Problem 1

Several types of orthogonal polynomials frequently occur in various physics problems. For instance, Hermite polynomials $H_n(x)$ arise in the quantum harmonic oscillator problem. We will try to find them here via an eigenvalue problem for the following differential operator:

$$\hat{L} = \frac{d^2}{dx^2} - 2x \frac{d}{dx}.$$ 

This operator is defined on a space of functions (in our case, polynomials) such that $\hat{L}\Psi(x) = \Psi''(x) - 2x\Psi'(x)$ for any function $\Psi(x)$. The Hermite polynomials are known to be the eigenvectors (or, more accurately, eigenfunctions) of this operator. We want to find a few of these eigenfunctions and the corresponding eigenvalues by calculating the matrix elements of this operator and then diagonalizing it.

a) For the formalism to work we have to ensure that the space of polynomials is a vector space and the operator is linear. Please prove that this is indeed the case.

This vector space is infinite-dimensional, so the matrix elements of $\hat{L}$ will also be infinite-dimensional. We don’t know how to diagonalize infinite-dimensional matrices. To circumvent this problem let us restrict our attention to a subspace of low order polynomials (up to order three). As our first step, let’s introduce a basis:

$$e_1 = 1, \quad e_2 = x, \quad e_3 = x^2, \quad e_4 = x^3.$$ 

b) Calculate the matrix elements $L_{mn}$ of $\hat{L}$ in this basis. The easiest way to do this is by simply expanding $\hat{L}e_n$ in the basis $\{e_k\}$, i.e., $\hat{L}e_n = \sum_k L_{kn}e_k$.

c) Find the eigenvalues and eigenvectors of $L_{mn}$ and thus determine the coefficients of the first four Hermite polynomials.

If you are feeling adventurous, you can find quite a few more higher order polynomials this way, but the procedure quickly becomes too cumbersome due to the necessity of evaluating the determinant and solving algebraic equations of high order. The next problem explores an alternative route for generating sets of orthogonal polynomials.

Problem 2

Orthogonal polynomials can be very conveniently generated by using the Gram-Schmidt orthogonalization procedure. For instance, Laguerre polynomials $L_n(x)$ arise in solutions of Schrödinger equation for the Hydrogen atom. The polynomials with different indices are orthogonal with respect to the scalar product defined in the following way:

$$\langle \Phi | \Psi \rangle = \int_0^\infty e^{-x} \Phi(x)\Psi(x)dx,$$

where $w(x) = e^{-x}$ is the weight function. Start with a set of non-orthogonal polynomials

$$e_1 = 1, \quad e_2 = a + x, \quad e_3 = b + cx + x^2, \quad e_4 = d + fx + gx^2 + x^3,$$

where $a, b, c, d, f, g$ are constants, and orthogonalize (with respect to the scalar product defined above) them sequentially beginning with $e_2, e_3$, and so on to generate the first four Laguerre polynomials.