamard representation of the *spectral determinant*. Consider first the case of maps, for which the evolution operator advances the densities by integer steps in time. In this case we can use the formal matrix identity

$$\ln\det(1-M) = \operatorname{tr}\,\ln(1-M) = -\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{tr}M^n\,, \tag{45}$$

to relate the spectral determinant of an evolution operator for a map to its traces, that is, periodic orbits:

$$\begin{aligned}\det(1-z\mathcal{L}) &= \exp\left(-\sum_{n}^{\infty}\frac{z^n}{n}\operatorname{tr}\mathcal{L}^n\right)\\ &= \exp\left(-\sum_{p}\sum_{r=1}^{\infty}\frac{1}{r}\frac{z^{n_p r}e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1}-\mathbf{J}_p^r\right)\right|}\right)\,.\end{aligned} \tag{46}$$

The inverse relation which recovers the trace formula (32) from the spectral determinant is given by the derivative

$$\operatorname{tr}\frac{z\mathcal{L}}{1-z\mathcal{L}} = -z\frac{d}{dz}\ln\det(1-z\mathcal{L})\,. \tag{47}$$

Spectral determinants of transfer operators

For a piecewise-linear map (20) with a finite Markov partition, an explicit formula for the spectral determinant follows by substituting the trace formula (34) into (46):

$$\det(1-z\mathcal{L}) = \prod_{k=0}^{\infty}\left(1-\frac{t_0}{\Lambda_0^k}-\frac{t_1}{\Lambda_1^k}\right)\,, \tag{48}$$

where $t_s = z/|\Lambda_s|$. The eigenvalues are simply

$$e^{s_k} = \frac{1}{|\Lambda_0|\Lambda_0^k}+\frac{1}{|\Lambda_1|\Lambda_1^k}\,.$$

If the map is a symmetric 2-branch repeller, with $\Lambda=\Lambda_1=-\Lambda_0$, all odd eigenvalues vanish, and the even eigenvalues are given by $e^{s_k}=2/\Lambda^{k+1}$, k even. If the system is bounded and no trajectories escape, the leading eigenvalue of $\mathcal{L}^t$ is exactly 1, that is

$s(0) = 0$. Asymptotically the spectrum is dominated by the lesser of the two fixed point slopes $\Lambda = \Lambda_0$ (if $|\Lambda_0| < |\Lambda_1|$, otherwise $\Lambda = \Lambda_1$), and the eigenvalues e^{s_k} fall off exponentially as $1/\Lambda^k$, just as in the single fixed-point examples§. The exponential spacing of eigenvalues leads to entire spectral determinant. It is this property that will generalize to piecewise smooth flows with finite Markov parititions and single out spectral determinants rather than the trace formulas or dynamical zeta functions as the tool of choice for evaluation of spectra.

Spectral determinant for flows

We write the formula for the spectral determinant for flows by analogy to (46), the determinant for maps

$$\det(s - \mathcal{A}) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{r(\beta \cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|}\right), \tag{49}$$

and then check that the trace formula (39) is the logarithmic derivative of the spectral determinant so defined

$$\mathrm{tr}\, \frac{1}{s - \mathcal{A}} = \frac{d}{ds} \ln \det(s - \mathcal{A}). \tag{50}$$

To recover $\det(s - \mathcal{A})$ integrate both sides $\int_{s_0}^{s}$. With z set to $z = e^{-s}$ in (46), the spectral determinant (49) has the same form for both maps and flows.

We now note that the r sum in (49) is close in form to the expansion of a logarithm. This observation enables us to recast the spectral determinant into an infinite product over periodic orbits as follows:

Let $\mathbf{J}_p$ be the p-cycle $[d \times d]$ transverse Jacobian matrix, with eigenvalues $\Lambda_{p,1}$, $\Lambda_{p,2}$, $\ldots$, $\Lambda_{p,d}$. Expanding $1/(1 - 1/\Lambda_{p,e})$, $1/(1 - \Lambda_{p,c})$ in (27) as geometric series, substituting back into (49), and resumming the logarithms, we find that the spectral determinant is formally given by the infinite product

$$\begin{aligned}
\det(s - \mathcal{A}) &= \prod_{k_1=0}^{\infty} \cdots \prod_{l_c=0}^{\infty} \frac{1}{\zeta_{k_1 \cdots l_c}(s)} \\
1/\zeta_{k_1 \cdots l_c}(s) &= \prod_p \left(1 - \frac{e^{\beta \cdot A_p - sT_p}}{|\Lambda_p|} \frac{\Lambda_{p,e+1}^{l_1} \Lambda_{p,e+2}^{l_2} \cdots \Lambda_{p,d}^{l_c}}{\Lambda_{p,1}^{k_1} \Lambda_{p,2}^{k_2} \cdots \Lambda_{p,e}^{k_e}}\right).
\end{aligned} \tag{51}$$

The observable being averaged contributes through the A_p term, which is the p cycle average of the multiplicative weight $e^{A^t(x)}$. By its definition (1), for maps the weight is a product along the cycle points

$$e^{A_p} = \prod_{j=0}^{n_p-1} e^{a(f^j(x_p))},$$

and for the flows the weight is an exponential of the integral (4) along the cycle

$$e^{A_p} = \exp\left(\int_0^{T_p} d\tau\, a(x(\tau))\right).$$

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§Here are some situations in which everything is mathematically rigorous, and the transfer operator is well defined: expanding Hölder case, weighted subshifts of finite type, expanding differentiable case, see Bowen[1]: expanding holomorphic case, see Ruelle[4]; piecewise monotone maps of the interval, see Hofbauer and Keller[5] and Baladi and Keller[6].

