For elastic solids:

$\vec{f} + \vec{\nabla}\sigma = 0$, $\sigma \cdot \hat{n} = \sigma_{ext} \cdot \hat{n}$

$\sigma \neq -p\,\mathbb{I}$ instead $\sigma = 2\mu\varepsilon + \lambda \,\mathrm{Tr}\,\varepsilon \cdot \mathbb{I}$

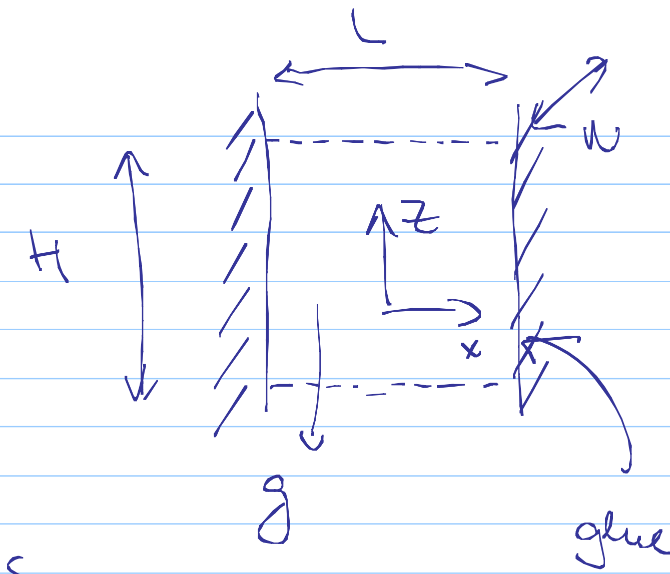
stress strain

$\varepsilon = \frac{1}{2}\left(\vec{\nabla}\vec{u} + (\vec{\nabla}u)^T\right)$

deformation tensor

Additional B.C. $\vec{u}$ at the boundary

Example:



B.C.'s: $\vec{u}(x=\frac{L}{2}) = \vec{u}(x=-\frac{L}{2}) = 0$

$dF = \sigma \cdot n \, dS \Rightarrow dF_z = \hat{z} \cdot \sigma \hat{n} \, dS$

at $x = \frac{L}{2}$, $\hat{n} = +\hat{x} \Rightarrow dF_z = \hat{z} \cdot \sigma \cdot \hat{x} \cdot dS$

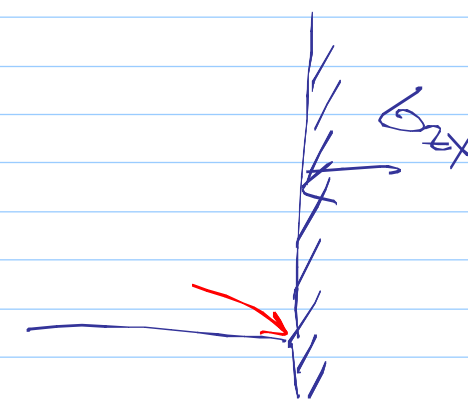
$\Rightarrow \hat{z} \cdot \sigma \cdot \hat{x} = \sigma_{zx} = \frac{dF_z}{dS} = \tau = \sigma_{xz}$

at $x = -\frac{L}{2}$, $\hat{n} = -\hat{x} \Rightarrow \hat{z} \cdot \sigma \cdot (-\hat{x}) = -\sigma_{xz} = -\sigma_{zx} = \frac{dF_z}{dS} = \tau$

$\Rightarrow \sigma_{xz} = \sigma_{zx} = -\tau$

$\vec{f} + \vec{\nabla} \sigma = 0$,

$\vec{f} = (0, 0, -\rho g)$



$$\hat{x}: \quad 0 + \partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz} = 0$$

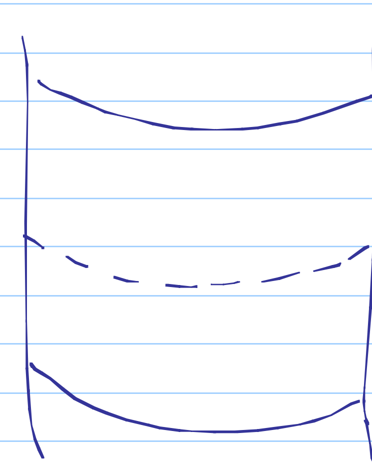
$$\cancel{\hat{y}: \quad 0 + \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} = 0}$$

$$\hat{z}: \quad -\rho g + \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} = 0$$

$$\sigma = 2\mu\varepsilon + \lambda \,\mathrm{Tr}\,\varepsilon \cdot \mathbb{1}, \quad \varepsilon = \frac{1}{2}\left(\vec{\nabla}\vec{u} + (\vec{\nabla}\vec{u})^T\right)$$

