Phys. 4421 Final Exam

- This is an open book exam. Feel free to use any reference literature.
- The exam is to be completed individually, absolutely no discussions allowed.
- Use can use any software (Maple, Mathematica, etc.) if you would like.
- The exam is due Thursday, December 11, at 3pm in my office. Slide you work under the door if I am not in.
- If you have any questions, let instructor know asap.

Problem 1

We have discovered in class that small axisymmetric perturbations with wave length $k > a^{-1}$ of a liquid cylinder with average radius a decrease the total surface area and hence are further amplified as a result of capillary forces. What we have not computed is for which wave length the perturbations grow the fastest. That's what we will try do here.

Consider a small axisymetric perturbation (of amplitude $\epsilon \ll a$, $\epsilon \ll k^{-1}$) of the surface of a liquid cylinder in vacuum (or gas of negligible density and viscosity). Consider the liquid to be incompressible, inviscid, with density ρ, surface tension α, and ignore gravity. The motion of the liquid resulting from the surface perturbation will either lead to an oscillating surface wave with constant amplitude or will produce a growing deformation of the cylinder, which will eventually lead to its breakup into separate droplets.

(a) Pick an appropriate coordinate system, write down the equations of motion for the liquid and the appropriate boundary conditions at the cylinder surface and the cylinder axis. The latter follow from symmetry. *Hint: Only keep leading order terms in ϵ in all of your calculations.*

(b) Find the general (axisymmetric) solution for the velocity and pressure fields, using separation of variables. *Hint: You should discover that your solution is expressed in terms of Bessel functions I_0 and I_1.*

(c) Find the perturbation in the free surface that corresponds to your velocity field using some of the boundary conditions.

(d) Using the remaining boundary conditions, derive the dispersion relation for the surface wave frequency $\omega(k)$. Confirm that $\omega(k)$ is real for $k > a^{-1}$ – this is the range of k for which the liquid cylinder is stable and does not break up into droplets.

(e) When $\omega(k)$ is complex, the surface perturbation is exponentially amplified, indicating an instability. Plot using your favorite software (or sketch) the growth rate $\gamma(k) = Im(\omega(k))$ of the instability for $0 < k < a^{-1}$. What is the growth rate at $k = 0$? At $k = a^{-1}$? At what k is the perturbation amplified the quickest? What is the corresponding droplet size (assuming all droplets are the same size)?

(Over for other problems)

Problem 2

Incompressible fluid with viscosity μ and density ρ occupies the region between to concentric cylinders of length L and radii a and $b > a$ (this setup is called the Taylor-Couette flow), with the inner cylinder rotating with angular velocity Ω and the outer cylinder stationary. This geometry suggests the use of cylindrical polar coordinates (r, ϕ, z). Assuming the flow is stationary, the symmetry suggests that the simplest (laminar) solution for the velocity has only one component (angular component, v_ϕ), which only depends on the distance r from the axis, i.e., $\mathbf{v} = \hat{\phi} v_\phi(r)$. Similarly, the pressure only depends on r, $p = p(r)$.

(a) Show that the ϕ component of the Navier-Stokes equation reduces to

$$\partial_r^2 v_\phi + \frac{1}{r}\partial_r v_\phi - \frac{1}{r^2} v_\phi = 0$$

and use it to find the velocity field subject to appropriate boundary conditions at the cylinder walls $r = a$ and $r = b$. *Hint: Use our assumptions about the velocity and pressure field to simplify your derivation, but don't forget that, for instance,* $\partial_\phi \hat{\phi} \neq 0$*!*

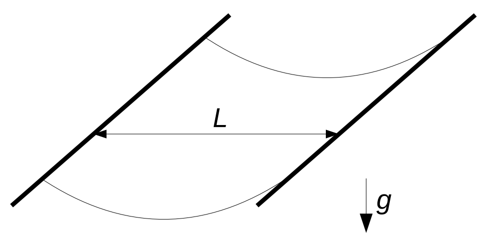
(b) Derive the r component of the Navier-Stokes equation and use it to express pressure as a function of the velocity. Leave the answer in quadratures, do not evaluate the integral.

(c) Compute all nonzero components of the stress tensor.

(d) Compute the torque of the fluid on the inner and outer cylinder and confirm that it takes equal and opposite values.

Problem 3

A thin elastic sheet with density ρ and elastic moduli μ and λ is suspended on two slender horizontal rails in vertical gravitational field $\mathbf{g} = -g\hat{z}$. The rails are parallel, with separation L.



(a) Assuming the displacement of the sheet is strictly vertical, write down the equation(s) governing the sheet deformation and solve them to compute the deflection $z(x, y)$ due to gravity.

(b) Compute all nonzero components of the stress tensor σ that corresponds to this deformation.

(c) Now, neglecting gravity, write down the equation(s) describing elastic vibrations of the sheet, again assuming the displacement of the sheet is strictly vertical. Solve the equation(s) and find the lowest vibrational frequency of the sheet.