Problem 1

(a) The Lagrangian describing the pendulum is $\mathcal{L} = T - V$, where the kinetic energy is

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right)$$

and the potential energy is

$$V = mg \cos \theta.$$  

(b) The Euler-Lagrange equations are:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = \dot{L}_z = 0,$$

where

$$L_z = ml^2 \sin^2 \theta \dot{\phi}$$

is the angular momentum around the $z$-axis and

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = ml^2 \ddot{\theta} + mg \sin \theta = 0.$$  

(c) The Lagrangian $L$ is independent of the angle $\phi$ and time $t$, so there are two conservation laws. The first represents the conservation of

$$L_z = \text{const},$$

and the second – the conservation of energy

$$E = \text{const},$$

where

$$E = \phi \frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \theta \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} = ml^2 \sin^2 \theta \dot{\phi}^2 + x ml^2 \ddot{\theta}^2 - \frac{1}{2} ml^2 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + mg \cos \theta = T + V.$$  

(d) In the special case $\phi = \phi_0 = \text{const}$, we have

$$\frac{1}{2} ml^2 \ddot{\theta}^2 + mg \cos \theta = E$$

such that

$$\frac{d\theta}{dt} = \left[ \frac{2(E - mg \cos \theta)}{ml^2} \right]^{1/2}$$

and

$$t = \int_{\theta_0}^{\theta} \left[ \frac{2(E - mg \cos \theta)}{ml^2} \right]^{-1/2} d\theta.$$  

This is the solution for a finite-amplitude oscillation of a planar pendulum.

(e) In the special case $\theta = \theta_0 = \text{const}$, we have

$$\dot{\phi} = \omega \equiv \frac{L_z}{ml^2 \sin^2 \theta_0}$$

such that

$$\phi = \phi_0 + \omega t.$$
Problem 2

The constrained minimization problem can be written in the form

\[ S' = \int L(x, v, t) dt + \int w \cdot (v - \dot{x}) dt = \int L'(x, \dot{x}, v, t) dt, \]

where \( L'(x, \dot{x}, v, t) = L(x, v, t) + \mu (v - \dot{x}) \) and \( w(t) \) is the (vector) Lagrange multiplier. The corresponding Euler-Lagrange equations are

\[
\frac{d}{dt} \frac{\partial L'}{\partial \dot{x}} - \frac{\partial L'}{\partial x} = -\dot{w} - \frac{\partial L}{\partial x} = 0
\]

and

\[
\frac{d}{dt} \frac{\partial L'}{\partial \dot{v}} - \frac{\partial L'}{\partial v} = -\frac{\partial L}{\partial v} - w = 0.
\]

Taking the time derivative of the second equation and substituting into the first one, we recover the standard Euler-Lagrange equation

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{v}} - \frac{\partial L}{\partial v} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.
\]