Problem 1 (van der Pol oscillator)
Find approximate solution \(x(\epsilon, t)\) to the nonlinear ODE
\[
\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{1}
\]
in the limit of small \(\epsilon\). Since the unperturbed version of this equation is simply a harmonic oscillator \(\ddot{x} + x = 0\), we should use the method of multiple scales to find the solution: \(x(\epsilon, t) = x_0(t) + \epsilon x_1(t) + \cdots\).

1. Derive the differential equations satisfied by \(x_0(t)\) and \(x_1(t)\).
2. Identify the resonant terms.
3. Compute the leading term \(x_0(t)\) which satisfies the initial conditions \(x(0) = 1\) and \(\dot{x}(0) = 0\).

Problem 2 (Confined BEC)
Steady states of confined 1D Bose-Einstein condensate are described by the Gross-Pitaevsky equation
\[
E\Psi = -\frac{\hbar^2}{2m}\Psi'' + U(r)\Psi + g|\Psi|^2\Psi, \tag{2}
\]
where \(g\) is a real constant describing the interaction strength (and sign). Let confining potential have depth \(U_0 > 0\) and width \(a\), so that \(E > U(x)\) for \(-a/2 < x < a/2\). Find the solution for highly excited energy states \((E \approx U_0)\) using the WKB method.

1. Rewrite the equation in nondimensional form by introducing a small parameter \(\epsilon\) and the following nondimensional functions and constants: \(Q^2(y) = (E - U(x))/U_0\) and \(G = g/U_0\), where \(y = x/a\).
2. Write the solution for \(\Psi(y)\) in the form
\[
\Psi(y) = e^{i\delta s_0(y)+s_1(y)} \tag{3}
\]
and find the scaling of the leading order term, \(\delta(\epsilon)\), where both unknown functions \(s_0(y)\) and \(s_1(y)\) are real.
3. Obtain and solve the differential equations for \(s_0(y)\) and \(s_1(y)\).
4. As you will find out, the solution for \(s_0(y)\) depends on \(s_1(y)\) and vice versa. Derive the equation for the particle density \(\rho(y) = |\Psi(y)|^2 = e^{2s_1(y)}\).
5. This equation has an analytical solution that is rather messy, so let us compute the approximate solution for a weakly interacting BEC, where \(G \ll 1\). Compute the leading term and one correction for \(\rho(y)\) and write down the corresponding solution for \(\Psi(y)\).