Problem 1:

1) Interaction of an electric charge and an electromagnetic wave suggests that we use $\varepsilon_0$ and $c$ along with $m$, $q$, and $\lambda$. Along with $\sigma_T$ there are 6 parameters and 4 basic dimensions ($L$, $T$, $M$, $A$), so according to the Pi-theorem we should have 2 nondimensional combinations. A standard (by now) calculation gives

$$\Pi_1 = \frac{\sigma_T}{\lambda^2}, \quad \Pi_2 = \frac{q^2/\varepsilon_0 c^2}{m c^2}$$

so that $\Pi_1 = f(\Pi_2)$ or

$$\sigma_T = \lambda^2 f\left(\frac{q^2}{\varepsilon_0 c^2 m \lambda}\right)$$

This result applied to a single charge $q$, and so cannot possibly depend on the wavelength $\lambda$, so that $f(x) = x^2$ and

$$\sigma_T \sim \left(\frac{q^2}{\varepsilon_0 c m c^2}\right)^2$$

The exact result is: $\sigma_T = \frac{8\pi}{3} \left(\frac{q^2}{4\pi^2 \varepsilon_0 m c^2}\right)^2$
2) Since there are no point charges, we should not use \( \varepsilon_0 \). Instead, we should use either \( \varepsilon \) or \( \bar{n} \) (which is a function of \( \varepsilon \)). Of the 5 parameters \((\sigma_p, d, A, \bar{n}, c)\) only \( c \) has dimension which involves \( T \), so \( c \) can be discarded, leaving 4 parameters and one dimension \((L)\). We therefore have 3 nondimensional combinations:

\[
\Pi_1 = \frac{\sigma_p}{\chi^2}, \quad \Pi_2 = \frac{\chi}{d}, \quad \Pi_3 = \bar{n}
\]

so that \( \Pi_1 = f(\Pi_2, \Pi_3) \) or

\[
\sigma_p = \chi^2 f\left(\frac{A}{d}, \bar{n}\right)
\]

Scattering cross section is proportional to the intensity of the scattered wave (and hence proportional to the square of the electric field \( E \) due to polarization of the dielectric). Polarization field \( E \approx \text{volume} \times d^3/1 \), so that \( \sigma_p \propto E^2 \times d^3/1 \), so that

\[
\sigma_p = \alpha^2 \left(\frac{d}{\lambda}\right)^6 f_1(\bar{n}) = \frac{d^6}{\lambda^4} f_1(\bar{n}) \quad \left[ f_1(\bar{n}) = \frac{2\pi}{3\lambda^2} \left(\frac{\bar{n}^2-1}{\bar{n}^2+2}\right)^2 \right]
\]
3) The dimensions are:
\[ [n] = \text{L}^{-3} \quad [6] = \text{L}^2 \quad [S_{L}] = \text{L} \]
so that \[ \frac{S_{L}^2}{6} = g(n^2 \sigma^3) \]
\[ \Rightarrow S_{L} = \sigma^{1/2} g(n^2 \sigma^3) \]

4) We have \[ \partial_{x} f = D \partial_{x}^2 f \Rightarrow \frac{f}{S_{L}} \sim D \]

Furthermore, scattering being a random walk process, we have \( \Delta \propto n \), since scattering events are uncorrelated. Collecting all the information, we find
\[ S_{L} \sim D^{-1} \propto n^{-1} \Rightarrow S_{L} \sim (\sigma n)^{-1} \]

**Problem 2:**

1) \( d \) and \( g \) only appear as a product \( dg \), so we have 6 parameters \( (\theta, k, d, \partial, \Delta T, p) \) and 4 dimensions \( (M, T, L, \theta) \), yielding \( 6 - 4 = 2 \) non-dimensional combinations.
2) Introducing the scales \( S_L = d \), \( S_\theta = \Delta T \), \( S_t = S_L / S_t \) and \( S_\rho = S_L / S_t \) we obtain

\[
\frac{s_v}{s_t} \left( \partial_t \bar{\nu} + \bar{\nu} \cdot \bar{\nabla} \bar{\nu} \right) = -\frac{s_p \bar{\rho}}{\rho s_L} \bar{\nabla} \bar{p} + \frac{\partial \nu}{s_L} \bar{\nabla}^2 \bar{\nu} + \partial \xi \bar{\delta} \bar{\theta} \bar{\nabla}^2 \bar{\theta} \\
\frac{s_\theta}{s_t} \left( \partial_t \bar{\theta} + \bar{\nu} \cdot \bar{\nabla} \bar{\theta} \right) = \frac{\Delta T}{d} s_v \nu \bar{\nu} + \frac{k}{s_L^2} \bar{\nabla}^2 \bar{\theta}
\]