A trial solution: $u_x = u_y = 0, \; u_z = v(x)$

$$\vec{u} = (0, 0, v(x))$$



$$\vec{\nabla}\vec{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v' & 0 & 0 \end{pmatrix}, \quad (\vec{\nabla}\vec{u})^T = \begin{pmatrix} 0 & 0 & v' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \varepsilon = \frac{1}{2} v' \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathrm{Tr}\,\varepsilon = 0$$

$$\rightarrow \sigma = 2\mu\varepsilon = \mu v' \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \sigma_{xz} = \sigma_{zx} = \mu v'$$

$\partial_z \sigma_{xz} = 0$ $\sigma_{xz} = \mu v'(x)$ $\rightarrow$ OK to choose $u_z = v(x)$

$-\rho g + \mu v'' = 0 \Rightarrow v'' = \frac{\rho g}{\mu} = \text{const} \rightarrow v(x) = \frac{1}{2}\frac{\rho g}{\mu}x^2 + \cancel{bx} + c$

$\vec{u} = 0$ at $x = \pm\frac{L}{2} \Rightarrow v(x) = \frac{\rho g}{2\mu}\left(x^2 - \frac{L^2}{4}\right)$

$\vec{u} = \left(0, 0, \frac{\rho g}{2\mu}\left(x^2 - \frac{L^2}{4}\right)\right)$

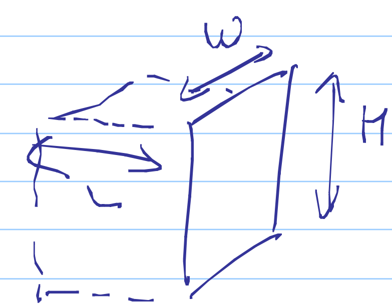
$\sigma_{zx} = \mu v'(x) = \rho g\, x$ at $x = \frac{L}{2} \Rightarrow \sigma_{zx} = \rho g \frac{L}{2} = \tau$

gravity: shear stress $x = -\frac{L}{2} \Rightarrow \sigma_{zx} = -\rho g \frac{L}{2} = -\tau$

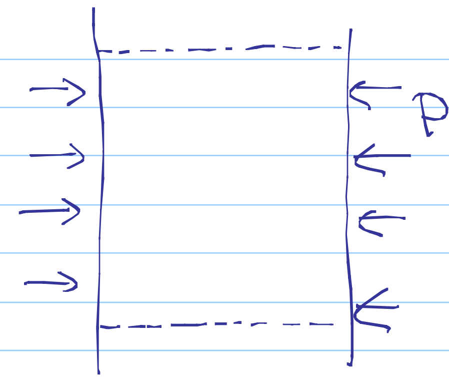
$\rho g \cdot \cancel{WH} \cdot L = 2\tau \cdot \cancel{WH}$

$\Updownarrow$

$\tau = \frac{\rho g}{2} L$



Example:



$$\vec{u} = (\overset{\neq 0}{u_x}, \overset{= 0}{u_y}, \overset{\neq 0}{u_z}) = (v(x), 0, \upsilon(x))$$

