OPTIMAL CONTROL OF SPATIOTEMPORAL CHAOS IN COUPLED MAP LATTICES

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Abstract. An optimal control approach is used to develop a localized linear control scheme for Coupled Map Lattices. The optimal arrangement of the control sites is shown to depend on the symmetry properties of the system and the controlled state, while the minimal density of controllers depends on the strength of noise present in the system. The method allows stabilization of a variety of steady and periodic states for arbitrary values of system parameters, which is essential for tracking problems.

Systematic application of control theory methods [1] to coupled map lattices (CML) demonstrating chaotic behavior seems very promising. There is a number of reasons for this. On the one hand, due to the explicit spatiotemporal structure, the behavior of CML's is substantially richer than that of the low-dimensional chaotic systems, although many analytic tools developed for the latter can still be successfully applied. On the other hand, CML's are highly symmetric. This property, important in practical applications, is usually missing in the theory of low-dimensional chaos.

The results obtained for the model systems described by a CML can prove highly beneficial for the study of high-dimensional chaotic [2, 3, 4] and, in particular, spatiotemporally chaotic systems [5], with applications to turbulence [6], instabilities in plasma [7], multi-mode lasers [8] and chemical reaction systems [9] and many other practical problems.

Below we will try to uncover the requirements on the control scheme imposed by different properties of the system studied and the state to be stabilized. We will see, that one of the easiest ways to control an unstable reference state is to apply local feedback through controllers placed at certain lattice sites (we will call them pinnings, following Qu and Hu [5]). In particular, we will obtain the limitations on the minimal number and density of controllers and on their possible arrangement.

Consider the following general coupled map lattice:

$$z_i^{t+1} = f((1-2\epsilon)z_i^t + \epsilon(z_{i-1}^t + z_{i+1}^t)), \tag{1}$$

with $i = 1, 2, \cdots, L$ and periodic boundary conditions $z_{i+L}^t = z_i^t$, or in vector notation:

$$\mathbf{z}^{t+1} = F(\mathbf{z}^t). \tag{2}$$

To be specific, we choose the local map f as a nonlinear function with parameter a,

$$f(z) = az(1-z), \tag{3}$$

but emphasize, that the only result affected by this particular choice is the set of existing periodic trajectories.

Our particular CML has a homogeneous steady state $\hat{z} = 1 - 1/a$, which is unstable for $a > 3.0$ and a variety of unstable time-periodic states embedded in the chaotic attractor. The period-τ trajectory can be found as a steady state solution of the τ-th iterate of the map (1):

$$\hat{\mathbf{z}}^k = F^\tau(\hat{\mathbf{z}}^k), \quad k = 1, 2, \cdots, \tau. \tag{4}$$

It can be easily shown, that due to the translational symmetry of the CML it is impossible to stabilize any unstable trajectory by using the internal parameters of the lattice system (a and ϵ in our case). We can however modify the evolution equation (1) by adding a perturbation $G(\mathbf{u})$, vanishing when no control is applied ($\mathbf{u} = 0$):

$$\mathbf{z}^{t+1} = F(\mathbf{z}^t) + G(\mathbf{u}^t). \tag{5}$$

Linearizing the modified equation around the period-τ periodic orbit $\hat{\mathbf{z}}^1, \hat{\mathbf{z}}^2, \cdots, \hat{\mathbf{z}}^\tau, \hat{\mathbf{z}}^1, \cdots$ we obtain

$$\mathbf{x}^{t+1} = A^t\mathbf{x}^t + B^t\mathbf{u}^t, \tag{6}$$

where we denoted the displacement from the reference trajectory $\mathbf{x}^t = \mathbf{z} - \hat{\mathbf{z}}^t$.

The Jacobian matrix $A_{ij}^k = \partial_j F_i(\hat{\mathbf{z}}^k)$ can be represented as the product $A^k = D^k E$ of the constant coupling matrix

$$E_{ij} = (1-2\epsilon)\delta_{i,j} + \epsilon(\delta_{i,j+1} + \delta_{i,j-1}) \tag{7}$$

and the state-dependent matrix

$$D_{ij}^k = f'((1-2\epsilon)\hat{z}_i^k + \epsilon(\hat{z}_{i-1}^k + \hat{z}_{i+1}^k))\,\delta_{i,j}. \tag{8}$$

The matrix $B_{ij}^k = \partial_j G_i(0)$ defines how the control is applied. If we choose to apply control locally (at the fixed pinning sites i_m), B could be rewritten in the form

$$B_{ij} = \sum_m \delta(j-m)\delta(i-i_m), \tag{9}$$

that depends only on how the pinning sites are placed throughout the lattice. The components u_m of the control vector $\mathbf{u}$ then define the strength of control applied at site i_m, $m = 1, \cdots, M$.

