Problem 1

(a) Consider the problem of finding the potential in the upper-half plane, if the potential along the $x$-axis is $\phi(x, 0) = V_0$, $|x| < a$, and $\phi(x, 0) = 0$, $|x| > a$. Show that this is the same problem as finding the potential due to two line charges, if we exchange the roles of potential and stream function. The solution is

$$\phi = \frac{V_0}{\pi} \left( \arctan \frac{x+a}{y} - \arctan \frac{x-a}{y} \right).$$

(b) With the aid of the solution to part a, and the conformal map

$$w = \cosh \frac{\pi z}{d} \quad \text{or} \quad w = \cosh \frac{\pi z}{2d}$$

find the temperature distribution in the semi-infinite metal plate whose cross-section is shown below, if the three faces are maintained at the indicated constant temperatures. Note that $\nabla^2 T = 0$ inside the metal.

Problem 2

Find the fringing field of a semi-infinite parallel plane capacitor using a conformal map

$$w = f(z) = \frac{a}{2\pi} (1 + z + e^z),$$

where $z = x + iy$ and $w = u + iv$.

(a) Show that $f$ maps the plates of the infinite plate capacitor in the $z$-plane into the plates of the semi-infinite-plane capacitor in the $w$-plane.

(b) What is the complex potential $g(z) = \phi(x, y) + i\psi(x, y)$ of the infinite-plane capacitor?

(c) The complex potential of the semi-infinite-plane capacitor is

$$h(w) = \phi'(u, v) + i\psi'(u, v) = g(f^{-1}(w))$$

where, unfortunately, we cannot compute the inverse $z = f^{-1}(w)$ exactly. Show that the space between the plates of the semi-infinite capacitor (away from the edge $u = 0$) corresponds to the limit $x \to -\infty$, find $f^{-1}(w)$ approximately, and thus determine the potential $\phi'$ and the electric field $E'$ inside the semi-infinite-plane capacitor.
(d) Show that the space outside the conductor (away from the edge $u = 0$) corresponds to the limit $x \to \infty$, find $f^{-1}(w)$ approximately, and thus determine the potential $\phi'$ and the electric field $E'$ outside the semi-infinite-plane capacitor.

(e) Using your results from parts (c) and (d) sketch the equipotentials and the lines of electric field in the $w$-plane. For the adventurous among you, these can be computed exactly in parametric form. We can easily find that

$$u = \frac{a}{2\pi} (1 + x + e^x \cos y), \quad v = \frac{a}{2\pi} (y + e^x \sin y)$$

To find equipotentials in the $w$ plane, fix $y$ and vary $x$ (this corresponds to equipotentials in the $z$ plane); to find the field lines in the $w$ plane, fix $x$ and vary $y$ (this corresponds to field lines in the $z$ plane).