Dividing by \( s_v / s_t \) (or \( s_\theta / s_t \)) we obtain:

\[
\partial_t \bar{\nu} + \bar{\nu} \cdot \bar{\nabla} \bar{\nu} = -\frac{s_p \bar{\rho}}{\rho s_v^2} \bar{\nabla} \bar{p} + \frac{\partial \nu}{s_v} \bar{\nabla}^2 \bar{\nu} + \frac{\partial \xi s_L}{s_v^2} \bar{\delta} \bar{\theta} \bar{\nabla}^2 \bar{\theta} \\
\partial_t \bar{\theta} + \bar{\nu} \cdot \bar{\nabla} \bar{\theta} = \frac{\Delta T}{s_\theta} s_L \nu \bar{\nu} + \frac{k}{s_L s_v} \bar{\nabla}^2 \bar{\theta}
\]

Choosing \( s_p = \rho s_v^2 \), \( s_v = d / S_L \) we obtain

\[
\partial_t \bar{\nu} + \bar{\nu} \cdot \bar{\nabla} \bar{\nu} = -\bar{\nabla} \bar{p} + \bar{\nabla}^2 \bar{\nu} + \text{Ra} \bar{\theta} \bar{\nabla}^2 \bar{\theta} \\
\partial_t \bar{\theta} + \bar{\nu} \cdot \bar{\nabla} \bar{\theta} = \nu \bar{\nu} + \text{Pr}^{-1} \bar{\nabla}^2 \bar{\theta}
\]

where \( \text{Pr} = \frac{\nu}{\kappa} \) is the Prandtl number

and \( \text{Ra} = \frac{\partial g \Delta T d^3}{\kappa} \) is the Rayleigh number
3) The timescale $S_t = S_L / S_v = S_L^2 / \nu = d^2 / \nu$ is the time it takes for momentum to "diffuse" across the entire fluid layer of depth $d$. This describes the action of viscosity.

The problem has another "diffusion constant" — thermal diffusivity $\kappa$, so there is another timescale $S_t = d^2 / \kappa = Pr \cdot (d^2 / \nu)$, which corresponds to the time temperature diffuses across the fluid layer.

These two timescales determine how quickly perturbations in $\bar{u}$ and $\Theta$ are equilibrated in this problem.

**Problem 3:**

Since $\mathbf{E} = i \hbar \partial_t \Psi$, we can always write

$$\psi(\bar{\mathbf{r}}, t) = \bar{\psi}(\bar{\mathbf{r}}) e^{i \frac{\hbar}{\nu} E t}$$

So from now on we will only concentrate on the spatial part of the wavefunction which is described by the following equation:
\[ E\psi = -\frac{\hbar^2}{2m}\nabla^2 \psi + U\psi + \frac{4\pi^2\hbar^2}{m} \delta(x - y)^2 \psi \]

1) Using the normalization condition we find \( |\psi|^2 = n \), so we can nondimensionalize this equation by introducing the scales \( s_L \) and \( s_\psi \)

\[ E\psi = -\frac{\hbar^2}{2m s_L^2} \nabla^2 \psi + \frac{4\pi^2\hbar^2}{m} \delta(x - y) \psi \]

Dividing by \( E \) and choosing \( s_L^2 = \frac{\hbar^2}{2mE} \), we find

\[ \psi = -\nabla^2 \psi + \varepsilon \psi \]

where \( \varepsilon = \frac{4\pi^2\hbar^2}{mE} \) is a nondimensional parameter which quantifies the ratio of interaction energy to the energy of non-interacting bosons.

In order to have \( |\psi| = \sqrt{n} \), we should have \( \psi \sim e^{i\alpha x} \), with \( \alpha \)-real. Hence, \( \alpha^2 = 1 - \varepsilon \).

In dimensional units

\[ \psi = \sqrt{n} e^{i\alpha x/s_L} = \sqrt{n} \exp \left(i \frac{\sqrt{1 - \frac{4\pi^2\hbar^2}{mE}}}{\hbar / \sqrt{2mE}} x \right) \]
2) It is easy to check that

\[ f = a \tanh \frac{x}{s} \Rightarrow f'' = \frac{2}{a^2 s^2} f^3 - \frac{2}{s^2} f \]

Using non-dimensionalization from part 1:

\[ \psi'' = \psi^4 = \beta |\psi|^3 \psi - \psi \]

and this equation has a solution \( \psi = a \tanh x/s \)
with \( 2/s^2 = 1 \) and \( 2/a^2 s^2 = \beta \) (\( s = \sqrt{2}, \ a = \beta^{-1/2} \)).