Below (unless otherwise noted) we consider the case of the steady state solution $\hat{\mathbf{z}}$ of (1), which yields constant matrices A and B. In the simplest case of synchronous linear feedback $\mathbf{u}^t = -K\mathbf{x}^t$ equation (6) reduces to

$$\mathbf{x}^{t+1} = (A - BK)\mathbf{x}^t. \tag{10}$$

The state $\mathbf{x} = 0$ can be made stable by a suitable choice of the feedback gain matrix K, if matrices A and B satisfy certain conditions (discussed below). The problem has infinitely many solutions, so in order to make the choice of K unique, we have to impose additional constraints.

Pole placement techniques based on Ackermann's method [10] are inapplicable to the problem of controlling spatially extended systems, because they are numerically unstable [11] and break down rapidly for problems of order greater than 10.

Instead we use the method of the linear-quadratic (LQ) control theory [1], applicable to unstable periodic trajectories as well as steady states. This method is not only numerically stable, but also allows one to *optimize* the control algorithm to increase convergence speed, and at the same time minimize the strength of control. The optimal solution is obtained by minimizing the cost functional

$$V(\mathbf{x}^0) = \sum_{t=0}^{\infty} (\mathbf{x}^{t\dagger} Q \mathbf{x}^t + \mathbf{u}^{t\dagger} R \mathbf{u}^t), \tag{11}$$

where Q and R are the weight matrices, that can be chosen as any positive-definite square matrices.

The minimum of (11) is reached when

$$K = (R + B^\dagger P B)^{-1} B^\dagger P A, \tag{12}$$

where P is the solution to the discrete-time algebraic Ricatti equation

$$P = (Q + A^\dagger P A) - A^\dagger P^\dagger B (R + B^\dagger P B)^{-1} B^\dagger P A, \tag{13}$$

which exists and is unique, if A and B satisfy the *controllability* condition $\mathrm{rank}(C) = L$. The controllability matrix C is defined via

$$C = (B \ AB \ \cdots \ A^{L-1}B). \tag{14}$$

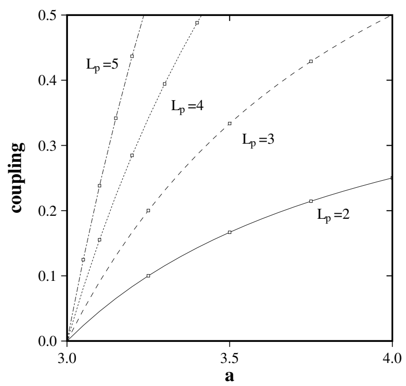

Figure 1. Periodic array of single pinning sites: critical coupling ϵ_{cr} as a function of parameter a. The dots represent the numerical results from figure 2 of Reference [5], with ϵ rescaled by a factor of two to make it compatible with our definition.

For periodic trajectories all the matrices become time-dependent and eq. (13) is replaced by a system of τ iteratively coupled equations.

The controllability condition ensures, that the unstable steady (or periodic) trajectory of the system can be stabilized for the whole range of internal system parameters, where that trajectory exists, using the same control arrangement. For instance, one can easily verify, that the matrix B calculated for a periodic array of single pinning sites $i_m = i_0 + (m-1)L/M$ does not satisfy the controllability condition and therefore spatially homogeneous states are not controllable. Such states could be stabilized however in a restricted area of the parameter space (fig. 1), requiring a prohibitively high density of controllers [12]:

$$L_p = \frac{L}{M} \leq \pi \left[\arccos \left(1 - \frac{a-3}{2\epsilon(a-2)} \right) \right]^{-1}. \tag{15}$$

The numerical results of Hu and Qu [5] support this conclusion completely.

It is possible to extend the limits of the proposed control scheme quite substantially by choosing a different matrix B, i.e. by placing the pinning sites differently. Doing so will enable us to control the system *anywhere* in the parameter space (a, ϵ) at the same time using a *smaller* density of controllers.