$$u_x = v(x) \quad , \quad u_z = \upsilon(x)$$

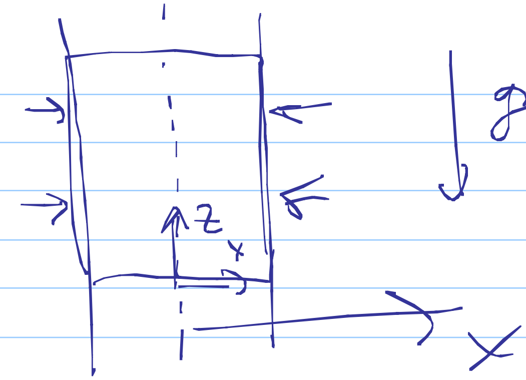
$$\vec{\nabla}\vec{u} = \begin{pmatrix} v' & 0 & 0 \\ 0 & 0 & 0 \\ \upsilon' & 0 & 0 \end{pmatrix} \quad , \quad (\vec{\nabla}\vec{u})^T = \begin{pmatrix} v' & 0 & \upsilon' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \varepsilon = \frac{1}{2}\left((\vec{\nabla}\vec{u}) + (\vec{\nabla}\vec{u})^T\right) = \frac{1}{2}\begin{pmatrix} 2v' & 0 & \upsilon' \\ 0 & 0 & 0 \\ \upsilon' & 0 & 0 \end{pmatrix} \quad , \quad \mathrm{Tr}\,\varepsilon = v'$$

$$\Rightarrow \sigma = 2\mu\varepsilon + \lambda\,\mathrm{Tr}\,\varepsilon \cdot \mathbb{I} = \begin{pmatrix} (2\mu+\lambda)v' & 0 & \mu\upsilon' \\ 0 & \lambda v' & 0 \\ \mu\upsilon' & 0 & \lambda v' \end{pmatrix}$$

$$\underline{\vec{f} + \vec{\nabla}\sigma = 0}$$

B.C.:



$$\bar{u} = (u_x, u_y, u_z) = (V, 0, 0), \quad x = \pm\frac{L}{2}$$

$$\sigma \cdot \hat{n} = p\,\hat{n} + \tau\,\hat{z}, \quad x = \pm\frac{L}{2}$$

$$\sigma \cdot \hat{n} = 0, \quad \text{top, bottom}$$

$$\hat{x}: \; 0 + \partial_x \sigma_{xx} + \cancel{\partial_y \sigma_{xy}} + \cancel{\partial_z \sigma_{xz}} = (2\mu + \lambda) V'' = 0$$

$$\cancel{\hat{y}: \; 0 + \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} = 0}$$

$$\hat{z}: \; -\rho g + \partial_x \sigma_{zx} + \cancel{\partial_y \sigma_{zy}} + \cancel{\partial_z \sigma_{zz}} = -\rho g + \mu v'' = 0$$

$$\Rightarrow \quad V = ax + \cancel{b}$$

$$v = \frac{\rho g}{2\mu} x^2 + \cancel{c}x + d$$

$$\bar{u} = (V, 0, 0), \quad x = \pm\frac{L}{2} \quad (\text{small deformation in } x)$$

$$\Rightarrow \quad v = \frac{\rho g}{2\mu}\left(x^2 - L^2/4\right)$$

$$V = ax = a\frac{L}{2}, \quad x = \frac{L}{2}$$

$$\hat{n}\cdot\sigma\cdot\hat{n} = -p, \text{ at } x=\frac{L}{2} \rightarrow \sigma_{xx} = -p = (2\mu+\lambda)v' = (2\mu+\lambda)a$$

$$\Rightarrow \quad a = -\frac{p}{(2\mu+\lambda)} \quad \Rightarrow \quad v(x) = -\frac{px}{2\mu+\lambda}$$

$$\vec{u} = \left(-\frac{px}{2\mu+\lambda}, 0, \frac{\rho g}{2\mu}\left(x^2 - \frac{L^2}{4}\right)\right)$$

$$\tau = \hat{z}\cdot\sigma\cdot\hat{n} = \sigma_{zx} \text{ at } x=\frac{L}{2}$$

$$\tau = \mu v' = \frac{\rho g}{\mu}\cdot\frac{L}{2}$$

at the sides ✓

~~At~~ the top / bottom:

$$\sigma\cdot\hat{n} = 0 \iff \sigma_{zz} \overset{?}{=} 0, \quad \sigma_{xz} \overset{?}{=} 0$$

$$\sigma_{zz} = \lambda v' = \lambda\left(-\frac{p}{2\mu+\lambda}\right) \overset{?}{=} 0$$

$$\sigma_{xz} = \mu v' = \mu\left(\frac{\rho g}{\mu}x\right) \overset{?}{=} 0 \qquad \hat{n} = ? \qquad \hat{x}\cdot\sigma\cdot\hat{n} \overset{?}{=} 0, \quad \hat{z}\cdot\sigma\cdot\hat{n} \overset{?}{=} 0$$