solution 1

Helmholtz eqn: \( \nabla^2 \psi + k^2 \psi = 0 \)

in spherical coordinates:

\[
\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}
\]

let \( \psi = R(r) \Theta(\theta) \Phi(\phi) \)

Dividing by \( \psi \)

\[
\frac{1}{Rr^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) + \frac{1}{\Theta r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) + \frac{1}{\Phi r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = \frac{2}{r^2} + \frac{k^2}{\Phi r^2 \sin^2 \theta} = 0
\]

multiply by \( r^2 \)

\[
\frac{2}{r^2} \left( r^2 \frac{\partial^2 R}{\partial r^2} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) + \frac{1}{\Phi \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0
\]

independent of \( \theta, \phi \)

\( k = \text{constant} \)

\[
\frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) + \frac{1}{\Phi \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = k
\]

multiply by \( \sin^2 \theta \)

\[
\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) - k \sin^2 \theta + \frac{1}{\Phi \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0
\]

independent of \( \phi \)

\( k = \text{constant} \)

\[
\frac{1}{\Phi \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = -m^2
\]

\[
\frac{1}{\sin^2 \theta} \frac{\partial^2 \Theta}{\partial \theta^2} + \frac{m^2}{\sin^2 \theta} \Theta + k \Theta = 0
\]
Problem 2

Separating the variables in the steady state solution $\Theta(x, y) = X(x)Y(y)$ to the heat equation $\frac{\partial^2 x}{\partial x} \Theta + \frac{\partial^2 y}{\partial y} \Theta = \alpha^{-1} \partial_t \Theta = 0$ we obtain two equations

$$\frac{X''}{X} = -\frac{Y''}{Y} = k^2,$$

with $k$-to be determined. We know that both of those equations admit solutions in the form of sines/cosines and exponentials. Since this a semi-infinite domain (in $x$), solutions to $X(x)$ can be a decaying (in $x$) exponentials.

However, $Y(y)$ cannot be expressed in terms of exponentials due to the specific boundary conditions (homogeneous Diriclet at both ends). The only type of solutions for $Y(y)$ that satisfies the boundary conditions $Y(0) = Y(1) = 0$ is $Y(y) \propto \sin(ky)$ with $k$ taking values in a discrete set $k_n = \pi n$ with $n$-any positive integer. After normalization we find $Y_n(y) = \sqrt{2} \sin(k_n y)$.

Since $k_n^2 > 0$, the $x$-dependence is given by $X_n(x) \propto e^{-k_n x}$, such that the general solution for the temperature is

$$\Theta(x, y) = \sum_n A_n e^{-k_n x} \sin(k_n y).$$

The only boundary conditions that has not been satisfied yet is that at $x = 0$, which gives

$$\sum_n A_n \sin(k_n y) = 1.$$ 

Multiplying this by $\sin(k_m y)$ and integrating from 0 to 1 we find $A_n = 4/k_n$ for $n$-odd and zero otherwise, so that

$$\Theta(x, y) = \sum_{n \text{ odd}} \frac{4}{k_n} e^{-k_n x} \sin(k_n y).$$