First one has to determine the dimensionality of the matrix B, in other words determine the minimal number of parameters required to control the CML (1) of an arbitrary length. It can be shown [13], that the minimal number of parameters required to control a system with degenerate Jacobian is equal to the dimension of the largest irreducible representation contained in the decomposition of the state space representation of the symmetry group of the linearized equation (6).

The coupling matrix (7) has parity symmetry as well as translational symmetry, reduced by the periodic boundary condition, reflecting the symmetries of the original lattice. Since the reference state $\hat{\mathbf{z}}$ in general has a lower symmetry, the symmetry group of (6) is isomorphic to one of the subgroups of C_{Lv} determined by the symmetry of (8).

C_{Lv} has only 1- and 2-dimensional irreducible representations, so the minimal number of control parameters yielding a controllable system for an arbitrary unstable periodic orbit (UPO) in our case is $M = 2$, meaning at least two pinning sites are required. One can easily verify, that the controllability condition is indeed satisfied with an $L \times 2$ matrix

$$B_{ij} = \delta(j-1)\delta(i-i_1) + \delta(j-2)\delta(i-i_2) \tag{16}$$

for a variety of reference states $\hat{\mathbf{z}}$ and for a number of arrangements $\{i_1, i_2\}$, e.g. $i_2 = i_1 + 1$. We choose $i_1 = 1$ and $i_2 = L$ below.

Another complication is noise, limiting our ability to locally control arbitrarily large systems with local interactions, using the minimal number of controllers. Rather simple arguments show [12], that there exist a number of independent theoretical bounds on the size of the controllable system in the presence of noise, such as

$$\begin{aligned} L^{(1)}_{max} &= -\frac{\log(\sigma)}{\lambda_{max}}, \\ L^{(2)}_{max} &= \frac{2\log(\sigma)}{\log(\epsilon) - \lambda_{max}}, \end{aligned} \tag{17}$$

where σ denotes the strength of noise and λ_{max} is the maximal Lyapunov exponent. Similar constraints were obtained by Aranson, Golomb and Sompolinsky for the lattices with asymmetric coupling [14].

Numerical simulations show, that the CML (1) of a moderate size L can indeed be stabilized by the LQ method using the minimal number of controllers (two) in a wide range of parameters a and ϵ (see fig. 3(a) for an example of control of the unstable steady uniform state). The maximal length $L_{max}(\sigma)$ of the system can be obtained by choosing the fixed point as the initial condition and monitoring the evolution of the system in the presence of noise under control (12), calculated by solving eq. (13) with e.g. $Q = I_{L\times L}$ and $R = I_{M\times M}$. This length is quite large for moderate noise level (fig. 2) and agrees with the estimate (17) rather well.

The problem of controlling a large 1-dimensional system with the length $L > L_{max}(\sigma)$, exceeding the maximum allowed for a given noise level, can be easily reduced to the problem of controlling a number of smaller systems with the length $L_p < L_{max}(\sigma)$ [12]. We partition the entire lattice

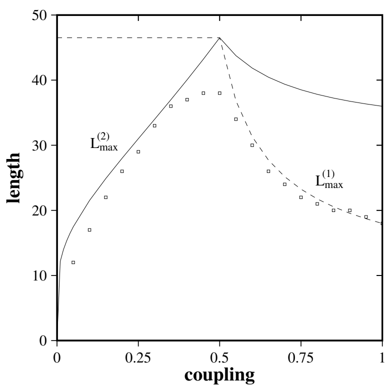

Figure 2. The largest length of the lattice which can be controlled with two pinning sites: theoretical estimates (solid lines) and numerical results (dots) obtained with the uniformly distributed noise of amplitude $\sigma = 10^{-14}$ as functions of coupling ϵ for $a = 4.0$.

$\{z_1^t, \cdots, z_L^t\}$ into $M = L/L_p$ subdomains $\{z^t_{(m-1)L_p+1}, \cdots, z^t_{mL_p}\}$, and control it with an array of pinning sites $i_{m1} = (m-1)L_p + 1$, $i_{m2} = mL_p$, $m = 1, \cdots M$, positioned periodically at the boundaries of subdomains.