In dimensional units

\[ \Psi = \beta^{-1/2} \tanh \left( \pm \frac{x}{\sqrt{2} s_e} \right) = \sqrt{\frac{mE}{4 \pi \hbar^2 a_s}} \tanh \left( \pm \frac{\sqrt{mE}}{\hbar} x \right) \]

This solution only works for \( \beta > 0 \) (\( a_s > 0, \ E > 0 \)).

For \( a_s < 0 \) we use another trial function

\[ f = \frac{a}{\cosh \frac{x}{s}} \Rightarrow f'' = \frac{1}{s^2} f - \frac{2}{a^2 s^2} f^3 \]

This is not a solution for \( E > 0 \) (the signs are wrong). For \( E < 0 \) and \( a_s < 0 \) again \( \beta > 0 \), but now \( \hbar^2/2mE = -1 \), not +1, so that
\[ \nabla^2 \psi = \psi' = -\left( \beta \frac{1}{2} \psi^3 \psi - \psi \right) \]

And \( \psi = a / \cosh(x/s) \) with \( s=1 \) and \( 2/a^2 = \beta \).

In dimensional form:

\[
\psi = \sqrt{\frac{2}{\beta}} \frac{1}{\cosh(\pm \frac{x}{sL})} = \sqrt{\frac{2mE}{4\pi^2 \hbar^2 \alpha}} \frac{1}{\cosh(\pm \frac{\sqrt{2mE}}{\hbar} x)}
\]

Problem 4:

1) The dimensions are:

\([R] = L^2 M T^{-3} A^{-2}, \ [L] = M L^2 T^{-2} A^2, \ [C] = M^{-1} L^{-2} T^4 A^2\)

It is easy to check that \( N = L / R^2 C \) is dimensionless.

2) Introducing scales \( S_Q \) and \( S_T \) we obtain

\[
\frac{L S^2 Q}{S_T^2} \ddot{Q} + \frac{R S_Q}{S_T} \dot{Q} + \frac{S_Q}{c} \dot{Q} = 0
\]

\[
\Rightarrow \ddot{Q} + \frac{R S_T}{L} \dot{Q} + \frac{S_T^2}{L C} \dot{Q} = 0
\]

Choosing \( S_Q = Q_0 \) and \( S_T = L/R \) we obtain

\[ \dot{q} + \dot{q} + 2q = 0, \ q = \eta \]
Alternatively choosing \( S_T = \sqrt{\frac{L}{C}} \), we find instead
\[
\ddot{q} + 3\dot{q} + q = 0, \quad \varepsilon = \frac{1}{2}
\]

Finally, multiplying the equation for \( q \) by \( 1/RST \) we find
\[
\frac{1}{RST} \ddot{Q} + \dot{Q} + \frac{S_T}{RC} Q = 0
\]

Setting \( S_T = RC \) we obtain
\[
3\ddot{q} + \dot{q} + q = 0, \quad \varepsilon = 1
\]

3. The three time scales correspond to:
- \( L/R \) - time to dissipate magnetic field energy into Joule heat.
- \( \sqrt{L/C} \) - period of oscillation in LC circuit (time to convert electric \( \leftrightarrow \) magnetic field energy)
- \( RC \) - time to dissipate electric field energy into Joule heat
4. The resistance is

\[ R = \frac{\text{length}}{\text{area}} \cdot \sigma = \frac{\pi D}{\varepsilon_0 \pi r^2} = \frac{D}{\varepsilon_0 r^2} \]

The capacitance is

\[ C = \varepsilon_0 \cdot \frac{\text{area}}{\text{gap}} = \varepsilon_0 \cdot \frac{\pi r^2}{a} \]

The inductance is

\[ L = \frac{\Phi}{\dot{I}}, \quad \Phi = \oint B \cdot ds \]

Using straight wire result for \( B \):

\[ \Phi = \int_0^{2\pi} dy \int_0^{D/2-a} x \cdot \frac{\mu_0 I}{2 \left( \frac{D}{2} - x \right)} = \frac{\pi}{2} \mu_0 I D \left( \ln \frac{D}{2a} - 1 \right) \]

\[ \Rightarrow L \sim \mu_0 D \ln \frac{D}{a} \]

In nondimensional form, \( \frac{L}{\mu_0 D} = f \left( \frac{D}{a} \right), \quad f(x) = \ln x \)

\[ \Rightarrow \eta = \frac{L}{R C} = \frac{\mu_0 D \ln \frac{D}{a}}{\left( \frac{D}{2a} \right)^2 \cdot \frac{\varepsilon_0 \pi r^2}{a}} \]