This formula is correct for scalar weighting functions; more general matrix valued weights (such at the stability matrix) would require a time-ordering prescription.

The infinite product formula (51) is in practice not much more than a shorthand notation for the periodic orbit weights contributing to the spectral determinant; more work will be needed to bring such cycle formulas into a tractable form.

Now we are finally poised to deal with the problem posed at the beginning of this chapter; how do we evaluate the averages of (7)? The eigenvalues of the dynamical averaging evolution operator are given by the values of s for which the spectral determinant

$$\det(s-\mathcal{A}(\beta)) \;=\; \exp\left(-\sum_p\sum_{r=1}^{\infty}\frac{1}{r}\frac{e^{(\beta\cdot A_p-sT_p)r}}{|\det\left(\mathbf{1}-\mathbf{J}_p^r\right)|}\right)\,. \tag{52}$$

of the evolution operator (18) vanishes. If we can compute the leading eigenvalue $s_0(\beta)$ and its derivatives, we are done.

Dynamical zeta functions

As we saw in the (42), if one is interested only in the leading eigenvalue of $\mathcal{L}^t$, the size of the p cycle neighborhood can be approximated by $1/|\Lambda_p|^r$, the dominant term in the $rT_p=t\to\infty$ limit, where $\Lambda_p=\prod_e\Lambda_{p,e}$ is the product of the expanding eigenvalues of the Jacobian matrix $\mathbf{J}_p$. With this replacement the spectral determinant (51) is approximated by the *dynamical zeta function*

$$1/\zeta(s)=\exp\left(-\sum_p\sum_{r=1}^{\infty}\frac{1}{r}t_p^r\right)\,. \tag{53}$$

Resumming the logarithms using $\sum_r t_p^r/r=-\ln(1-t_p)$ we obtain the *Euler product representation* of the dynamical zeta function:

$$1/\zeta(s)\;=\;\prod_p(1-t_p)\,,\qquad t_p=\frac{1}{|\Lambda_p|}e^{\beta\cdot A_p-sT_p}\,. \tag{54}$$

The approximate trace formula (42) plays the same role vis-a-vis the dynamical zeta function

$$\Gamma(s)=\frac{d}{ds}\zeta(s)^{-1} \tag{55}$$

as the exact trace formula (39) plays vis-a-vis the spectral determinant (49), see (50) and (47). The heuristically derived dynamical zeta function now reemerges as the $1/\zeta_{0\cdots0}(z)$ part of the *exact* spectral determinant; other factors in the infinite product (51) affect the non-leading eigenvalues of $\mathcal{L}$. The dynamical zeta function is useful because it also vanishes at e^s equal to e^{s_0}, the leading eigenvalue of $\mathcal{L}^t$, defined implicitly as the largest solution of either of the equations

$$\det(s-\mathcal{A})=0\,,\quad 1/\zeta(s)=0\,. \tag{56}$$

Dynamical zeta functions for transfer operators

Ruelle constructed the first dynamical zeta function as a generalization of the topological zeta function

$$\zeta(z)=\exp\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{tr}\,T^n\,,\quad \mathrm{tr}\,T^n=\sum_{x_i\in\mathrm{Fix}f^n}\mathrm{tr}\prod_{j=0}^{n-1}g(f^j(x_i))\,.$$

Here the sum goes over all periodic points x_i of period n, and $g(x)$ is a (matrix valued) weighting function, with weight evaluated multiplicatively along the trajectory of x_i.

By the chain rule the stability of any n-cycle of a 1-d map factorizes as $\Lambda = \prod_{j=1}^{n} f'(x_i)$, so the cycle stability is the simplest example of a multiplicative cycle weight, and indeed - via the Perron-Frobenius evolution operator - the historical motivation for Ruelle's more abstract construction.