The stabilization can be achieved by adjusting the feedback gain

$$\begin{aligned}
u^t_{i_{m1}} &= F(z^t_{i_{m2}}, z^t_{i_{m1}}, z^t_{i_{m1}+1}) - F(z^t_{i_{m1}-1}, z^t_{i_{m1}}, z^t_{i_{m1}+1}) \\
+ & \prod_{i=1}^{L_p} \theta(\delta x_i - |x^t_{(m-1)L_p+i}|) \sum_{i=1}^{L_p} K_{1i} x^t_{(m-1)L_p+i}, \\
u^t_{i_{m2}} &= F(z^t_{i_{m2}-1}, z^t_{i_{m2}}, z^t_{i_{m1}}) - F(z^t_{i_{m2}-1}, z^t_{i_{m2}}, z^t_{i_{m2}+1}) \\
+ & \prod_{i=1}^{L_p} \theta(\delta x_i - |x^t_{(m-1)L_p+i}|) \sum_{i=1}^{L_p} K_{2i} x^t_{(m-1)L_p+i}
\end{aligned} \tag{18}$$

to nonlinearly decouple adjacent subdomains (first two terms). Here $\theta(x)$ is a step-function. One can easily see, that in order to calculate the perturbations $u_{i_{m1}}$ and $u_{i_{m2}}$ one needs to know the state of the system only in the immediate vicinity (up to L_p sites away) of the respective controllers. Eqs. (17) give the minimal density of pinning sites.

We demonstrate this approach by stabilizing a number of UPO's of the CML, defined by equations (1,3) with $a = 4.0$ and $\epsilon = 0.33$. In particular, we studied the steady homogeneous state (S1T1), the time-period-2 homogeneous (S1T2), and the time-period-3, space-period-8 (S8T3) orbits. $L = 128$ sites were divided into $M = 16$ subdomains of length $L_p = 8$, each controlled by two pinning sites. The results presented in figs. 3(b) and 4 show the evolution of the system from the initial condition, chosen to be a collection of random numbers in the interval $[0, 1]$, with the nonlinear decoupling control (18).

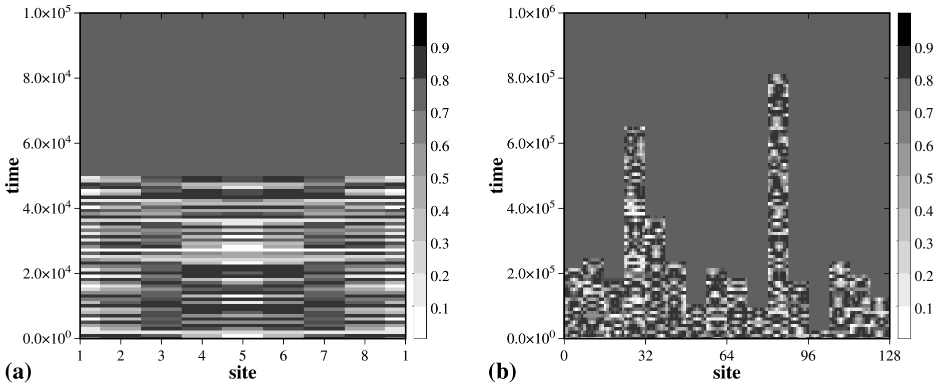

Figure 3. Stabilizing uniform steady state (S1T1): (a) lattice with the length $L = 8$ is controlled by two controllers placed at the sites $i_1 = 1$ and $i_2 = 8$; (b) lattice with the length $L = 128$ is controlled by an array of double pinning sites, placed at the boundaries of subdomains with length $L_p = 8$. The state of the system is plotted at each 10^4-th step.

To summarize, we have shown, that symmetries of the controlled system impose a number of specific constraints on the control scheme, rendering the control more difficult. In particular, it is not possible to control the system perturbing its internal parameters, such as coupling ϵ and local stretching factor a. The dynamics of the system should be modified to effectively break the symmetry, enabling the control of symmetric unstable trajectories.

This can be achieved by placing an array of controllers throughout the lattice. The versatility and efficiency of the control scheme can be improved significantly, if one uses double pinnings instead of single ones. The UPO's become controllable for any values of the internal system parameters and the minimal density of pinning sites is reduced substantially. It is shown, that the maximal distance between pinnings depends on the strength of noise in the system and can be estimated analytically.

Tracking the UPO's in the phase space and/or switching between different orbits is accomplished by readjusting the feedback gain K, without changing the placement of controllers. This stresses the importance of the optimal choice of the control matrix B for systems with symmetries.

