APPENDIX C

THEOREMS USED IN THE PROOFS

We use successive approximation method to prove the existence of modulated amplitude waves. Below are listed several theorems from the theory of nonlinear oscillations taken from Hale's monograph [113].

Consider the system of equations

$$\dot{z} = Az + \epsilon Z(\tau, z, \epsilon) \tag{166}$$

where A is a constant matrix, $\epsilon, \tau \in \mathbb{R}$, and $z, Z \in \mathbb{R}^n$. Z is a continuous function of τ, z, ϵ, periodic in τ of period T. In the following, we only consider the case that Z is a smooth function. Without loss of generality, A can always be assumed to have the standard form

$$A = \mathrm{diag}(0_p, B),$$

Where 0_p is a $p \times p$ zero matrix and B is a constant matrix with the property that the equation $\dot{y} = By$ has no nontrivial periodic solution of period T. Under these settings, if the successive approximation is applied to (166), we have

Theorem C.0.1 *Given $d > b > 0$, there is an $\epsilon_1 > 0$ such that for any given constant p vector $a, \|a\| < b$ and real $\epsilon, |\epsilon| < \epsilon_1$, there is a unique function*

$$z^*(\tau) = z(\tau, a, \epsilon), \textit{with} \sup_\tau \|z^*(\tau)\| < d$$

which has continuous first derivative with respect to τ and satisfies

$$\dot{z^*} = Az^* + \epsilon Z(\tau, z^*, \epsilon) - \epsilon P_0 Z(\tau, z^*, \epsilon).$$

Furthermore, $z(\tau, a, 0) = a^*$, $a^* = \mathrm{col}(a, 0)$, $P_0(z^*) = a^*$, and $z(\tau, a, \epsilon)$ has continuous first derivatives with respect to a, ϵ.

P_0 is defined as a projection operator on the Banach space S of continuous periodic functions of period T. If $f \in S$, write $f = \mathrm{col}(g, h)$ where g is a p vector and h is a $n - p$ vector, then

$$P_0(f) = \mathrm{col}\left(T^{-1} \int_0^T g(t)\, dt, 0\right)$$

So, P_0 brings an element f in S to a constant vector which has the average values of g over one period as the first p components and zeros as the rest components. The equation satisfied by z^* is different from (166) by a constant vector. By a proper choice of the starting vector a, we may make this constant vector zero to obtain a solution for the system (166). The mathematical statement is give by the following theorem.

Theorem C.0.2 *Let $z(\tau, a, \epsilon)$ be the function given by the Theorem C.0.1 for all $\|a\| \le b < d, |\epsilon| \le \epsilon_1$. If there exist an $\epsilon_2 \le \epsilon_1$ and a continuous function $a(\epsilon)$ such that*

$$P_0 Z(\tau, z(\tau, a(\epsilon), \epsilon), \epsilon) = 0, \textit{ with } \|a(\epsilon)\| \le b \textit{ for } |\epsilon| \le \epsilon_2 \tag{167}$$

then $z(\tau, a(\epsilon), \epsilon)$ is a periodic solution of system (166) for $\|\epsilon\| \le \epsilon_2$. Conversely, if system (166) has a periodic solution $\bar{z}(\tau, \epsilon)$, of period T, $\|\bar{z}(\tau, \epsilon)\| \le d, |\epsilon| \le \epsilon_2$, then $\bar{z}(\tau, \epsilon) = z(\tau, a(\epsilon), \epsilon)$.

Therefore, the existence of a continuous function $a(\epsilon)$ satisfying (167) is a necessary and sufficient condition for the existence of a periodic solution of system (166) of period T. As we do not know the exact functional form of the periodic solution, the condition (167) could not be solved explicitly. But by using implicit function theorem, we can show that the substitution into (167) of a proper approximate function of $z(\tau, a, \epsilon)$ leads to the existence condition for periodic solutions.

Theorem C.0.3 *In the system (166), let*

$$Z = \mathrm{col}(X, Y), \quad z = \mathrm{col}(x, y)$$

where X, x are p vectors and define

$$X_0(x, y, \epsilon) = \frac{1}{T} \int_0^T X(\tau, x, y, \epsilon) d\tau.$$

If there is a p vector $a_0, \|a_0\| < d$, such that

$$X_0(a_0, 0, 0) = 0, \quad \det \left[\frac{\partial X_0(a_0, 0, 0)}{\partial x}\right] \ne 0 \tag{168}$$

then there exists an $\epsilon_1 > 0$ and a periodic function $z(\tau, \epsilon), |\epsilon| \le \epsilon_1$, of system (166) of period T with $z(\tau, 0) = \mathrm{col}(a_0, 0)$.

If we need to determine other parameters as a function of ϵ in practical applications, similar theorems could be derived. Specifically, in the main text we consider the period T as a function of ϵ. It is clear that theorem C.0.3 applies if we suppose $T(\epsilon)$ is continuous in ϵ and bounded for $|\epsilon| \le \epsilon_1$. Furthermore, despite the use of the zeroth approximation in the above theorem, the nth approximation could be used instead. If *simple* (non-vanishing determinant) solutions to the determining equations can be found for ϵ in the neighborhood of 0 then system (166) has a periodic solution.

If the system which we are studying possesses certain symmetries, we can prove the existence of particular symmetric solutions by a simplified version of determining equations. Let us define the symmetry first.

Definition C.0.1 *Let $\dot{z} = f(\tau, z)$, where $z, f \in \mathbb{R}^n$, be a system of differential equations. It is said to have the property E with respect to Q if there exists a nonsingular matrix Q such that*

$$Q^2 = I \quad Qf(-\tau, Qz) = -f(\tau, z) \quad QP_0 = P_0 Q$$

where P_0 is the projection operator defined before.

Under this symmetry assumption the following theorems apply:

Theorem C.0.4 *Suppose* $Q = \mathrm{diag}(Q_1, Q_2)$ *where* Q_1 *is a* $p \times p$ *matrix. If system (166) has property E with respect to this* Q *for all* ϵ*. If* $a, \|a\| \le b$*, is a* p *vector and* $a^* = \mathrm{col}(a, 0)$ *is a* n *vector, chosen in such a way that* $Qa^* = a^*$*, then the solution* $z(\tau, a, \epsilon)$ *satisfies the relation*

$$Qz(-\tau, a, \epsilon) = z(\tau, a, \epsilon)$$

and consequently,

$$Z(-\tau, z(-\tau, a, \epsilon), \epsilon) = -Qz(\tau, z(\tau, a, \epsilon), \epsilon)$$

Theorem C.0.5 *If the* j*-th element of the diagonal of the matrix* Q_1 *in Theorem C.0.4 is* $+1$*, then the* j*-th equation in the determining equations is equal to zero for every vector* a^* *in Theorem C.0.4.*

The system (94) derived here from the 1-D CGLe has this symmetry, so the number of determining equations can be reduced using these two theorems.