We now show that for a piecewise-linear map with a finite Markov partition, the dynamical zeta function is given by a finite polynomial, a straightforward generalization of the topological transition matrix. For a finite $[N \times N]$ dimensional matrix the determinant is given by

$$\prod_p (1 - z^{n_p} t_p) = \sum_{n=1}^{N} z^n c_n \,,$$

where c_n = sum over all non-self-intersecting closed paths of length n together with products of all non-intersecting closed paths of total length n. We illustrate this by the piecewise linear repeller (20). Due to the piecewise linearity, the stability of any n-cycle factorizes as $\Lambda_{s_1 s_2 \ldots s_n} = \Lambda_0^m \Lambda_1^{n-m}$, so the sum (42) takes a particularly simple form

$$\Gamma_n = \left(\frac{1}{|\Lambda_0|} + \frac{1}{|\Lambda_1|} \right)^n \,.$$

The dynamical zeta function (53) evaluated from the trace via (55) is indeed the $1/\zeta_{00\ldots0}$ part of the corresponding spectral determinant (48)

$$1/\zeta(z) = 1 - z/|\Lambda_0| - z/|\Lambda_1| \,. \tag{57}$$

More generally, piecewise-linear approximations to dynamical systems yield polynomial or rational polynomial cycle expansions, provided that the symbolic dynamics is a subshift of finite type.

We see that the dreaded exponential proliferation of cycles is a bogus anxiety; all that really matters in this example are the stabilities of the two fixed points. Clearly the information carried by individual cycles is highly redundant.

Topological zeta function for flows

The leading zero of a topological zeta function describes the growth rate of the number of topologically distinct trajectories with time. We can apply the method we used in deriving (39) to the problem of deriving the topological zeta functions for flows. We construct the time-weighted density of prime cycles of period t by analogy to (38):

$$\Gamma(t) = \sum_p \sum_{r=1} T_p \delta(t - rT_p) \tag{58}$$

A Laplace transform smoothes the sum over Dirac delta spikes and yields the *topological trace formula* and *topological zeta function* for the continuous time flows:

$$\sum_p \sum_{r=1} T_p \int_{0_+}^{\infty} dt\, e^{-st}\, \delta(t - rT_p) = -\frac{\partial}{\partial s} \ln 1/\zeta_{\mathrm{top}}(s)$$

$$1/\zeta_{\mathrm{top}}(s) = \prod_p \left(1 - e^{-sT_p}\right) \,. \tag{59}$$

Zeta functions, historical remarks

For a function to be deserving of the appellation “zeta function”, one expects it to have an Euler product type representation, and perhaps also satisfy a functional equation. Various kinds of zeta functions are reviewed by A. Voros in ref. 7. Historical antecedents of the dynamical zeta function are the fixed-point counting functions introduced by Weil[8], Lefschetz[9], Artin and Mazur[10], and the determinants of transfer operators of statistical mechanics[11]. Ref. 12 contains a concise survey of the many zeta functions of mathematics.

In his review article Smale[13] already intuited, by analogy to the Selberg Zeta function, that the spectral determinant is the right generalization for continuous time flows. Ruelle was motivated by the Artin-Mazur zeta function and the statistical mechanics analogy, so he did not consider the spectral determinant a more natural object. In dynamical systems theory dynamical zeta functions arise naturally only for piecewise linear mappings; for smooth flows the natural object for study of classical and quantal spectra are the spectral determinants introduced above.

The nomenclature has not settled down yet; what we call evolution operators here is called transfer operators[14], Perron-Frobenius operators[15] and/or Ruelle-Araki operators elsewhere. We follow Ruelle in our usage of the term “dynamical zeta function”, but elsewhere in the literature function (54) is often called the Ruelle zeta function. Ruelle[16] points out the corresponding transfer operator T was never considered by either Perron or Frobenius; a more appropriate designation would be the Ruelle-Araki operator. Here we have consistently refered to kernels such as (18) as evolution operators. Determinants similar to or identical with our spectral determinants are sometimes called Selberg Zetas, Selberg-Smale zetas[17], functional determinants, Fredholm determinants, or even - to maximize confusion - dynamical zeta functions[18]. A Fredholm determinant is a notion that applies to trace class operators only - as we consider here a somewhat wider class of determinants, we prefer to refer to them losely as “spectral determinants”.

Resumé

The description of a chaotic dynamical system in terms of cycles can be visualized as a tessellation of the dynamical system, with a smooth flow approximated by its *periodic orbit skeleton*, each region $\mathcal{M}_i$ centered on a periodic point x_i of the topological length n, and the size of the region determined by the linearization of the flow around the periodic point. The integral over such topologically partitioned phase space yields a trace formula, and via a Laplace transform of this trace formula the spectral problem is recast into a problem of determining zeros of a spectral determinant.

Spectral determinants and dynamical zeta functions arise in classical and quantum mechanics because in both the dynamical evolution can be described by the action of linear evolution operators on infinite-dimensional vector spaces. The critical step in the derivation of spectral determinants and dynamical zeta functions is the hyperbolicity assumption, that is the assumption that all cycle stability eigenvalues are bounded away from unity, $|\Lambda_{p,i}| \neq 1$. By dropping the prefactors in this limit we have given up on any possibility of recovering the precise distribution of starting x (which should anyhow be impossible due to the exponential growth of errors), but in exchange we gain an effective description of the asymptotic behavior of the system. The pleasant surprise is that the infinite time behavior of an unstable system turns out to be as easy to determine as its short time behavior. To learn how this is accomplished the reader should turn to more comprehensive expositions of the subject, refs. 17, 19 in particular.

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