Problem 1

(a) Compute the covariant basis vectors $\mathbf{q}_i$, contravariant basis vectors $\mathbf{q}^i$, and normalized basis vectors $\hat{i}$ corresponding to the (primed) spherical polar coordinates with polar angle $\theta$ and azimuthal angle $\phi$, i.e., write down their components in (unprimed) Cartesian coordinates.

(b) Compute the metric $g'_{ij}$ and the Lamé coefficients $h'_{i}$ (in spherical polar coordinates).

(c) Compute the components $u'_i$ of the vector $\mathbf{u} = \mathbf{z} \times \mathbf{r}$ in all three bases ($\mathbf{q}_i$, $\mathbf{q}^i$, and $\hat{i}$) associated with spherical polar coordinates.

(d) For a circular ring of radius $R$ centered around the origin and lying in the plane normal to the $z$-axis, compute the matrix elements of the electric quadrupole tensor $Q = \int_V [3\mathbf{rr} - \mathbf{r}^2 \mathbf{1}] \rho(\mathbf{r}) dV$ (1)
in the covariant basis $\mathbf{q}_r$, $\mathbf{q}_\phi$, $\mathbf{q}_\theta$, assuming the total charge $q$ is distributed uniformly around the ring. You can either compute $Q^i_j$ in Cartesian coordinates first and then use coordinate transformation to obtain matrix elements $Q'^i_j$ in spherical polar coordinates or you can compute the matrix elements $Q'^i_j$ directly from the definition (1).

(e) Compute the matrix elements $Q'_{ij}$ of $Q$ in the normalized basis $\hat{r}$, $\hat{\phi}$, $\hat{\theta}$ using any method you want.

Problem 2

Beer bubbles rise due to buoyancy (the vapor inside the bubbles is much lighter than the beer itself). Let’s compute the force on a bubble directly, instead of using the Archimedes’ law. The force on the bubble has two components: the body force

$$\mathbf{F}_b = \int_V \mathbf{f}_b dV,$$

where $\mathbf{f}_b = -\rho_{\text{vapor}} g \mathbf{z}$ is the body force density (gravity acting on vapor inside) and the surface force due to the pressure $p$ in the liquid. The surface force is

$$\mathbf{F}_s = \int_S \sigma dS$$

where $dS = \hat{n} dS$, $\hat{n}$ is the surface normal and $\sigma$ is the (symmetric) stress tensor in the liquid. For a stationary fluid (beer), the stress tensor itself can be computed using local force equilibrium for the liquid

$$\mathbf{f}_b + \nabla \sigma = 0,$$

where $\mathbf{f}_b = -\rho_{\text{beer}} g \mathbf{z}$ and $\nabla \sigma = \hat{e}_j \partial_i \sigma_{ij}$ in any orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$. Since $\rho_{\text{vapor}} \ll \rho_{\text{beer}}$, the total force $\mathbf{F}_b + \mathbf{F}_s \approx \mathbf{F}_s$.

a) In Cartesian coordinates, it can be shown that $\sigma_{ij} = q(x,y,z) \delta_{ij}$. Compute the function $q$ using (4) and give it a physical interpretation.

b) Compute the matrix elements $\sigma_{ij}$ of the stress tensor in the normalized basis $\hat{r}$, $\hat{\phi}$, $\hat{\theta}$ by using a coordinate transformation from Cartesian to spherical coordinates.

c) Use (3) and the result of part (b) to compute the force of beer on spherical bubble of radius $a$. Does your answer agree with the Archimedes’ force?