And finally, we may suggest, that the dynamics of extended spatiotemporally chaotic systems with local interactions can generically be controlled in such a way, that the information about the state of the system is collected and applied locally. The size of this local region though may depend on the strength of noise as well as the strength of interactions in the system.

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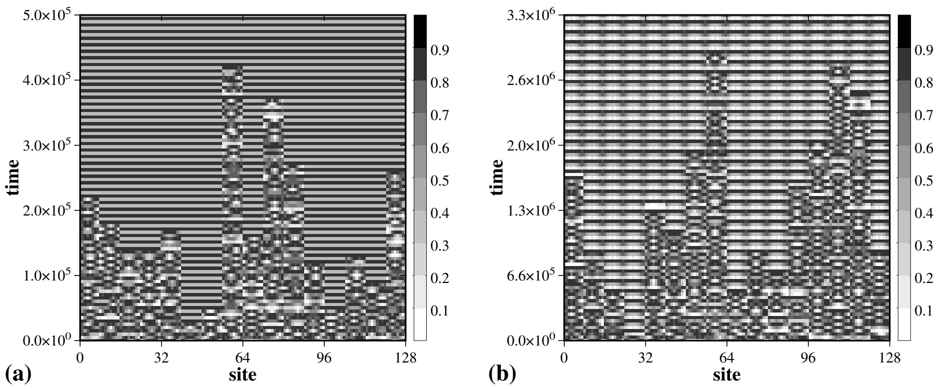

Figure 4. Stabilizing periodic trajectories: lattice with the length $L = 128$ is controlled by an array of double pinning sites, placed at the boundaries of subdomains with length $L_p = 8$. (a) period-2 UPO (S1T2), 2 consecutive states of the system are plotted after skipping each 10^4 steps; (b) period-3 UPO (S8T3), 3 consecutive states of the system are plotted after skipping each 10^5 steps.

References

1. P. Dorato, C. Abdallah and V. Cerrone, (1995) *Linear-Quadratic Control: An Introduction*, Prentice Hall, New Jersey.
2. M. Ding, W. Yang, V. In, W. L. Ditto, M. L. Spano and B. Gluckman (1996) Controlling chaos in high dimensions: Theory and Experiment, *Phys. Rev. E* **53**, 4334-4344.
3. Y. C. Lai and C. Grebogi (1994) Synchronization of spatiotemporal chaotic systems by feedback control, *Phys. Rev. E* **50**, 1894-1899.
4. J. Warncke, M. Bauer and W. Martienssen (1994) Multiparameter control of high-dimensional chaotic systems, *Europhys. Lett.* **25**, 323-328.
5. G. Hu and Z. Qu (1994) Controlling spatiotemporal chaos in coupled map lattice systems, *Phys. Rev. Lett.* **72**, 68-71.
6. C. Lee, J. Kim, D. Bobcock and R. Goodman (1997) Applications of neural networks to turbulence control for drag reduction, *Physics of Fluids* **9**, 1740-1747.
7. A. Pentek, J. B. Kadtke and Z. Toroczkai (1996) Stabilizing chaotic vortex trajectories - An example of high-dimensional control, *Phys. Lett. A* **224**, 85-92.
8. P. Colet, R. Roy and K. Weisenfeld (1994) Controlling hyperchaos in a multimode laser model, *Phys. Rev. E* **50**, 3453-3457.
9. V. Petrov, M. J. Crowley and K. Showalter (1994) Tracking unstable periodic orbits in the Belousov-Zhabotinsky reaction, *Phys. Rev. Lett.* **72**, 2955-2958.
10. E. Barreto and C. Grebogi (1995) Multiparameter control of chaos, *Phys. Rev. E* **52**, 3553-3557.
11. J. Kautsky, N. K. Nichols and P. Van Dooren (1985) Robust pole assignment in linear state feedback, *Intl. J. Control* **41**, 1129-1155.
12. R. O. Grigoriev, M. C. Cross and H. G. Schuster (1997) Pinning control of spatiotemporal chaos, submitted to *Phys. Rev. Lett.*
13. R. O. Grigoriev and M. C. Cross (1997) Controlling physical systems with symmetries, submitted to *Phys. Rev. E.*
14. I. Aranson, D. Golomb and H. Sompolinsky (1992) Spatial coherence and temporal chaos in macroscopic systems with asymmetric couplings, *Phys. Rev. Lett.* **68**, 3495-3498.